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Dynamics and stability results for impulsive type integro-differential equations with generalized fractional derivative

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Abstract

In this paper, we investigate the existence, uniqueness, and Ulam stability of solutions for impulsive type integro-differential equations with generalized fractional derivative. The arguments are based upon the Banach contraction principle and Schaefer's fixed point theorem.

Keywords: Integro-differential equations, impulsive differential equations, generalized fractional derivative, existence, Ulam-Hyers stablity.

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1. Introduction

Fractional differential equations (FDEs) have attracted greater interest in latest years. Fractional derivative is the exceptional tool for describing memory and the hereditary properties of numerous materials and processes, and plenty of FDEs applications are found in viscoelasticity, manage, porous media, electromagnetism, and so forth; see the books [8, 15]. These days generalization of the derivatives of both Riemann-Liouville and Caputo types are introduced and proven the impact of utilizing it in equations of mathematical physics or related to probability. This turned into completed using the definition of generalized fractional derivatives given through Katugampola [12]. The author initiated a new fractional integral, which generalizes the Riemann-Liouville and the Hadamard integrals into a single form. Later, Katugampola [10] delivered a new fractional derivative, which generalizes the two derivatives in query. For more details on the generalized fractional derivative, on can refer to the papers [11, 16, 17]. Inspired through the papers [7, 16], we take a look at generalized fractional derivative (or Katugampola-Caputo derivative) for impulsive type integro-differential equations.

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We consider the following impulsive type integro-differential equation with generalized fractional derivative of the form

$$ {}_{c}^{\rho}D_{0^+}^{\alpha} x(t) = f\left(t, x(t), \int_0^t h(t,s,x(s))ds\right), \quad t \in J := [0,T], \quad t \neq t_k, \tag{1.1} $$

$$ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \tag{1.2} $$

$$ x(0) = x_0, \tag{1.3} $$

where $\alpha \in \mathbb{R}^+$, $\rho > 0$, ${}_{c}^{\rho}D_{0^+}^{\alpha}$ is the generalized fractional derivative in Caputo sense, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : \Delta \times \mathbb{R} \to \mathbb{R}$ are given functions, $I_k : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, $0 < t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, and $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Here, $\Delta = \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$. For sake of brevity, we take

$$ Hx(t) = \int_0^t h(t,s,x(s))ds. $$

On the other hand, impulsive differential equations have received much attention; we refer the reader to the books [4, 6] and the papers [13, 21], and the references therein. Some very recent results concerning stability of ordinary differential equations and FDEs can be found in [5, 19] and the references therein. In latest years, many researchers have focused at the examine of the Hyers-Ulam stability of FDEs; one can refer to [14, 18, 19]. However, the study of the Hyers-Ulam stability of FDEs is still in the initial stage. Motivated by the works mentioned above, we explore the existence and Ulam-stability of the problem (1.1)-(1.3) by adopting the idea of [2, 3, 9] and present some sufficient conditions in order that these equations have the Ulam-Hyers stability.

2. Preliminaries

In this section, we will introduce some basic definitions, notations, lemmas and theorems that will be used in the proofs of main results. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$ \|x\|_\infty := \sup\{|x(t)| : t \in J\}. $$

Consider the set of functions

$$ \begin{aligned} PC(J,\mathbb{R}) = \{x \in J \to \mathbb{R} : \ & x \in C\left((t_k, t_{k+1}], \mathbb{R}\right), \ k = 0, \dots, m \quad \text{and there exist} \\ & x(t_k^-) \quad \text{and} \quad x(t_k^+), \ k = 1, \dots, m \quad \text{with} \quad x(t_k^-) = x(t_k)\big\}. \end{aligned} $$

This set is a Banach space with the norm

$$ \|x\|_{PC} = \sup_{t \in J} |x(t)|. $$

Set $J' := [0,T]|\{t_1, \dots, t_m\}$.

Definition 2.1. The Riemann-Liouville fractional integral and derivative of order $\alpha \in \mathbb{C}$, $Re(\alpha) \geqslant 0$ are given by

$$ \left(I_{0^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds $$

and

$$ \left(D_{0^+}^{\alpha} f\right)(t) = \left(\frac{d}{dt}\right)^n \left(I_{0^+}^{n-\alpha} f\right)(t), \quad t > 0, $$

respectively, where $n = [Re(\alpha)]$ and $\Gamma(\alpha)$ is the Gamma function.

Definition 2.2. The Hadamard fractional integral and derivative are given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}$$

and

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_0^t \left(\log \frac{t}{s}\right)^{n-\alpha+1} f(s) \frac{ds}{s},$$

respectively, for $t > 0$ and $Re(\alpha) > 0$.

Now we give the definitions of the generalized fractional operators introduced in [10, 12].

Definition 2.3. The generalized left-sided fractional integral ${}^{\rho}I_{0^+}^{\alpha} f$ of order $\alpha \in \mathbb{C}$ $(Re(\alpha))$ is defined by

$$\left({}^{\rho}I_{0^+}^{\alpha} f\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s) ds \tag{2.1}$$

for $t > 0$, if the integral exists.

The generalized fractional derivative, corresponding to the generalized fractional integral (2.1), is defined for $t > 0$, by

$$\left({}^{\rho}D_{0^+}^{\alpha} f\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-1)} \left(t^{1-\rho} \frac{d}{dt}\right)^n \int_0^t (t^{\rho} - s^{\rho})^{n-\alpha-1} s^{\rho-1} f(s) ds, \tag{2.2}$$

if the integral exists.

Definition 2.4. The Caputo-type generalized fractional derivative ${}_{c}^{\rho}D_{0^+}^{\alpha}$ is defined via the above generalized fractional derivative (2.2) as follows

$${}_{c}^{\rho}D_{0^+}^{\alpha} f(t) = \left({}^{\rho}D_{a^+}^{\alpha} \left[f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} (s)^k\right]\right)(t),$$

where $n = \lceil Re(\alpha) \rceil$.

Definition 2.5. A function $x \in PC(J, \mathbb{R})$ whose α-derivative exists on J' is said to be a solution of (1.1)-(1.3) if x satisfies the equation ${}_{c}^{\rho}D_{0^+}^{\alpha} x(t) = f(t, x(t), Hx(t))$ on J' and satisfy the conditions

$$\Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \dots, m, \quad x(0) = x_0.$$

Lemma 2.6. *Let $\alpha \in \mathbb{R}^+$, $\rho > 0$, then the differential equation*

$${}_{c}^{\rho}D_{0^+}^{\alpha} h(t) = 0$$

has solutions $h(t) = c_0 + c_1(\frac{t^{\rho}}{\rho}) + c_2(\frac{t^{\rho}}{\rho})^2 + \cdots + c_{n-1}(\frac{t^{\rho}}{\rho})^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

Lemma 2.7. *Let $\alpha \in \mathbb{R}^+$, $\rho > 0$, then*

$${}^{\rho}I_{0^+}^{\alpha}\, {}_{c}^{\rho}D_{0^+}^{\alpha} h(t) = h(t) + c_0 + c_1 \left(\frac{t^{\rho}}{\rho}\right) + c_2 \left(\frac{t^{\rho}}{\rho}\right)^2 + \cdots + c_{n-1} \left(\frac{t^{\rho}}{\rho}\right)^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n-1$, $n = [\alpha] + 1$.

As a consequence of Lemma 2.6 and Lemma 2.7 we have the following results which is useful in what follows.

Lemma 2.8. *Let $\alpha \in \mathbb{R}^+$, $\rho > 0$, and let $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h : \Delta \times \mathbb{R} \to \mathbb{R}$ be continuous. A function x is a solution of the integral equation*

$$x(t) = \begin{cases} x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds, & \text{if } t \in [0, t_1], \\ x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds & \\ \quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds + \sum_{i=1}^k I_i(x(t_i^-)), & \text{if } t \in (t_k, t_{k+1}], \end{cases} \tag{2.3}$$

where $k = 1, \dots, m$ if and only if x is a solution of the problem (1.1)-(1.3).

Proof. Assume x satisfies (1.1)-(1.3). If $t \in [0, t_1]$, then

$$ {}^\rho_c D^\alpha_{0^+} x(t) = f(t, x(t), Hx(t)).$$

Lemma 2.7 implies

$$x(t) = x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds.$$

If $t \in (t_1, t_2]$, then Lemma 2.7 implies

$$\begin{aligned} x(t) &= x(t_1^+) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds \\ &= \Delta x|_{t=t_1} + x(t_1^-) + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds \\ &= I_1(x(t_1^-)) + x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds. \end{aligned}$$

If $t \in (t_2, t_3]$, then from Lemma 2.7 we get

$$\begin{aligned} x(t) &= I_2(x(t_2^-)) + I_1(x(t_1^-)) + x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t_1^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_2}^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 2.7 we get (2.3).

Converse is also true. Hence, we omit the proof. □

Lemma 2.9 ([20]). *Let $V : [0, T] \to [0, +\infty)$ be a real function and $w(\cdot)$ is a non-negative, locally integrable function on $[0, T]$ and there are constants $a > 0$, $\alpha \in \mathbb{R}^+$, $\rho > 0$ such that*

$$V(t) \leqslant w(t) + a \int_0^t (t^\rho - s^\rho)^{-\alpha} s^{-\rho} v(s) ds.$$

Then there exists a constant $K = K(\alpha)$ such that

$$V(t) \leqslant w(t) + K(a) \int_0^t (t^\rho - s^\rho)^{-\alpha} s^{-\rho} w(s) ds \quad \textit{for every } t \in [0, T].$$

Balnov and Hristova [1] introduced the following inequality of Gronwall type for piecewise continuous which can be used in the sequel.

Lemma 2.10. *Let for* $t \geqslant t_0 \geqslant 0$ *the following inequality holds*

$$x(t) \leqslant a(t) + \int_{t_0}^{t} g(t,s)x(s)ds + \sum_{0<t_k<t} \beta_k(t)x(t_k),$$

where $\beta_k(t)$ $(k \in \mathbb{N})$ *are nondecreasing functions for* $t \geqslant t_0$*,* $a \in PC([0,\infty),\mathbb{R}_+)$*,* a *is nondecreasing and* $g(t,s)$ *is a continuous nonnegative function for* $t, s \geqslant t_0$ *and nondecreasing with respect to* t *for any fixed* $s \geqslant t_0$*. Then, for* $t \geqslant t_0$*, the following inequality is valid:*

$$x(t) \leqslant a(t)\Pi_{t_0<t_k<t}(1+\beta_k(t)) \exp\left(\int_0^t g(t,s)ds\right).$$

Theorem 2.11 (Schaefer's fixed point theorem)**.** *Let* X *be a Banach space, and* $N : X \to X$ *completely continuous operator. If the set* $\zeta = \{x \in X : \ x = \delta N(x), \text{ for some } \delta \in (0,1)\}$ *is bounded, then* N *has fixed points.*

3. Existence results

This section deals with the existence of solutions for the problem (1.1)-(1.3). Before stating and proving the main results, we introduce the following assumptions.

(A1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(A2) There exists a constant $l > 0$ such that $|f(t,u,v) - f(t,\overline{u},\overline{v})| \leqslant l(|u-\overline{u}| + |v-\overline{v}|)$ for each $t \in J$, and each $u, v, \overline{u}, \overline{v} \in \mathbb{R}$.

(A3) There exists a constant $l^* > 0$ such that $|I_k(u) - I_k(\overline{u})| \leqslant l^* |u-\overline{u}|$ for each $u, \overline{u} \in \mathbb{R}$ and $k = 1, \ldots, m$.

(A4) There exists a constant $M > 0$ such that $|f(t,u,v)| \leqslant M$ for each $t \in J$ and each $u, v \in \mathbb{R}$.

(A5) The functions $I_k : \mathbb{R} \to \mathbb{R}$ are continuous and there exists a constant $M^* > 0$ such that $|I_k(u)| \leqslant M^*$ for each $u \in \mathbb{R}$, $k = 1, \ldots, m$.

(A6) The function $h : \Delta \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a constant $H_1 > 0$ such that

$$|h(t,s,u) - f(t,s,\overline{u})| \leqslant H_1 |u-\overline{u}|, \quad \forall\, u, \overline{u} \in \mathbb{R}.$$

Now, we transform the problem (1.1)-(1.3) into a fixed point problem. Let $Z = PC(J,\mathbb{R})$. Consider the operator $N : Z \to Z$ defined by

$$\begin{aligned} N(x)(t) = x_0 &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s,x(s),Hx(s))ds \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s,x(s),Hx(s))ds + \sum_{0<t_k<t} I_k(x(t_k^-)). \end{aligned} \tag{3.1}$$

Firstly we prove that the operator N defined by (3.1) verifies the condition of Theorem 2.11.

Lemma 3.1. *The operator* N *is continuous.*

Proof. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in Z. Then for each $t \in J$

$$\begin{aligned} |N(x_n)(t) - N(x)(t)| \leqslant &\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |f(s,x_n(s),Hx_n(s)) - f(s,x(s),Hx(s))|\, ds \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |f(s,x_n(s),Hx_n(s)) - f(s,x(s),Hx(s))|\, ds \\ &+ \sum_{0<t_k<t} \left|I_k(x_n(t_k^-)) - I_k(x(t_k^-))\right|. \end{aligned}$$

Since f and I_k, $k = 1, \dots, m$ are continuous functions, we have

$$\|N(x_n) - N(x)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

This proves the continuity of N. □

Lemma 3.2. *The operator N maps bounded sets into bounded sets in Z.*

Proof. We need to prove that for any $\eta^* > 0$, there exists a positive constant r such that for each $x \in B_{\eta^*} = \{x \in Z : \ \|x\| \leqslant \eta^*\}$, we have $\|N(x)\|_\infty \leqslant r$. By (A4) and (A5) we have for each $t \in J$,

$$\begin{aligned}|N(x)(t)| &\leqslant |x_0| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |f(s, x(s), Hx(s))| \, ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |f(s, x(s), Hx(s))| \, ds + \sum_{0<t_k<t} \left| I_k(x(t_k^-)) \right| \\ &\leqslant |x_0| + \frac{MT^{\rho\alpha}(m+1)}{\rho^\alpha \Gamma(\alpha+1)} + mM^*.\end{aligned}$$

Thus

$$\|N(x)\|_\infty \leqslant |x_0| + \frac{mMT^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{MT^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} + mM^* := r,$$

that is $N(x)$ is bounded. □

Lemma 3.3. *The operator N maps bounded sets into equicontinuous sets of Z.*

Proof. Let $t_1, t_2 \in J$, $t_1 < t_2$, B_{η^*} be a bounded set of Z as in Lemma 3.2, and let $x \in B_{\eta^*}$. Then

$$\begin{aligned}|N(x)(t_1) - N(x)(t_2)| &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_1^\rho - s^\rho)^{\alpha-1} s^{\rho-1} - (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right| |f(s, x(s), Hx(s))| \, ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left| (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right| |f(s, x(s), Hx(s))| \, ds + \sum_{0<t_k<t_2-t_1} \left| I_k(x(t_k^-)) \right| \\ &\leqslant \frac{M\rho^{1-\alpha}}{\Gamma(\alpha)} \left(\int_0^{t_1} \left| (t_1^\rho - s^\rho)^{\alpha-1} s^{\rho-1} - (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right| ds + \int_{t_1}^{t_2} \left| (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right| ds \right) \\ &\quad + \sum_{0<t_k<t_2-t_1} \left| I_k(x(t_k^-)) \right|.\end{aligned}$$

The second integral in the right-hand side of the last inequality has the value $\frac{1}{\rho\alpha}(t_2^\rho - t_1^\rho)^\alpha$.

For the first integral, consider the three cases $\alpha < 0$, $\alpha = 0$, and $\alpha > 1$, separately. In the case $\alpha = 1$, the integral has the zero value.

For $\alpha < 1$, we have $(t_1^\rho - s^\rho)^{\alpha-1} \geqslant (t_2^\rho - s^\rho)^{\alpha-1}$. Thus,

$$\begin{aligned}\int_0^{t_1} \left| (t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right| s^{\rho-1} ds &= \int_0^{t_1} \left[(t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} ds \\ &= \frac{1}{\rho\alpha}(t_1^\rho \alpha - t_2^\rho \alpha) + \frac{1}{\rho\alpha}(t_2^\rho - t_1^\rho)^\alpha = \frac{1}{\rho\alpha}(t_2^\rho - t_1^\rho)^\alpha.\end{aligned}$$

Finally, if $\alpha > 1$, then $(t_1^\rho - s^\rho)^{\alpha-1} \leqslant (t_2^\rho - s^\rho)^{\alpha-1}$ and hence

$$\int_0^{t_1} \left| (t_1^\rho - s^\rho)^{\alpha-1} - (t_2^\rho - s^\rho)^{\alpha-1} \right| s^{\rho-1} ds = \int_0^{t_1} \left[(t_1^\rho - s^\rho)^{\alpha-1} - (t_1^\rho - s^\rho)^{\alpha-1} \right] s^{\rho-1} ds$$

$$= \frac{1}{\rho\alpha}(t_2^{\rho}\alpha - t_1^{\rho}\alpha) - \frac{1}{\rho\alpha}(t_2^{\rho} - t_1^{\rho})^{\alpha} = \frac{1}{\rho\alpha}(t_2^{\rho}\alpha - t_1^{\rho}\alpha).$$

Combining these results, we have

$$|N(x)(t_1) - N(x)(t_2)| \leqslant \begin{cases} \frac{2C_0}{\rho^{\alpha}\Gamma(\alpha+1)}(t_2^{\rho} - t_1^{\rho})^{\alpha}, & \text{if} \quad \alpha \leqslant 1, \\ \frac{C_0}{\rho^{\alpha}\Gamma(\alpha+1)}\left[(t_2^{\rho} - t_1^{\rho})^{\alpha} + t_2^{\rho\alpha} - t_1^{\rho\alpha}\right], & \text{if} \quad \alpha > 1, \end{cases}$$

As $t_1 \to t_2$, the right-hand side of above inequality tends zero. As a consequence of Lemmas 3.1-3.3 together with Arzela-Ascoli theorem, we conclude that $N : Z \to Z$ is completely continuous. □

Now, we have the possibility to prove the main results of this section.

Theorem 3.4 (Existence of solution)**.** *Assume that* (A1), (A4), (A5) *hold. Then, the problem* (1.1)-(1.2) *has at least one solution* $x \in Z$ *and the set of the solutions of the problem* (1.1)-(1.3) *is bounded in* Z.

Proof. Let $N : Z \to Z$ be the operator defined in the beginning of this section. It is continuous and bounded (see Lemmas 3.1-3.3) .

Set $\zeta = \{x \in Z : \ x = \delta N(x) \quad \text{for some} \quad 0 < \delta < 1\}$. Next, we prove that N is bounded in Z. Let $x \in \zeta$, then $x = \delta N(x)$ for some $0 < \delta < 1$. Thus, for each $t \in J$, we have

$$\begin{aligned} \|x\|_{\infty} \leqslant \delta \Bigg[& x_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s, x(s), Hx(s)) ds + \sum_{0<t_k<t} I_k(x(t_k^-)) \Bigg] \\ \leqslant & |x_0| + \frac{MT^{\rho\alpha}(m+1)}{\rho^{\alpha}\Gamma(\alpha+1)} + mM^* := R. \end{aligned}$$

This shows that the set ζ is bounded. As a consequence of Theorem 2.11, we deduce that N has a fixed point which is a solution of the problem (1.1)-(1.3). □

Finally, we give the following uniqueness results.

Theorem 3.5. *Assume that* (A1), (A2)*, and* (A6) *hold. If*

$$\left(\frac{l(1+H_1)(m+1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + ml^* \right) < 1, \tag{3.2}$$

then the problem (1.1)-(1.3) *has a unique solution on* J.

Proof. Consider the operator $N : Z \to Z$ defined by (3.1). Clearly, the fixed points of the operator N are solutions of the problem (1.1)-(1.3). We shall use the Banach contraction principle to prove that N has a fixed point. We shall show that N is a contraction. Let $x, y \in Z$. Then, for each $t \in J$ we have

$$\begin{aligned} |N(x)(t) - N(y)(t)| \leqslant & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} |f(s, x(s), Hx(s)) - f(s, y(s), Hy(s))| ds \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} |f(s, x(s), Hx(s)) - f(s, y(s), Hy(s))| ds \\ & + \sum_{0<t_k<t} \left| I_k(x(t_k^-)) - I_k(y(t_k^-)) \right| \end{aligned}$$

$$\leqslant \frac{l(1+H_1)\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} (t_k^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} |x(s) - y(s)| \, ds$$
$$+ \frac{l(1+H_1)\rho^{1-\alpha}}{\Gamma(\alpha)} + \int_{t_k}^{t} (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} |x(s) - y(s)| \, ds + \sum_{k=1}^{m} l^* \left| x(t_k^-) - y(t_k^-) \right|$$
$$\leqslant \frac{l(1+H_1)mT^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \|x-y\|_{\infty} + \frac{l(1+H_1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \|x-y\|_{\infty} + ml^* \|x-y\|_{\infty}.$$

Therefore,

$$\|N(x) - N(y)\|_{\infty} \leqslant \left(\frac{l(1+H_1)(m+1)T^{\rho\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + ml^* \right) \|x-y\|_{\infty}.$$

Consequently by (3.2), N is a contraction. As a consequence of Banach fixed point theorem, we deduce that N has fixed point which is a solution of the problem (1.1)-(1.3). □

4. Stability analysis

In this section, we adopt the concepts in [19] and introduce Ulam's type stability concepts for the problem (1.1)-(1.3). Let ϵ be a positive real number and $\varphi : J \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities

$$\begin{cases} \left| {}^{\rho}_{c}D^{\alpha}_{0^+} y(t) - f(t, y(t), Hy(t)) \right| \leqslant \epsilon, & t \in J', \\ \left| \Delta y(t_k) - I_k(y(t_k^-)) \right| \leqslant \epsilon, & k = 1, 2, \ldots, m, \end{cases} \tag{4.1}$$

$$\begin{cases} \left| {}^{\rho}_{c}D^{\alpha}_{0^+} y(t) - f(t, y(t), Hy(t)) \right| \leqslant \varphi(t), & t \in J', \\ \left| \Delta y(t_k) - I_k(y(t_k^-)) \right| \leqslant \varphi(t), & k = 1, 2, \ldots, m, \end{cases} \tag{4.2}$$

$$\begin{cases} \left| {}^{\rho}_{c}D^{\alpha}_{0^+} y(t) - f(t, y(t), Hy(t)) \right| \leqslant \epsilon\varphi(t), & t \in J', \\ \left| \Delta y(t_k) - I_k(y(t_k^-)) \right| \leqslant \epsilon\varphi(t), & k = 1, 2, \ldots, m. \end{cases} \tag{4.3}$$

Definition 4.1. The problem (1.1)-(1.2) is Ulam-Hyers stable if there exists a real number $C_{f,m} > 0$ such that for each ϵ and for each solution $y \in Z$ of the inequality (4.1) there exists a solution $x \in Z$ of the problem (1.1)-(1.2) with

$$|y(t) - x(t)| \leqslant C_{f,m}\epsilon, \quad t \in J.$$

Definition 4.2. The problem (1.1)-(1.2) is generalized Ulam-Hyers stable if there exists $\theta_{f,m} \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\theta_{f,m}(0) = 0$ such that for each solution $y \in Z$ of the inequality (4.1) there exists a solution $x \in Z$ of the problem (1.1)-(1.2) with

$$|y(t) - x(t)| \leqslant \theta_{f,m}(\epsilon), \quad t \in J.$$

Definition 4.3. The probelm (1.1)-(1.2) is Ulam-Hyers-Rassias stable with respect to φ if there exists $C_{f,m,\varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $y \in Z$ of the inequality (4.3) there exists a solution $x \in Z$ of the problem (1.1)-(1.2) with

$$|y(t) - x(t)| \leqslant C_{f,m,\varphi}\epsilon\varphi(t), \quad t \in J.$$

Definition 4.4. The problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists $C_{f,m,\varphi} > 0$ such that for each solution $y \in Z$ of the inequality (4.2) there exists a solution $x \in Z$ of the problem (1.1)-(1.2) with

$$|y(t) - x(t)| \leqslant C_{f,m,\varphi}\varphi(t), \quad t \in J.$$

Remark 4.5. It is clear that

(i) Definition 4.1 ⇒ Definition 4.2;

(ii) Definition 4.3 $\Rightarrow$ Definition 4.4;
(iii) Definition 4.3 for $\varphi(t) = 1 \Rightarrow$ Definition 4.1.

Remark 4.6. A function $y \in Z$ is a solution of the inequality (4.1) if and only if there exists a function $g \in Z$ and a sequence g_k, $k = 1, 2, \dots, m$ (which depends on y) such that

(i) $|g(t)| \leqslant \epsilon$ and $|g_k| \leqslant \epsilon$, $k = 1, \dots, m$;
(ii) ${}^{\rho}_{c}D^{\alpha}_{0^+} y(t) = f(t, y(t), Hy(t)) + g(t)$, $t \in J'$;
(iii) $\Delta y(t_k) = I_k(y(t_k^-)) + g_k$, $k = 1, 2, \dots, m$.

One can have similar remarks for the inequalities (4.2), (4.3).

Remark 4.7. Let $\alpha \in \mathbb{R}^+$, $\rho > 0$, if $y \in Z$ is a solution of the inequality (4.1), then y is a solution of the following integral inequality

$$\begin{aligned}&\Bigg|y(t) - y_0 - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y(s), Hy(s)) ds \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y(s), Hy(s)) ds\Bigg| \leqslant \left(m + \frac{T^{\rho\alpha}(m+1)}{\rho^\alpha \Gamma(\alpha+1)}\right)\epsilon, \quad t \in J.\end{aligned}$$

Indeed, by Remark 4.6 we have that

$$\begin{cases} {}^{\rho}_{c}D^{\alpha}_{0^+} y(t) = f(t, y(t), Hy(t)) + g(t), & t \in J', \\ \Delta y(t_k) = I_k(y(t_k^-)) + g_k, & k = 1, 2, \dots, m. \end{cases}$$

Then

$$\begin{aligned} y(t) &= y_0 + \sum_{i=1}^{k} I_i(y(t_i^-)) + \sum_{i=1}^{k} g_i + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y(s), Hy(s)) ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} g(s) ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y(s), Hy(s)) ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} g(s) ds, \quad t \in (t_k, t_{k+1}]. \end{aligned}$$

From this it follows that

$$\begin{aligned} &\Bigg|y(t) - y_0 - \sum_{i=1}^{k} I_i(y(t_i^-)) - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y(s), Hy(s)) ds \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s, y(s), Hy(s)) ds\Bigg| \\ &= \left|\sum_{i=1}^{k} g_i + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} g(s) ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} g(s) ds\right| \\ &\leqslant \sum_{i=1}^{k} |g_i| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |g(s)| ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |g(s)| ds \\ &\leqslant \left(m + \frac{T^{\rho\alpha}(m+1)}{\rho^\alpha \Gamma(\alpha+1)}\right)\epsilon. \end{aligned}$$

We have similar remarks for the solutions of the inequality (4.2)-(4.3)
Now, we give the main results, generalized Ulam-Hyers-Rassias stable results, in this section.

Theorem 4.8. *Assume that* (A1)-(A6), *and* (3.2) *hold. Suppose there exists* $\lambda_\varphi > 0$ *such that*

$$\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}\varphi(s)ds \leqslant \lambda_\varphi \varphi(t), \text{ for each } t \in J,$$

where $\varphi \in C(J, \mathbb{R}^+)$ *is nondecreasing. Then, the problem* (1.1)-(1.2) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y \in Z$ be a solution of the inequality (4.2). By Theorem 2.8, there exists a unique solution x of the impulsive type integro-differential equation with generalized fractional derivative

$$\begin{cases} {}^\rho_c D^\alpha_{0+} x(t) = f(t, x(t), Hx(t)), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k^-)), & k = 1, 2, \dots, m, \\ x(0) = y(0) = y_0. \end{cases}$$

Then we have

$$x(t) = \begin{cases} y_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, x(s), Hx(s))ds, & \text{if } t \in [0, t_1], \\ y_0 + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, x(s), Hx(s))ds & \\ \quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, x(s), Hx(s))ds + \sum_{i=1}^k I_i(x(t_i^-)), & \text{if } t \in (t_k, t_{k+1}], \end{cases}$$

By integration of the inequality (4.2), for each $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &\Bigg| y(t) - y_0 - \sum_{i=1}^k I_i(y(t_i^-)) - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, y(s), Hy(s))ds \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, y(s), Hy(s))ds \Bigg| \\ &\leqslant \sum_{i=1}^k |g(t_k^-)| + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1}s^{\rho-1}\varphi(s)ds + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}\varphi(s)ds \\ &\leqslant (m + (m+1)\lambda_\varphi)\, \varphi(t), \quad t \in J. \end{aligned}$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows

$$\begin{aligned} |y(t) - x(t)| &\leqslant \Bigg| y(t) - y_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, x(s), Hx(s))ds \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, x(s), Hx(s))ds - \sum_{i=1}^k I_i(x(t_i^-)) \Bigg| \\ &\leqslant \Bigg| y(t) - y_0 - \sum_{i=1}^k I_i(y(t_i^-)) - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, y(s), Hy(s))ds \\ &\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1}f(s, y(s), Hy(s))ds \Bigg| + \sum_{i=i}^k \left| I_i(y(t_i^-)) - I_i(x(t_i^-)) \right| \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1}s^{\rho-1} |f(s, y(s), Hy(s)) - f(s, x(s), Hx(s))| ds \\ &\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_k}^t (t^\rho - s^\rho)^{\alpha-1}s^{\rho-1} |f(s, y(s), Hy(s)) - f(s, x(s), Hx(s))| ds \end{aligned}$$

$$
\begin{aligned}
&\leqslant (m+(m+1)\lambda_\varphi)\,\varphi(t) + \frac{l(1+H_1)\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} (t_i^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |y(s)-x(s)|\, ds \\
&\quad + \frac{l(1+H_1)\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{t_k}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |y(s)-x(s)|\, ds + \sum_{k=1}^{m} l^* \left|y(t_k^-) - x(t_k^-)\right| \\
&\leqslant (m+(m+1)\lambda_\varphi)\,\varphi(t) + \sum_{k=1}^{m} l^* \left|y(t_k^-) - x(t_k^-)\right| \\
&\quad + \frac{l(1+H_1)(m+1)\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{0}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |y(s)-x(s)|\, ds.
\end{aligned}
$$

We consider the function V_1 defined by

$$V_1(t) = \sup\{|y(s)-x(s)| : \ 0 \leqslant s \leqslant T\}, \quad 0 \leqslant t \leqslant T.$$

If $t \in J'$, then by previous inequality, we have

$$V_1(t) \leqslant (m+(m+1)\lambda_\varphi)\,\varphi(t) + \sum_{k=1}^{m} l^* V_1(t_k^-) + \frac{l(1+H_1)(m+1)\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{0}^{t} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} V_1(s) ds.$$

Applying Lemma 2.10, we get

$$
\begin{aligned}
V_1(t) &\leqslant (m+(m+1)\lambda_\varphi)\,\varphi(t) \times \left[\Pi_{0<t_i<t}(1+l^*) \exp\left(\frac{l(1+H_1)(m+1)\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds\right)\right] \\
&\leqslant C_{f,m,\varphi}\,\varphi(t),
\end{aligned}
$$

where

$$
\begin{aligned}
C_{f,m,\varphi}\,\varphi(t) &= (m+(m+1)\lambda_\varphi)\,\varphi(t) \times \left[\Pi_{0<t_i<t}(1+l^*) \exp\left(\frac{l(1+H_1)(m+1)T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)}\right)\right] \\
&= (m+(m+1)\lambda_\varphi)\,\varphi(t) \times \left[(1+l^*) \exp\left(\frac{l(1+H_1)(m+1)T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)}\right)\right]^m.
\end{aligned}
$$

Thus, the problem (1.1)-(1.2) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi(t)$. This proof is complete. □

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