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Subordination properties of p-valent functions involving the generalized hypergeometric functions

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Abstract

In the present paper, using principle of differential subordination, we investigate some interesting properties of certain subclasses of p-valent functions which are defined by linear operator involving the generalized hypergeometric functions.

Keywords: Differential subordination, p-valent functions, Hadamard product, hypergeometric functions.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (p \in \mathbb{N} = \{1, 2, \cdots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}_1 = \mathcal{A}$ a well-known class of normalized analytic functions in $\mathbb{U}$.

Given two functions $f, g \in \mathcal{A}_p$, $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$ their Hadamard product (convolution) $f(z) * g(z)$ is defined by

$$f(z) * g(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{U}). \tag{1.2}$$

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If f and g are analytic functions in $\mathbb{U}$, we say that f is subordinate to g, (or g is superordinate to f), and write $f(z) \prec g(z)(z \in \mathbb{U})$, if there exists the Schwarz function $w(z)$, analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1$ then $f(z) = g(w(z))(z \in \mathbb{U})$. In particular if g is univalent in $\mathbb{U}$ then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For parameters $\alpha_i \in \mathbb{C}(i = 1, ..., q)$, and $\beta_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}(j = 1, ..., s)$, the generalized hypergeometric functions ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ is defined as:

$$
{}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k, ..., (\alpha_q)_k}{(\beta_1)_k, ..., (\beta_s)_k} \frac{z^k}{k!},
$$
$$
(q \leqslant s + 1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}),
$$

where $(\nu)_k$ is the Pochhammer symbol (or the shifted factorial) defined (in terms of the Gamma function) by

$$
(\nu)_k = \frac{\Gamma(\nu + k)}{\Gamma(\nu)} = \begin{cases} 1, & k = 0, \nu \in \mathbb{C}\backslash\{0\}; \\ \nu(\nu + 1)(\nu + 2)....(\nu + k - 1), & k \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}
$$

Dziok and Srivastava [4] defined the linear operator under the multivalent analytic functions

$$
H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)f(z) : \mathcal{A}_p \to \mathcal{A}_p,
$$

defined by the convolution

$$
\begin{aligned}
H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)f(z) &= z^p \; {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) * f(z) \\
&= z^p + \sum_{k=p+1}^{\infty} \Upsilon_s^{q,p} a_k z^k,
\end{aligned} \tag{1.3}
$$

where

$$
\Upsilon_s^{q,p} = \frac{(\alpha_1)_{k-p}, ..., (\alpha_q)_{k-p}}{(\beta_1)_{k-p}, ..., (\beta_s)_{k-p}(k - p)!}. \tag{1.4}
$$

For convenience, we write

$$
H_p(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)f(z) = H^p_{q,s}[\alpha_i, \beta_j]f(z).
$$

Let $\mathcal{M}^{m,p}_{\lambda_1,\lambda_2,\ell,d} \in \mathcal{A}_p$ be defined by

$$
\mathcal{M}^{m,p}_{\lambda_1,\lambda_2,\ell,d}(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{\ell(p + (\lambda_1 + \lambda_2)(k - p)) + d}{\ell(p + \lambda_2(k - p)) + d} \right]^m z^k, \quad p \in \mathbb{N}, \tag{1.5}
$$

where $p \in \mathbb{N}, m, d \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \lambda_2 \geqslant \lambda_1 \geqslant 0, \ell \geqslant 0$, and $\ell p + d > 0$.

Corresponding to $H^p_{q,s}[\alpha_i, \beta_j]f(z), \mathcal{M}^{m,p}_{\lambda_1,\lambda_2,\ell,d}$ and using Hadamard product, we define a new generalized hypergeometric derivative operator $\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]$ as follows:

Definition 1.1. Let the function $f \in \mathcal{A}_p$, then the generalized hypergeometric derivative operator

$$
\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] : \mathcal{A}_p \to \mathcal{A}_p
$$

is given by

$$
\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z) = H^p_{q,s}[\alpha_i, \beta_j]f(z) * \mathcal{M}^{m,p}_{\lambda_1,\lambda_2,\ell,d}(z).
$$

Then from (1.3) and (1.5) we get

$$
\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{\ell(p + (\lambda_1 + \lambda_2)(k - p)) + d}{\ell(p + \lambda_2(k - p)) + d} \right]^m \Upsilon_s^{q,p} a_k z^k. \tag{1.6}
$$

It follows from the above definition that

$$\begin{aligned}[\ell(p+\lambda_2(k-p))+d]\mathcal{D}^{m+1,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z) =&[\ell(p+\lambda_2(k-p)-p\lambda_1)+d]\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)\\ &+\ell\lambda_1 z\big(\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)\big)',\end{aligned} \tag{1.7}$$

$$\alpha_i\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i+1,\beta_j]f(z)=z(\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z))'+(\alpha_i-p)\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z),\quad (i=1,2,\cdots,q). \tag{1.8}$$

Remark 1.2. It should be remarked that $\mathcal{D}^{0,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z) = H^p_{q,s}[\alpha_i,\beta_j]f(z)$ where the linear operator $H^p_{q,s}[\alpha_i,\beta_j]f(z)$ was introduced by Dziok and Srivastava [4]. Also, the linear operator $\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)$ includes various other linear operators which were considered in earlier works. We list a few of them.

(i) For $\ell=1$, we get the operator $\mathcal{D}^{m,b}_{\lambda_1,\lambda_2,p}$ given by El-Yagubi [7].

(ii) For $q=s+1, \alpha_i=1(i=1,\cdots,s+1)$, $\beta_j=1(j=1,\cdots,s)$, $\lambda_2=d=0, \ell=\lambda_1=1$, we get the operator $I_p(m,\ell)$ given by Kumar et al. [9].

(iii) For $q=s+1, \alpha_i=1(i=1,\cdots,s+1)$, $\beta_j=1(j=1,\cdots,s), \lambda_2=0$ and $\ell=1$, we get the operator $I^m_p(\lambda,\ell)$ given by Cătaş [2].

(iv) For $q=s+1, \alpha_i=1(i=1,\cdots,s+1)$, $\beta_j=1(j=1,\cdots,s)$, $\lambda_2=d=0, p=1$ and $\lambda_1=\ell=1$, we get salagean operator D^m [13] .

(v) For $q=s+1, \alpha_i=1(i=1,\cdots,s+1)$, $\beta_j=1(j=1,\cdots,s)$, $\lambda_2=d=0, p=1$ and $\ell=1$, we get the operator D^m_λ given by Al-Oboudi [1].

(vi) For $q=2, s=1, \alpha_1=n+1, \alpha_2=1$ and $\beta_1=1, p=1$, we get derivative operator $\mathcal{D}^{n,m}_{\lambda_1,\lambda_2,\ell,d}$ given by Oshah and Darus [11].

(vii) For $q=2, s=1, \alpha_1=\alpha+1, \alpha_2=1, \beta_1=1, \ell=1, p=1$ and $d=0$, we get derivative operator $D^{m,\alpha}_{\lambda_1,\lambda_2}$ given by Eljamal and Darus [5].

(viii) For $q=2, s=1, \alpha_1=\delta+1, \alpha_2=1, \beta_1=1, p=1$ and $\ell=1$, we get derivative operator $D^{m,b}_{\lambda_1,\lambda_2,\delta}$ given by El-Yagubi and Darus [6].

(ix) For $q=2, s=1, \alpha_1=n+1, \alpha_2=1, \beta_1=1, \lambda_1=1$ and $\lambda_2=0$, we get derivative operator $I^m_{\alpha,\beta}$ given by Swamy [15]. In the case when $p=1$, $I^m_{\alpha,\beta}$ is the derivative operator defined also by Swamy [14].

(x) For $q=s+1, \lambda_2=0, p=1$ and $\lambda_1=\ell=1, d=\lambda$, we get derivative operator I^m_λ given by Cho and Srivastava [3].

Now, for $f(z)\in\mathcal{A}_p$, the integral operator $I_{\mu,p}$ defined by

$$I_{\mu,p}(z)=I_{\mu,p}(f)(z))=\frac{\mu+p}{z^\mu}\int_0^z t^{\mu-1}f(t)dt,\quad (\mu>-p). \tag{1.9}$$

It can easily be verified from (1.9) that

$$z(\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z))'=(\mu+p)\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)-\mu\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z). \tag{1.10}$$

2. A set of preliminary lemmas

To prove our main result, we need the following lemmas.

Lemma 2.1 ([8]). *Let h be a convex function with $h(0) = 1$ and let $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\Re\{\gamma\} \geqslant 0$. Suppose also that the function p given by*

$$p(z) = 1 + p_1 + p_2 z^2 + \cdots , \tag{2.1}$$

is analytic in $\mathbb{U}$. If

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \ (z \in \mathbb{U}),$$

then

$$p(z) \prec q(z) \prec h(z), \ (z \in \mathbb{U}),$$

where

$$q(z) = \gamma z^{-\gamma} \int_0^z h(t) t^{\gamma-1} dt, \ (z \in \mathbb{U}).$$

The function q is convex and is the best dominant.

For real or complex numbers $\alpha_1, \alpha_2, \beta_1 (\beta_1 \neq 0, -1, -2, \cdots)$ the hypergeometric function is defined by

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = 1 + \frac{\alpha_1 \alpha_2}{\beta_1} \frac{z}{1!} + \frac{\alpha_1(\alpha_1+1)\alpha_2(\alpha_2+1)}{\beta_1(\beta_1+1)} \frac{z^2}{2!} + \cdots . \tag{2.2}$$

We note the series in (2.2) converges absolutely for all $z \in \mathbb{U}$ and hence represents an analytic function in the unit disk $\mathbb{U}$ (see, for details, [16]). Each of the identities (asserted by Lemma 2.2 below) is well-known.

Lemma 2.2. *For real or complex parameters $\alpha_1, \alpha_2, \beta_1 (\beta_1 \neq 0, -1, -2, \cdots)$, $\Re e(\beta_1) > \Re e(\alpha_2) > 0$, we have*

$$\int_0^1 t^{\alpha_2 - 1}(1-t)^{\beta_1 - \alpha_2 - 1}(1 - tz)^{-\alpha_1} dt = \frac{\Gamma(\alpha_1)\Gamma(\beta_1 - \alpha_1)}{\Gamma(\beta_1)} \, {}_2F_1(\alpha_1, \alpha_2; \beta_1; z); \tag{2.3}$$

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = {}_2F_1(\alpha_2, \alpha_1; \beta_1; z); \tag{2.4}$$

$${}_2F_1(\alpha_1, \alpha_2; \beta_1; z) = (1-z)^{-\alpha} \, {}_2F_1(\alpha_1, \beta_1 - \alpha_2; \beta_1; z/z - 1); \tag{2.5}$$

$$(\alpha_1 + 1) {}_2F_1(1, \alpha_1; \alpha_1 + 1; z) = (\alpha_1 + 1) + \alpha_1 z {}_2F_1(1, \alpha_1 + 1; \alpha_1 + 2; z). \tag{2.6}$$

3. Main results

Unless otherwise mentioned, we will suppose in the remainder of this paper that $z \in \mathbb{U}$, the powers are understood as principle values and the parameters $p, m, A, B, \sigma, \alpha_i, \beta_j, \mu, \lambda_1, \lambda_2, \ell$, and d are constrained as follows:
$p \in \mathbb{N}, \sigma > 0, \mu > -p, \alpha_i \in \mathbb{C}(i = 1, ..., q), \beta_j \in \mathbb{C} \setminus \{0, -1, -2, ...\}(j = 1, ..., s), m, d \in \mathbb{N}_0, \lambda_2 \geqslant \lambda_1 > 0, \ell > 0,$ such that $\ell p + d > 0, -1 \leqslant B < A \leqslant 1$.

Theorem 3.1. *Let $f \in \mathcal{A}_p$, satisfy the following subordination*

$$(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z)}\right)^{\sigma} - \beta \left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z)}\right)^{\sigma} \left(\frac{\mathcal{D}^{m+1,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z)}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z)}\right) \prec \frac{1+Az}{1+Bz}, \tag{3.1}$$

then

$$\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j]f(z)}\right)^{\sigma} \prec q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.2}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1} \, {}_2F_1\left(1, 1; \frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta \ell \lambda_1} + 1; \frac{Bz}{Bz+1}\right), & (B \neq 0) \\ 1 + \frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\sigma(\ell(p+\lambda_2(k-p))+d) + \beta \ell \lambda_1} Az, & (B = 0) \end{cases},$$

and $q(z)$ *is the best dominant. Furthermore*

$$\Re e\left\{\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma}\right\} > \rho,$$

where

$$\rho = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}\, {}_2F_1\left(1,1;\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}+1;\frac{B}{B-1}\right), & (B\neq 0) \\ 1-\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\sigma(\ell(p+\lambda_2(k-p))+d)+\beta\ell\lambda_1}A, & (B=0) \end{cases}. \tag{3.3}$$

This result is sharp.

Proof. Set

$$p(z) = \left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma}, \tag{3.4}$$

then $p(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Using (1.7), (3.1) and (3.4) we get

$$p(z) + \frac{\beta\ell\lambda_1}{\sigma(\ell(p+\lambda_2(k-p))+d)} zp'(z) \prec \frac{1+Az}{1+Bz}.$$

Thus, by Lemma 2.1 for $\gamma = \frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}$, we obtain

$$\begin{aligned}\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma} \prec q(z) &= \left(\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}\right) \\ &\quad\times z^{-\left(\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}\right)}\int_0^z t^{\left(\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}\right)-1}\left(\frac{1+At}{1+Bt}\right)dt \\ &= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1,1;\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}+1;\frac{Bz}{Bz+1}\right), & (B\neq 0) \\ 1+\frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\sigma(\ell(p+\lambda_2(k-p))+d)+\beta\ell\lambda_1}Az, & (B=0) \end{cases},\end{aligned}$$

by change of variables followed by the use of the identities (2.3),(2.4) and (2.5) from Lemma 2.1 with $\alpha_1 = 1, \beta_1 = \alpha_2 + 1, \alpha_2 = \frac{\sigma(\ell(p+\lambda_2(k-p))+d)}{\beta\ell\lambda_1}$. This proves the assertion (3.2). Following the same lines as in Theorem 4 [12], we can prove that $\inf\{\Re e(q(z))\} = q(-1)$. The proof of Theorem 3.1 is thus completed. □

Theorem 3.2. *Let* $f \in \mathcal{A}_p$, *satisfy the following subordination*

$$(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma} - \beta\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma}\left(\frac{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i+1,\beta_j]f(z)}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right) \prec \frac{1+Az}{1+Bz}, \tag{3.5}$$

then

$$\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma} \prec q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.6}$$

where

$$q(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1,1;\frac{\sigma\alpha_i}{\beta}+1;\frac{Bz}{Bz+1}\right), & (B\neq 0) \\ 1+\frac{\sigma\alpha_i}{\sigma\alpha_i+\beta}Az, & (B=0) \end{cases},$$

and $q(z)$ *is the best dominant. Furthermore*

$$\Re e\left\{\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{\sigma}\right\} > \kappa,$$

where

$$\kappa = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\, {}_2F_1\left(1, 1; \frac{\sigma\alpha_i}{\beta} + 1; \frac{B}{B-1}\right), & (B \neq 0) \\ 1 - \frac{\sigma\alpha_i}{\sigma\alpha_i + \beta} A, & (B = 0) \end{cases}. \tag{3.7}$$

This result is sharp.

Proof. Set

$$p(z) = \left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] f(z)}\right)^{\sigma}, \tag{3.8}$$

then $p(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Differentiating both sides of (3.8), and using (1.8), we have

$$p(z) + \frac{\beta}{\sigma\alpha_i} z p'(z) \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of the proof follows by employing the techniques that we used in proof of Theorem 3.1. □

Theorem 3.3. *Let $f \in \mathcal{A}_p$, satisfy the following subordination*

$$(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] I_{\mu,p}(z)}\right)^{\sigma} - \beta\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] I_{\mu,p}(z)}\right)^{\sigma+1}\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] f(z)}\right)^{-1} \prec \frac{1+Az}{1+Bz}, \tag{3.9}$$

then

$$\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] I_{\mu,p}(z)}\right)^{\sigma} \prec q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.10}$$

where $I_{\mu,p}$ is defined by (1.9) and $q(z)$ is given by

$$q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1, 1; \frac{\sigma(\mu+p)}{\beta} + 1; \frac{Bz}{Bz+1}\right), & (B \neq 0) \\ 1 + \frac{\sigma(\mu+p)}{\sigma(\mu+p)+\beta} Az, & (B = 0) \end{cases},$$

and $q(z)$ is the best dominant. Furthermore

$$\Re e\left\{\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] I_{\mu} f(z)}\right)^{\sigma}\right\} > \xi,$$

where

$$\xi = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}\, {}_2F_1\left(1, 1; \frac{\sigma(\mu+p)}{\beta} + 1; \frac{B}{B-1}\right), & (B \neq 0) \\ 1 - \frac{\sigma(\mu+p)}{\sigma(\mu+p)+\beta} A, & (B = 0) \end{cases}. \tag{3.11}$$

This result is sharp.

Proof. Set

$$p(z) = \left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i, \beta_j] I_{\mu,p}(z)}\right)^{\sigma}, \tag{3.12}$$

then $p(z)$ is of the form (2.1) and is analytic in $\mathbb{U}$. Differentiating both sides of (3.12), and using (1.10), we have

$$p(z) + \frac{\beta}{\sigma(\mu+p))} z p'(z) \prec \frac{1+Az}{1+Bz}.$$

Using Lemma 2.2 for $\gamma = \frac{\sigma(\mu+p)}{\beta}$, we obtain

$$p(z) \prec q(z) = \left(\frac{\sigma(\mu+p)}{\beta}\right) z^{-\left(\frac{\sigma(\mu+p)}{\beta}\right)} \int_0^z t^{\left(\frac{\sigma(\mu+p)}{\beta}\right)-1} \left(\frac{1+At}{1+Bt}\right) dt$$
$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}\, {}_2F_1\left(1,1;\frac{\sigma(\mu+p)}{\beta}+1;\frac{Bz}{Bz+1}\right), & (B \neq 0) \\ 1 + \frac{\sigma(\mu+p)}{\sigma(\mu+p)+\beta} Az, & (B = 0) \end{cases}.$$

The remaining part of the proof in Theorem 3.3 is similar to that of Theorem 3.1 and so we omit it. □

Now we prove the partial converse of Theorem 3.3, for $A = 1-2\rho, 0 \leqslant \rho < 1$ and $B = -1$.

Theorem 3.4. *If $f \in \mathcal{A}_p$, satisfies*

$$\Re e\left\{\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma}\right\} > \rho, \quad 0 \leqslant \rho < 1, \tag{3.13}$$

then

$$\Re e\left\{(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma} - \beta\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma+1}\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{-1}\right\} > \rho,\ |z| > R,$$

where

$$R = \frac{\sqrt{\beta^2 + [\sigma(\mu+p)]^2} - \beta}{\sigma(\mu+p)}. \tag{3.14}$$

The bound R is the best possible.

Proof. From (3.13), we have

$$\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma} = \rho + (1-\rho)p(z). \tag{3.15}$$

We see that $p(z)$ of the form (2.1) is analytic in $\mathbb{U}$ and $\Re e\{p(z)\} > 0,\ z \in \mathbb{U}$. Differentiating both sides of (3.15), and using (1.10), we get

$$\begin{aligned} &\Re e\left\{(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma} \right.\\ &\qquad \left. -\beta\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma+1}\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{-1} - \rho\right\} \\ &= (1-\rho)\Re e\left\{p(z) + \frac{\beta}{\sigma(\mu+p)} zp'(z)\right\} \\ &\geqslant (1-\rho)\left[\Re e(p(z)) - \frac{\beta}{\sigma(\mu+p)}|zp'(z)|\right]. \end{aligned} \tag{3.16}$$

Now, by applying the well-known estimate (see [10])

$$\frac{|zp'(z)|}{\Re e(p(z))} \leqslant \frac{2r}{1-r^2}, \quad (|z| = r < 1)$$

in (3.16), we deduce that

$$\begin{aligned}\Re\mathfrak{e}\Bigg\{&(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma}\\&\qquad-\beta\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma+1}\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{-1}-\rho\Bigg\}\\&\geqslant(1-\rho)\Re\mathfrak{e}\,(p(z))\left(1-\frac{\beta}{2r\beta\sigma(\mu+p)(1-r^2)}\right),\end{aligned}$$

which is positive if $r<R$, where R is given by (3.14).

In order to show that the bound R is the best possible, we consider the function $f\in\mathcal{A}_p$ defined by

$$\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma}=\rho+(1-\rho)\frac{1+z}{1-z}. \tag{3.17}$$

Noting that

$$\begin{aligned}\Re\mathfrak{e}\Bigg\{&(1+\beta)\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_j,\beta_j]I_{\mu,p}(z)}\right)^{\sigma}\\&\qquad-\beta\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]I_{\mu,p}(z)}\right)^{\sigma+1}\left(\frac{z^p}{\mathcal{D}^{m,p}_{\lambda_1,\lambda_2,\ell,d}[\alpha_i,\beta_j]f(z)}\right)^{-1}-\rho\Bigg\}\\&=(1-\rho)\Re\mathfrak{e}\left(\frac{1+z}{1-z}+\frac{\beta z}{\sigma(\mu+p)}\frac{2}{(1-z)^2}\right)=0\end{aligned}$$

for $z=R$, we complete the proof of Theorem 3.4. □

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