Available online at www.isr-publications.com/mns
Math. Nat. Sci., 1 (2017), 18–25
Research Article

Mathematics in Natural Science
An International Journal

Journal Homepage:www.isr-publications.com/mns

New direction in fractional differentiation

Abdon Atangana[a,*], Ilknur Koca[b]

[a] *Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, 9300, Bloemfontein, South Africa.*

[b] *Department of Mathematics, Faculty of Sciences, Mehmet Akif Ersoy University, 15100, Burdur, Turkey.*

Communicated by O. K. Matthew

Abstract

Based upon the Mittag-Leffler function, new derivatives with fractional order were constructed. With the same line of idea, improper derivatives based on the Weyl approach are constructed in this work. To further model some complex physical problems that cannot be modeled with existing derivatives with fractional order, we propose, a new derivative based on the more generalized Mittag-Leffler function known as Prabhakar function. Some new results are presented together with some applications.

Keywords: Atangana-Baleanu fractional derivatives, Weyl approach, Prabhakar Mittag-Leffler function, integral transform.
2010 MSC: 26A33, 33E12, 65R10.

1. Introduction

The concept of fractional calculus has been attracting attention of many mathematicians around the world [5, 7–9, 12, 13]. Many PhD, Master thesis have been written. Many conferences and symposiums have been organized around this topic. Many research papers have been published. However, it is important to note that, the commonly used fractional derivatives namely the Riemann-Liouville and Caputo derivatives have been mis-used [10, 14]. When looking at the literature, nowadays there exist a lot of papers in which these derivatives were used to model real world problems with no clear explanation. It also appears that, many people believe the Caputo derivative can be used to solve all the problems in real world. The concept of power law that is used to justify the use of fractional derivative in solving real world problem cannot always be observed in nature. In nature sometime the exponential decay law is observed meaning the power law cannot be adapted in this case. In addition to this, one can mostly observed the generalized exponential decay law. To describe these last two cases, the Caputo-Fabrizio derivative was introduced with exponential kernel, but was immediately criticized, due to the fact that the used kernel was not non-linear and the anti-derivative associate was only the average of the function and it integral [1, 6]. To solve this problem and maintain the concept of exponential law, Atangana-Baleanu derivatives were introduced [2–4]. In this paper we aim to introduce further definitions based on the Weyl approach and also the more generalized Mittag-Leffler function.

*Corresponding author
Email addresses: abdonatangana@yahoo.fr (Abdon Atangana), ikoca@mehmetakif.edu.tr (Ilknur Koca)

doi:10.22436/mns.01.01.02

Received 2016-10-28

2. Atangana-Baleanu derivative in Caputo sense with improper integral

In this section, we present the definitions of the new fractional derivative with improper integral.

Definition 2.1. Let $f \in H^1(a,\infty)$, and $0 < \alpha \leqslant 1$, then the improper fractional derivative based upon Atangana-Baleanu derivative in Caputo and Riemann-Liouville sense are given as:

$$ {}_{0}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\int_t^{\infty}\frac{df(\tau)}{d\tau}E_{\alpha}\left[\frac{-\alpha}{1-\alpha}\left(t-\tau\right)^{\alpha}\right]d\tau, $$

where f is differentiable on (a,∞).

$$ {}_{0}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_t^{\infty}f(\tau)E_{\alpha}\left[\frac{-\alpha}{1-\alpha}\left(t-\tau\right)^{\alpha}\right]d\tau, $$

here f is not necessarily differentiable.

Definition 2.2. Let $f \in H^1(-\infty,b)$ and $\alpha \in [0,1]$ then the improper fractional derivative of f based on Atangana-Baleanu derivative in Caputo and Riemann-Liouville sense are given as:

$$ {}_{0}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\int_{-\infty}^{t}\frac{df(\tau)}{d\tau}E_{\alpha}\left[\frac{-\alpha}{1-\alpha}\left(t-\tau\right)^{\alpha}\right]d\tau, $$

where $t \leqslant b$ and f is differentiable.

$$ {}_{0}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{-\infty}^{t}f(\tau)E_{\alpha}\left[\frac{-\alpha}{1-\alpha}\left(t-\tau\right)^{\alpha}\right]d\tau, $$

here f is not necessarily differentiable.

Definition 2.3. Let $f \in H^1(a,b)$, and $0 < \alpha \leqslant 1$, then the AK derivative with fractional order in Caputo sense is given as:

$$ {}_{0}^{AKC}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{1}{g(\alpha)}\int_0^{t}\frac{df(\tau)}{d\tau}E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)\left(t-\tau\right)^{\alpha}\right]d\tau, $$

where the function is well-defined such that

$$ \lim_{\alpha\to 0}\frac{1}{g(\alpha)}\int_0^{t}\frac{df(\tau)}{d\tau}E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)\left(t-\tau\right)^{\alpha}\right]d\tau = \int_0^{t}\frac{df(\tau)}{d\tau}d\tau = f(t)-f(0). $$

In Riemann-Liouville sense we have

$$ {}_{0}^{AKR}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{1}{g(\alpha)}\frac{d}{dt}\int_0^{t}f(\tau)E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)\left(t-\tau\right)^{\alpha}\right]d\tau. $$

Also $g(\alpha)$ is chosen such that

$$ \lim_{\alpha\to 0}\frac{1}{g(\alpha)}\frac{d}{dt}\int_0^{t}f(\tau)E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)\left(t-\tau\right)^{\alpha}\right]d\tau = \frac{d}{dt}\int_0^{t}f(\tau) $$

and

$$\lim_{\alpha\to 1}\frac{1}{g(\alpha)}\frac{d}{dt}\int_0^t f(\tau)E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)(t-\tau)^{\alpha}\right]d\tau=\frac{d}{dt}f(t).$$

Based upon the Weyl approach of derivative, we present the following definitions.

Definition 2.4. Let $f\in H^1(a,\infty)$, $0<\alpha\leqslant 1$, then the AK fractional derivative in Weyl sense is given as if the function f is differentiable

$${}_0^{AKW}D_t^{\alpha}(f(t))=\frac{1}{g(\alpha)}\int_t^{\infty}\frac{df(\tau)}{d\tau}E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)(t-\tau)^{\alpha}\right]d\tau.$$

If f is not differentiable,

$${}_0^{AKW}D_t^{\alpha}(f(t))=\frac{1}{g(\alpha)}\frac{d}{dt}\int_t^{\infty}f(\tau)E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)(t-\tau)^{\alpha}\right]d\tau.$$

The function $g(\alpha)$ is chosen to satisfy the criteria of a fractional derivative.

Definition 2.5. Let $f\in H^1(-\infty,b)$, $0<\alpha<1$, then for the differentiable function f the AK fractional derivative in Weyl sense is given as

$${}_0^{AKW}D_t^{\alpha}(f(t))=\frac{1}{g(\alpha)}\int_{-\infty}^{t}\frac{df(\tau)}{d\tau}E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)(t-\tau)^{\alpha}\right]d\tau.$$

If f is not differentiable, then the AK derivative in Weyl sense is given as

$${}_0^{AKW}D_t^{\alpha}(f(t))=\frac{1}{g(\alpha)}\frac{d}{dt}\int_{-\infty}^{t}f(\tau)E_{\alpha,\alpha}^{\alpha}\left[-g(\alpha)(t-\tau)^{\alpha}\right]d\tau.$$

Remark 2.6. In this paper and in all definitions above $E_{\alpha,\alpha}^{\alpha}$ is generalized Mittag-Leffler function. Let us give the generalized Mittag-Leffler function definition below with $E_{\alpha,\beta}^{\gamma}(z)$ form [11, 15]:

$$E_{\alpha,\beta}^{\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!},\qquad(\alpha,\ \beta,\ \gamma\in\mathbb{C},\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,\ \mathrm{Re}(\gamma)>0),$$

where $(\gamma)_n$ is the Pochhammer symbol

$$(\gamma)_n=\frac{\Gamma(\gamma+n)}{\Gamma(\gamma)},\ (\gamma)_0=1,\quad(\gamma)_n=\gamma(\gamma+1)(\gamma+1)\cdots\Gamma(\gamma+n-1),\ n\geqslant 1.$$

The AK derivative has the following properties.

$$\begin{aligned}{}_0^{AKC}D_t^{\alpha}(f(t))&=\frac{1}{g(\alpha)}\int_0^t\frac{df(\tau)}{d\tau}E_{\alpha,\beta}^{q}\left[-g(\alpha)(t-\tau)^{\alpha}\right]d\tau\\&=\frac{1}{g(\alpha)}\int_0^t\frac{df(\tau)}{d\tau}\frac{1}{\Gamma(q)}\Psi\left[\frac{(q,1)}{(\beta,\alpha)}\middle|-g(\alpha)\tau^{\alpha}\right]d\tau,\end{aligned}$$

where Ψ is the Wright function define as

$${}_p\Psi_q(z)=\sum_{k=0}^{\infty}\frac{\Pi_{i=1}^{p}\Gamma(a_i+A_ik)}{\Pi_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{z^k}{k!}$$

with of course $a_i,\ b_j\in\mathbb{C}$, $A_i,\ B_j\in\mathbb{R}$.

We now present some relationships with some integral transform. The Laplace transform of AKC derivative is given as

$$L({}_0^{AKC}D_t^{\alpha}f(t)) = \frac{1}{g(\alpha)}L\left(\frac{df(t)}{dt}\right)L\left(E_{\alpha,\beta}^{q}(-g(\alpha)t^{\alpha})\right) = \frac{1}{g(\alpha)}\{pF(p)-f(0)\}L\left(E_{\alpha,\beta}^{q}(-g(\alpha)t^{\alpha})\right).$$

Nevertheless we have the following relation if $\beta > 0$, $\mathrm{Re}(p) > 0$, $\lambda \in \mathbb{C}$, and $\left|\frac{\lambda}{p^{\alpha}}\right| < 1$, then

$$L\left(t^{\beta-1}E_{\alpha,\beta}^{q}(\lambda t^{\alpha})\right)_{(p)} = \frac{p^{\alpha q-\beta}}{(p^{\alpha}-\lambda)^{q}}.$$

Therefore when $\beta = 1$ we have

$$L\left(E_{\alpha,1}^{q}(\lambda t^{\alpha})\right)_{(p)} = \frac{p^{-n\alpha-1}}{(1-g(\alpha))^{q}}.$$

Thus if $\beta = 1$ the Laplace transform of AKC is given as

$$L({}_0^{AKC}D_t^{\alpha}f(t))_{(p)} = \frac{1}{g(\alpha)}\left(pF(p)-f(0)\right)\frac{p^{-n\alpha-1}}{(1-g(\alpha))^{q}} = \frac{1}{g(\alpha)}\frac{p^{-n\alpha}F(p)}{(1-g(\alpha))^{q}} - \frac{f(0)}{g(\alpha)}\frac{p^{-n\alpha-1}}{(1-g(\alpha))^{q}}.$$

Then AKR is given as

$$L({}_0^{AKR}D_t^{\alpha}f(t))_{(p)} = \frac{1}{g(\alpha)}\frac{p^{-n\alpha}F(p)}{(1-g(\alpha))^{q}}.$$

Theorem 2.7. *If $\beta = 1$ then the following ordinary differential equation with AKR fractional derivative has a unique solution*

$${}_0^{AKR}D_t^{\alpha}f(t) = u(t).$$

Proof. To solve this equation, we apply on both sides the Laplace transform to have

$$\frac{1}{g(\alpha)}\frac{p^{\alpha q}F(p)}{(p^{\alpha}+g(\alpha))^{q}} = U(p)F(p) = g(\alpha)\left(\frac{p^{\alpha}+g(\alpha)}{p^{\alpha}}\right)^{q}U(p)f(t) = g(\alpha)\int_0^t \beta(t-\tau)u(\tau)d\tau,$$

with

$$\beta(t) = L^{-1}\left(\left(\frac{p^{\alpha}+g(\alpha)}{p^{\alpha}}\right)^{q}\right) = L^{-1}\left(\left(1+\frac{g(\alpha)}{p^{\alpha}}\right)^{q}\right).$$

□

Theorem 2.8. *Let $f \in H^1(a,b)$, and $0 < \alpha \leqslant 1$, then the Sumudu transform of AK derivative with fractional order in Caputo sense is given as:*

$$ST\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(u)} = \frac{F(u)}{g(\alpha)\,(1-g(\alpha))^{q}} - \frac{f(0)}{g(\alpha)\,(1-g(\alpha))^{q}}.$$

Proof. When $\beta = 1$, the Sumudu transform of AK derivative with fractional order in Caputo sense is given as

$$ST\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(u)} = ST\left\{\frac{1}{g(\alpha)}\int_0^t \frac{df(\tau)}{d\tau}E_{\alpha,1}^{q}\left[\lambda\,(t-\tau)^{\alpha}\right]d\tau\right\},$$

where $-g(\alpha) = \lambda$.

Using the convolution properties of Sumudu transform we have the following

$$\begin{aligned}ST\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(u)} &= \frac{1}{g(\alpha)}\left(F(u)-f(0)\right)ST\left(E_{\alpha,1}^{q}\left[\lambda t^{\alpha}\right]\right)\\ &= \frac{1}{g(\alpha)}\left(F(u)-f(0)\right)\left(1-g(\alpha)^{-q}\right)\\ &= \frac{F(u)}{g(\alpha)\left(1-g(\alpha)\right)^{q}}-\frac{f(0)}{g(\alpha)\left(1-g(\alpha)\right)^{q}}.\end{aligned}$$

Then the Sumudu transform of AK derivative with fractional order in Riemann-Liouville sense is given with Theorem 2.9. □

Theorem 2.9. *Let* $f \in H^1(a,b)$, *and* $0 < \alpha \leqslant 1$, *then the Sumudu transform of AK derivative with fractional order in Riemann-Liouville sense is given as:*

$$ST\left({}_0^{AKR}D_t^{\alpha}f(t)\right)_{(u)} = \frac{1}{g(\alpha)}\frac{F(u)}{\left(1-g(\alpha)\right)^{q}}.$$

Proof. Using the same rules from Theorem 2.8, the requested result is obtained. □

Theorem 2.10. *Let* $f \in H^1(a,b)$, *and* $0 < \alpha \leqslant 1$, *then the Mellin transform of AK derivative with fractional order in Caputo sense is given as:*

$$M\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(s)} = \frac{1}{g(\alpha)}\frac{(1-s)F(s-1)(s+n\alpha)^{-1}}{\left(1-g(\alpha)\right)^{q}\,\Gamma(n\alpha+1)}.$$

Proof. The Mellin transform of AK derivative with fractional order in Caputo sense is given as

$$M\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(s)} = M\left\{\frac{1}{g(\alpha)}\int_0^t \frac{df(\tau)}{d\tau}E_{\alpha,1}^{q}\left[\lambda\left(t-\tau\right)^{\alpha}\right]d\tau\right\}.$$

Using the convolution properties of Mellin transform we have equality below:

$$M\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(s)} = \frac{1}{g(\alpha)}M\left(\frac{df(t)}{dt}\right)M(E_{\alpha,1}^{q}\left[\lambda t^{\alpha}\right]).$$

So we have

$$M\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(s)} = \frac{1}{g(\alpha)}\frac{(1-s)F(s-1)(s+n\alpha)^{-1}}{\left(1-g(\alpha)\right)^{q}\,\Gamma(n\alpha+1)}.$$

□

Theorem 2.11. *Let* $f \in H^1(a,b)$, *and* $0 < \alpha \leqslant 1$, *then the Mellin transform of AK derivative with fractional order in Riemann-Liouville sense is given as:*

$$M\left({}_0^{AKR}D_t^{\alpha}f(t)\right)_{(s)} = \frac{1}{g(\alpha)}\frac{F(s)(s+n\alpha)^{-1}}{\left(1-g(\alpha)\right)^{q}\,\Gamma(n\alpha+1)}.$$

Proof. Using the same rules from Theorem 2.10, the requested result is obtained. □

Theorem 2.12. *Let* $f \in H^1(a,b)$, *and* $0 < \alpha \leqslant 1$, *then the Fourier transform of AK derivative with fractional order in Caputo sense is given as:*

$$F\left({}_0^{AKC}D_t^{\alpha}f(t)\right)_{(w)} = \frac{1}{g(\alpha)}(jw)F(w)\frac{1}{\sqrt{2\pi}}(1-\lambda)^{-q}(\mathrm{sgn}(w)-1).\exp(\frac{1}{2}i\pi(n\alpha))\sin(\pi(n\alpha))\,|w|^{-(n\alpha-1)}$$

for $(\lambda = 1 \text{ or } \mathrm{Re}(q) < 0)$ *and* $|\lambda| < 1$.

Proof. From the definition of Fourier transform, we get equalities below:

$$\begin{aligned}
F\left({}_{0}^{AKC}D_{t}^{\alpha}f(t)\right)_{(w)} &= M\left\{\frac{1}{g(\alpha)}\int_{0}^{t}\frac{df(\tau)}{d\tau}E_{\alpha,1}^{q}\left[\lambda(t-\tau)^{\alpha}\right]d\tau\right\} \\
&= \frac{1}{g(\alpha)}F\left(\frac{df(t)}{dt}\right)F(E_{\alpha,1}^{q}\left[\lambda t^{\alpha}\right]) \\
&= \frac{1}{g(\alpha)}(jw)F(w)F(E_{\alpha,1}^{q}\left[\lambda t^{\alpha}\right]) \\
&= \frac{1}{g(\alpha)}(jw)F(w)\frac{1}{\sqrt{2\pi}}(1-\lambda)^{-q}(\operatorname{sgn}(w)-1).\exp(\frac{1}{2}i\pi(n\alpha))\sin(\pi(n\alpha))\,|w|^{-(n\alpha-1)}
\end{aligned}$$

for ($\lambda = 1$ or $\operatorname{Re}(q) < 0$) and $|\lambda| < 1$. □

Theorem 2.13. *Let $f \in H^1(a,b)$, and $0 < \alpha \leqslant 1$, then the Fourier transform of AK derivative with fractional order in Riemann-Liouville sense is given as:*

$$F\left({}_{0}^{AKR}D_{t}^{\alpha}f(t)\right)_{(w)} = \frac{1}{g(\alpha)}F(w)\frac{1}{\sqrt{2\pi}}(1-\lambda)^{-q}(\operatorname{sgn}(w)-1).\exp(\frac{1}{2}i\pi(n\alpha))\sin(\pi(n\alpha))\,|w|^{-(n\alpha-1)}$$

for $\lambda = 1$ v $\operatorname{Re}(q) < 0$ and $|\lambda| < 1$.

Proof. Using the same rules from Theorem 2.12, the requested result is obtained. □

3. Partial derivative

Since many physical problems are sometimes required the time and space component, that is to say the physical problem has to be evaluated in time and space. It is therefore important to extend the new definitions to the concept of partial derivative. In this section, we present some definitions of partial derivatives associated to the new derivative.

Definition 3.1. Let f be a function of two variable x, y such that $f \in H^1(a_1,\infty) \times H^1(a_2,\infty)$, let in addition $\alpha \in (0,1]$, then the partial derivative of Atangana-Baleanu with fractional order α is given as if f is differentiable in x-direction

$${}_{l}^{ABC}D_{x}^{\alpha}f(x,y) = \frac{1}{g(\alpha)}\int_{l}^{x}\frac{\partial f(\tau,y)}{\partial\tau}E_{\alpha}(-g(\alpha)(x-\tau)^{\alpha})d\tau.$$

If the $\frac{\partial f(\tau,y)}{\partial\tau}$ is differentiable in y-direction then we have below:

$${}_{l}^{ABC}D_{x,y}^{\alpha}f(x,y) = \frac{1}{(g(\alpha))^2}\int_{l}^{x}\int_{l}^{y}\frac{\partial^2 f(\tau,\lambda)}{\partial\tau\partial\lambda}.E_{\alpha,\beta}^{\gamma,q}(-g(\alpha)(x-\tau)^{\alpha})E_{\alpha,\beta}^{\gamma,q}(-g(\alpha)(y-\lambda)^{\alpha})d\tau d\lambda.$$

Theorem 3.2. *Let $f(x,y) \in H^1(a_1,b_1) \times H^1(a_2,b_2)$ be a function such that $\frac{\partial^2 f(\tau,\lambda)}{\partial\tau\partial\lambda}$ and $\frac{\partial^2 f(\tau,\lambda)}{\partial\lambda\partial\tau}$ exist and both are continuous in $(a_1,b_1)\times(a_2,b_2)$ then the following relationship is obtained*

$${}_{l}^{ABC}D_{x,y}^{\alpha}f(x,y) \;=\; {}_{l}^{ABC}D_{y,x}^{\alpha}f(x,y).$$

Proof. By definition we have

$${}_{l}^{ABC}D_{x,y}^{\alpha}f(x,y) = \frac{1}{(g(\alpha))^2}\int_{l}^{x}\int_{l}^{y}\frac{\partial^2 f(\tau,\lambda)}{\partial\tau\partial\lambda}.E_{\alpha}(-g(\alpha)(x-\tau)^{\alpha})E_{\alpha}(-g(\alpha)(y-\tau)^{\alpha})d\tau d\lambda.$$

Since $\frac{\partial^2 f(\tau,\lambda)}{\partial\tau\partial\lambda}$ and $\frac{\partial^2 f(\tau,\lambda)}{\partial\lambda\partial\tau}$ are continuous on $(a_1, b_1) \times (a_2, b_2)$ then using the Clairaut theorem we obtain $\frac{\partial^2 f(\tau,\lambda)}{\partial\tau\partial\lambda}$ and $\frac{\partial^2 f(\tau,\lambda)}{\partial\lambda\partial\tau}$ thus

$$\begin{aligned}
{}^{ABC}_{l}D^{\alpha}_{x,y}f(x,y) &= \frac{1}{(g(\alpha))^2}\int_l^x\int_l^y \frac{\partial^2 f(\tau,\lambda)}{\partial\tau\partial\lambda}.E_\alpha(-g(\alpha)(x-\tau)^\alpha)E_\alpha(-g(\alpha)(y-\tau)^\alpha)d\tau d\lambda \\
&= \frac{1}{(g(\alpha))^2}\int_l^y\int_l^x \frac{\partial^2 f(\tau,\lambda)}{\partial\lambda\partial\tau}.E_\alpha(-g(\alpha)(y-\tau)^\alpha)E_\alpha(-g(\alpha)(x-\tau)^\alpha)d\lambda d\tau \\
&= {}^{ABC}_{l}D^{\alpha}_{y,x}f(x,y).
\end{aligned}$$

This completes the proof. □

Definition 3.3. Let f be a function that is not necessarily differentiable in (a, ∞) then for $\alpha \in (0, 1]$ the partial derivative associated to Atangana-Baleanu in Riemann-Liouville sense is given as

$$ {}^{ABR}_{0}D^{\alpha}_{x}f(x,y) = \frac{B(\alpha)}{1-\alpha}\int_x^\infty f(\tau,y)E_\alpha(-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha)d\tau.$$

For second order we have

$$ {}^{ABR}_{0}D^{\alpha}_{x,y}f(x,y) = \frac{\partial^2}{\partial x\partial y}\frac{B(\alpha)}{(1-\alpha)^2}\int_x^\infty\int_y^\infty f(\tau,\lambda).E_\alpha(-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha)E_\alpha(-\frac{\alpha}{1-\alpha}(\lambda-y)^\alpha)d\tau d\lambda.$$

Definition 3.4. Let f be a function defined in $H^1(a_1, b_1) \times H^1(a_2, b_2)$ but not necessarily differentiable. Let in addition $\alpha \in (0, 1)$ then, the partial fractional derivative in Atangana-Baleanu type of f in x-direction is given as

$$ {}^{ABR}_{0}D^{\alpha}_{x}f(x,y) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dx}\int_l^x f(\tau,y)E_\alpha(-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha)d\tau.$$

Then the partial derivative in x, y-direction is given as

$$ {}^{ABR}_{0}D^{\alpha}_{x,y}f(x,y) = \frac{B(\alpha)}{(1-\alpha)^2}\frac{\partial^2}{\partial x\partial y}\int_l^x\int_l^y f(\tau,\lambda).E_\alpha(-\frac{\alpha}{1-\alpha}(x-\tau)^\alpha)E_\alpha(-\frac{\alpha}{1-\alpha}(\lambda-y)^\alpha)d\tau d\lambda.$$

Definition 3.5. Let f be a function defined in $H^1(a_1, b_1) \times H^1(a_2, b_2)$ but not necessarily differentiable. Let α, β, $\gamma \in (0, 1)$ then the AK fractional partial derivative in Caputo sense is given as

$$ {}^{AKC}_{0}D^{\alpha}_{x}f(x,y) = \frac{1}{g(\alpha)}\int_0^x \frac{\partial}{\partial\tau}f(\tau,y)E^{\gamma,q}_{\alpha,\beta}(-g(\alpha)(x-\tau)^\alpha)d\tau.$$

If $\frac{\partial}{\partial x}f(x,y)$ is differentiable in y-direction then we have

$$ {}^{AKC}_{0}D^{\alpha}_{y,x}f(x,y) = \frac{1}{(g(\alpha))^2}\int_0^y\int_0^x \frac{\partial^2 f(\tau,\lambda)}{\partial\lambda\partial\tau}.E^{\gamma,q}_{\alpha,\beta}(-g(\alpha)(x-\tau)^\alpha)E^{\gamma,q}_{\alpha,\beta}(-g(\alpha)(y-\lambda)^\alpha)d\tau d\lambda.$$

Definition 3.6. Let $f \in H^1(a_1, b_1) \times H^1(a_2, b_2)$ be a function not necessarily differentiable in x- and y-directions. Let α, β, $\gamma \in (0, 1)$ then the AK fractional partial derivative in Riemann-Liouville sense is given as

$$ {}_0^{AKR}D_x^{\alpha} f(x,y) = \frac{1}{g(\alpha)} \frac{\partial}{\partial x} \int_0^x f(\tau,y) E_{\alpha,\beta}^{\gamma,q}(-g(\alpha)(x-\tau)^{\alpha}) d\tau. $$

Then

$$ {}_0^{AKR}D_{x,y}^{\alpha} f(x,y) = \frac{1}{(g(\alpha))^2} \frac{\partial^2}{\partial x \partial y} \int_0^x \int_0^y f(\tau,\lambda) E_{\alpha,\beta}^{\gamma,q}(-g(\alpha)(x-\tau)^{\alpha}) E_{\alpha,\beta}^{\gamma,q}(-g(\alpha)(y-\lambda)^{\alpha}) d\tau d\lambda. $$

4. Conclusion

The main aim of the paper was to promote the idea of fractional derivative with generalized exponential decay law that is mostly observed in nature in our daily live. We introduced as modified version of Atangana-Baleanu derivatives the Weyl approach of derivative. Some useful properties are presented and some theorems are given. Another concepts of derivative called AKC and AKR are introduced based upon the Prabhakar Mittag-Leffler function are introduced.

References

[1] A. Atangana, *On the new fractional derivative and application to nonlinear Fisher's reaction-diffusion equation*, Appl. Math. Comput., **273** (2016), 948-956. 1

[2] A. Atangana, D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Therm. Sci., **20** (2016), 763–769. 1

[3] A. Atangana, I. Koca, *Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order*, Chaos Solitons Fractals, **89** (2016), 447–454.

[4] A. Atangana, I. Koca, *On the new fractional derivative and application to nonlinear Baggs and Freedman model*, J. Nonlinear Sci. Appl., **9** (2016), 2467–2480. 1

[5] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent*, II, Reprinted from Geophys. J. R. Astr. Soc., **13** (1967), 529–539, Fract. Calc. Appl. Anal., **11** (2008), 4–14. 1

[6] M. Caputo, M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl., **1** (2015), 73–85. 1

[7] A. V. Chechkin, R. Gorenflo, I. M. Sokolov, *Fractional diffusion in inhomogeneous media*, J. Phys. A, **38** (2005), L679–L684. 1

[8] A. H. Cloot, J. P. Botha, *A generalised groundwater flow equation using the concept of non-integer order derivatives*, Water SA, **32** (2006), 1–7.

[9] M. Davison, C. Essex, *Fractional differential equations and initial value problems*, Math. Sci., **23** (1998), 108–116. 1

[10] G. Jumarie, *Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results*, Comput. Math. Appl., **51** (2006), 1367–1376. 1

[11] A. A. Kilbas, M. Saigo, *On Mittag-Leffler type function, fractional calculus operators and solutions of integral equations*, Integral Transform. Spec. Funct., **4** (1996), 355–370. 2.6

[12] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, (2006). 1

[13] I. Koca, *A method for solving differential equations of q-fractional order*, Appl. Math. Comput., **266** (2015), 1–5. 1

[14] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Dedicated to the 60th anniversary of Prof. Francesco Mainardi, Fract. Calc. Appl. Anal., **5** (2002), 367–386. 1

[15] T. R. Prabhakar, *A singular integral equation with a generalized Mittag Leffler function in the kernel*, Yokohama Math. J., **19** (1971), 7–15. 2.6