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# Chaos control and synchronization of a new chaotic financial system with integer and fractional order

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# Abstract

Synchronization of chaotic dynamical systems with fractional order is receiving great attention in recent literature because of its applications in a variety of fields including optics, secure communications of analog and digital signals, and cryptographic systems. In this paper, chaos control of a new financial system, and chaos synchronization between two identical financial systems, and non-identical financial systems with integer and fractional order are investigated. Chaos control is based on a linear feedback controller for stabilizing chaos to unstable equilibrium. In addition, chaos synchronization, not only between two identical new chaotic financial systems, but also between the new financial system and an another financial system given in the literature is realized by using active control technique. The synchronization is done for integer and fractional order in each case. It is shown that chaotic behavior can be controlled easily to any unstable equilibrium point of the new financial system. Also, it is observed that synchronization is enhanced when the fractional order increases and approximates to one. Numerical simulations are used to verify the proposed methods.

**Keywords:** Chaos control, synchronization, chaotic financial systems, fractional order systems, active control, feedback controller.

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## 1. Introduction

Many life-like phenomena from ancient times to the present day, are based on nonlinearities. The knowledge and mastery of these phenomena require a good understanding of the related laws of nonlinear sciences [31]. To get their hands on these complex behaviors, several researchers invest themselves in the development of approximate solutions and others in numerical and/or experimental analysis. As an example we can cite the recent works [37] where, is found the solutions of the fractional kinetic equation related with the (p, q)-Mathieu-type series through the procedure of Sumudu and Laplace transforms, [35] that a modified Laplace transform and its properties are analyzed, [3] where a novel approach termed as fractional iteration algorithm-I for finding the numerical solution of nonlinear noninteger order partial

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differential equations is presented. Another example is a new analyzing tool, called the fractional iteration algorithm I for finding numerical solutions of nonlinear time fractional-order Cauchy reaction-diffusion model equations proposed in [4]. One of the complex phenomena is chaotic behavior in nonlinear systems [31]. Since the first chaotic attractor found in a weather forecast model known as the Lorenz chaotic system [24], chaotic behaviors have been also observed in a variety of fields, and the issue of chaos control and synchronization have been extensively investigated since early for integer order and fractional order systems. For example, see [7] for some review of several types of synchronization features, in [42] chaos control of Lorenz, Chen, and Lü systems has been performed. In [13] some methods of chaos control in several research branches are given, in [1] the authors deal with chaos control of a financial system with fractional order has been performed, and in [39] chaos control of a modified coupled dynamos system with fractional order has been investigated.

Fractional calculus was initiated 300 years ago, but fractional order systems have been recently widely used to describe the dynamics of many systems in many interdisciplinary fields. The fractional derivative takes into account the memory effect which is very important when we want to describe systems with long memories. The systems described with fractional derivative show chaotic behavior, see for example, reference [21] for Chen's system with fractional order, reference [26] for fractional order Arneodo's system, reference [8] for fractional order financial system, and reference [10] for the fractional Liu one. Those studies have been made about chaos synchronization due to its applications in a variety of fields including optics, secure communications of analog and digital signals [16], and cryptographic systems [15]. A variety of approaches including the OGY method [32], linear feedback control [19, 39], nonlinear feedback control [18], adaptive control [22, 41], backstepping design [42], active control [2, 5], nonlinear control [33], etc., have been proposed to control, and synchronize chaotic systems with integer and fractional order.

Since the first works of Strotz et al. [36] in the economic and financial area, various economics chaotic models have been proposed, such as the Kaldorian model [25], the hyperchaotic finance system [43], and other nonlinear dynamical models [27, 28]. Researchers in the field of nonlinear systems use chaos theory to study the complex behavior of these different economic and financial systems. It is well known that chaos in financial systems may lead to the destabilization of the economy and also to a financial crisis [11]. Thus, the market can be out of control [17, 29]. So, it is necessary to study the global stabilization of financial chaotic systems. The control of financial systems provides stabilization in the market and at the same time removes unpredictable trajectories. Therefore, it has great importance for financial systems. Realize the synchronization of financial chaotic systems is very useful. From a management point of view, the master system states are considered as the desired states, and the slave system states are considered as the actual states of the system. In this case, synchronization consists of making the real states of the system converge towards the desired states. In the literature, several works have been carried out to control the chaos in the economic and financial systems and to proceed to their synchronization. For example, in reference [1], the authors controlled the fractional order finance system proposed by Chen [8] by using the nonlinear feedback control to stabilize periodic orbits and unstable fixed points. Hajipour et al. studied the dynamic behavior in a financial system with incommensurate order and carry out adaptive control of chaos in the system [14]. In [9], Chen et al. designed simple linear feedback controllers to control and synchronize a fractional order financial system. In 2020, Liao et al. [23] presented an alternative model to consider the interplay between the interest rate x(t), the investment demand y(t), and the price index z(t). The authors investigated this model using numerical simulations. Their investigation indicated that the interactions among three factors in the financial model cause complex behaviors. This model exhibits a variety of complex dynamic behaviors including chaos and period-doubling [23]. Therefore, chaotic behaviors have been observed in this new financial system. To the best of the authors' knowledge, the control of chaos, and chaos synchronization of this new financial system have not been investigated yet. Inspired by the above works, in this work, chaos control using linear feedback control and chaos synchronization using active control of this new financial system are investigated. The main contributions in this work can be divided into three aspects: (1) the chaos is removed from this system with integer,

and fractional order by applying a simple, but effective linear feedback controller to stabilize the unstable equilibria of the new financial system; (2) furthermore, chaos synchronization between two identical new financial systems, and also between two non-identical financial systems is realized only using the active control technique for integer and fractional order systems; (3) the influence of the fractional order on synchronization is shown by numerical simulations, and it is shown that synchronization is faster when the fractional order q tends to one.

The rest of the paper is organized as follows. In Section 2, a description, and some properties of the new chaotic financial system including the equilibria, the Lyapunov exponents, and Kaplan-Yorke dimension are given. Chaos control via a linear feedback controller is studied in Section 3. Section 4 deals with the chaos synchronization between two identical new financial systems, and also between the new financial system, and an another financial system given in the literature for integer and fractional order. The conclusion is done in Section 5 to give some remarks.

#### 2. System description and some properties

Liao et al. [23] presented an alternative model to consider the interplay between the interest rate x(t), the investment demand y(t), and the price index z(t). This new chaotic financial system is described by:

$$\begin{cases} \dot{x} = ez + (y - d)x, \\ \dot{y} = -my^2 - nx^2 + l, \\ \dot{z} = -\alpha z - \gamma x - \delta y, \end{cases}$$
(2.1)

in which the parameters d, m,  $\alpha$ , n, l,  $\gamma$  and  $\delta$  are constants. In [23], it is assumed that in system (2.1), there is a proportional relationship between the rate of change of interest rate and the price index. The interest rate is significantly influenced by investment demand and the rate of change of investment demand is negatively correlated with the square of investment demand. It is also assumed that the interest rate has negative correlation on the change rate of the price index with rate  $\gamma$ .



Figure 1: Phase diagrams and time histories of system (2.1): (a) projected onto x-y phase plane, (b) projected onto x-z phase plane, (c) time history of x, and (d) time history of y.



Figure 2: Bifurcation diagrams and largest Lyapunov exponents for system (2.1) when *e*, l and  $\gamma$  vary: (a)  $y_{max}$  and  $\lambda_{max}$  when *e* varies, (b)  $y_{max}$  and  $\lambda_{max}$  when l varies, and (c)  $y_{max}$  and  $\lambda_{max}$  when  $\gamma$  varies.

In [23] when d = 0.3, m = 0.02,  $\alpha$  = 1, l = 1, n = 0.1,  $\delta$  = 0.05, e = 1.2,  $\gamma$  = 1, and initial conditions are taken as (1.2, 1.5, 1.6), this system exhibits chaotic behavior. Figures 1 (a)-1 (b) show the chaotic phase diagrams and Figures 1 (c)-1 (d) show the chaotic time histories of the system. Bifurcation diagrams of system (2.1) and largest Lyapunov exponents for parameters *e*, *l*, and  $\gamma$  are also shown in Figure 2. From

Figure 2 (a), it can be observed that for  $e \in [0.85, 1.5]$ , and when the values of the other parameters of the system remain the same as those in Figure 1, the largest Lyapunov exponent is positive, indicating that the system has chaotic behavior in this interval. For other values of e, the largest Lyapunov exponent is negative, so the system exhibits regular behavior. Similarly, in Figure 2 (b) for  $l \in [0.6, 1.8]$ , the largest Lyapunov exponent is positive, so the system exhibits a chaotic behavior for this range of values. Finally, in Figure 2 (c) for  $\gamma \in [0.72, 1.26]$ , the largest Lyapunov exponent is positive which shows that the system exhibits a chaotic behavior when  $\gamma$  is chosen in that interval. Also for initial conditions taken as (0.2, 0.5, 0.6) this system exhibit chaotic attractor [23].

The Kaplan-Yorke dimension [20], which shows the complexity of attractor, is given by:

$$D_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|},$$

where j is the largest integer satisfying  $\sum_{i=1}^{j} \lambda_i \ge 0$  and  $j \sum_{i=1}^{j+1} \lambda_i < 0$ . We use Wolf Algorithm [40] to calculate the Lyapunov exponents in order to determine the Kaplan-Yorke dimension for the system with initial conditions (1.2, 1.5, 1.6). The three Lyapunov exponents found for the attractor are as follows:  $\lambda_1 = 0.1910$ ,  $\lambda_2 = 0$  and  $\lambda_3 = -0.4969$ . The calculated fractal dimension of the system from these exponents is given by:

$$D_{KY} = 2 + 0.1910 / 0.4969 \approx 2.384$$

which shows that we can find a strange attractor in the system (see Figure 1).

The Jacobian matrix of system (2.1) at any of its equilibrium points  $E_* = (x^*, y^*, z^*)$  is given by:

$$\mathrm{J}=egin{pmatrix} \mathrm{y}^*-\mathrm{d} & \mathrm{x}^* & \mathrm{e} \ -2\mathrm{n}\mathrm{x}^* & -2\mathrm{m}\mathrm{y}^* & 0 \ -\gamma & -\delta & -lpha \end{pmatrix}.$$

When the values of parameters are chosen as above, the equilibrium points of system (2.1) can be obtained by solving the equations  $\dot{x} = 0$ ,  $\dot{y} = 0$  and  $\dot{z} = 0$ . The system has four equilibrium points given as follows:

 $E_1 = (0.049498497, -7.070201517, 0.304011579), E_2 = (0.076160842, 7.069016737, -0.429611679),$ 

 $\mathsf{E}_3 = (3.087391472, \ 1.529728564, \ -3.163877901), \quad \mathsf{E}_4 = (-3.093050811, \ 1.471456216, \ 3.019478000).$ 

The corresponding eigenvalues, their nature and index are given in Table 1.

rable 1. Equilibrium points, corresponding eigenvalues and then nature.			
Equilibria $(x_0, y_0, z_0)$	Eigenvalues	Nature	Index
E <sub>1</sub>	$\lambda_1 = -7.1758,  \lambda_2 = -1.1944,  \lambda_3 = 0.2828$	Saddle point	1
E <sub>2</sub>	$\lambda_1 = 6.6418,  \lambda_2 = -0.8428,  \lambda_3 = -0.2828$	Saddle point	1
E <sub>3</sub>	$\lambda_1 = -0.7378$ , $\lambda_{2,3} = 0.4532 \pm 1.5251$ i	Saddle point	2
E <sub>4</sub>	$\lambda_1 = -0.7548,  \lambda_{2,3} = 0.4337 \ \pm \ 1.5487$ i	Saddle point	2

Table 1: Equilibrium points, corresponding eigenvalues and their nature

#### 3. Chaos control via linear feedback controller

In this section, linear feedback controllers are designed to guide the chaotic trajectories of the system (2.1) to two of its unstable equilibrium points  $E_1$  and  $E_3$ . The controlled system is given by:

$$\begin{cases} \dot{x} = ez + (y - d)x + k_1(x - \bar{x}), \\ \dot{y} = -my^2 - nx^2 + l + k_2(y - \bar{y}), \\ \dot{z} = -\alpha z - \gamma x - \delta y + k_3(z - \bar{z}), \end{cases}$$
(3.1)

where  $k_i(i = 1, 2, 3)$  have to be determined and represent the feedback gains,  $(\bar{x}; \bar{y}; \bar{z})$  are any equilibrium point of the system (2.1). The Jacobian matrix of the controlled system (3.1) at any of its equilibrium points  $E_* = (x^*, y^*, z^*)$  is given by:

$$J_{(E^*)} = \begin{pmatrix} k_1 + y^* - d & x^* & e \\ -2nx^* & k_2 - 2my^* & 0 \\ -\gamma & -\delta & k_3 - \alpha \end{pmatrix}.$$
 (3.2)

# 3.1. Controlling chaos to the equilibrium point $E_1$



Figure 3: Controlling chaos to the equilibrium point  $E_1$ : (a) time history of x, (b) time history of y, and (c) time history of z. The control inputs are activated at t = 200.

The Jacobian matrix (3.2) of the controlled system (3.1) evaluated at the equilibrium point  $E_1 = (0.0495, -7.0702, 0.3040)$  is given by:

$$J_{(E_1)} = \begin{pmatrix} k_1 - 7.3702 & 0.0495 & 1.2 \\ -0.0099 & k_2 + 0.2828 & 0 \\ -1 & -0.05 & k_3 - 1 \end{pmatrix}.$$
(3.3)

To simplify our task, let  $k_1 = k_2 = k_3 = k$  and then, the characteristic equation of the Jacobian matrix (3.3) is:

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0$$

with

$$\begin{cases} a_1 = -3k + 8.0874, \\ a_2 = 3k^2 - 16.1748k + 6.2036, \\ a_3 = -k^3 + 8.0874k^2 - 6.2036k - 2.4238. \end{cases}$$

By using the Routh-Hurwitz criterion, we have found that the controlled system (3.1) is asymptotically stable at the equilibrium point  $E_1$  if the following conditions:

a)  $a_1 > 0;$ 

b)  $a_1a_2 > a_3, a_3 > 0,$ 

are satisfied. After some calculations, we get that to satisfy those two conditions, the control gain has to satisfy the following inequality: k < -0.2828. Thus, when k is chosen such as the condition k < -0.2828 is satisfied, the characteristic equation has three eigenvalues with negative real parts. For simulations, we choose  $k_1 = k_2 = k_3 = k = -0.30$  and initial conditions as (1.2, 1.5, 1.6) and then the controlled system is stabilized and converge towards the equilibrium point  $E_1$  (see Figures 3 (a)-3 (c)). Thus, with this choice, the chaos can be controlled to the equilibrium point  $E_1$  easily. In all simulations, the control inputs are activated at t = 200.

3.2. Controlling chaos to the equilibrium point E<sub>3</sub>



Figure 4: Controlling chaos to the equilibrium point  $E_3$ : (a) time history of x, (b) time history of y, and (c) time history of z. The control inputs are activated at t = 200.

The Jacobian (3.2) of the controlled system evaluated at the equilibrium point  $E_3 = (3.0874, 1.5297, -3.1639)$  is given by:

$$J_{(E_3)} = \begin{pmatrix} k_1 + 1.2297 & 3.0874 & 1.2 \\ -0.6175 & k_2 - 0.0612 & 0 \\ -1 & -0.05 & k_3 - 1 \end{pmatrix}.$$
 (3.4)

To simplify our task, let  $k_1 = k_2 = k_3 = k$  and then, the characteristic equation of the Jacobian matrix (3.4) is:

$$\lambda^3 + \mathfrak{a}_1 \lambda^2 + \mathfrak{a}_2 \lambda + \mathfrak{a}_3 = 0$$

with

$$\begin{cases} a_1 = -3k - 0.1685, \\ a_2 = 3k^2 + 0.337k + 1.8627, \\ a_3 = -k^3 - 0.1685k^2 - 1.8627k + 1.8676. \end{cases}$$

By using the Routh-Hurwitz criterion mentioned above and after some calculations, it is found that the controlled system (3.1) is asymptotically stable at the equilibrium point  $E_3$  if the control gain satisfies the following inequality: k < -0.4531.

Thus, when k is chosen such as the condition k < -0.4531 is satisfied, the characteristic equation has three eigenvalues with negative real parts. For simulations, we choose  $k_1 = k_2 = k_3 = k = -0.5$ , and then the controlled system (3.1) is stabilized and converge towards the equilibrium point  $E_3$  (see Figures 4 (a)-4 (c)). Therefore, the numerical simulations confirm all the analytical results. In all simulations, the control inputs are activated at t = 200. With an appropriate choice of control gains  $k_1$ ,  $k_2$ ,  $k_3$  such that all eigenvalues of the equilibrium point have negative real parts, we can stabilize the remainder equilibrium points. Moreover, for the control of the fractional order version of the controlled system (3.1), the same feedback gains can be used to stabilize the chaotic fractional order controlled system to the equilibrium points  $E_1$  and  $E_3$  (not shown).

## 4. Synchronization of the new financial system

#### 4.1. Integer order case

In this section, the active control method is used to achieve chaos synchronization between two identical new chaotic financial systems, and also between this system and an another financial system given in the literature [27, 28].

## 4.1.1. Chaos synchronization between identical systems

Chaos synchronization between two identical new chaotic financial systems is realized in this section. We assume that the drive system (master system) denoted by index 1 drives the response system (slave system) denoted by index 2. The systems are:

$$\begin{cases} \dot{x}_{1} = ez_{1} + (y_{1} - d)x_{1}, \\ \dot{y}_{1} = -my_{1}^{2} - nx_{1}^{2} + l, \\ \dot{z}_{1} = -\alpha z_{1} - \gamma x_{1} - \delta y_{1}, \end{cases}$$
(4.1)

and

$$\begin{cases} \dot{x}_{2} = ez_{2} + (y_{2} - d)x_{2} + u_{1}, \\ \dot{y}_{2} = -my_{2}^{2} - nx_{2}^{2} + l + u_{2}, \\ \dot{z}_{2} = -\alpha z_{2} - \gamma x_{2} - \delta y_{2} + u_{3}, \end{cases}$$
(4.2)

where  $u_i(i = 1, 2, 3)$  represent the active control functions and have to be designed. Let us define the error functions between the response system (4.2) and the drive system (4.1) as follows:

$$e_1 = x_2 - x_1, \qquad e_2 = y_2 - y_1, \qquad \text{and} \qquad e_3 = z_2 - z_1.$$
 (4.3)

Eq. (4.3) with systems (4.1) and (4.2) give the error dynamical system as follows:

$$\begin{cases} \dot{e}_{1} = ee_{3} - de_{1} + e_{1}y_{1} + e_{2}(e_{1} + x_{1}) + u_{1}, \\ \dot{e}_{2} = -me_{2}^{2} - 2me_{2}y_{1} - ne_{1}(e_{1} + 2x_{1}) + u_{2}, \\ \dot{e}_{3} = -\alpha e_{3} - \gamma e_{1} - \delta e_{2} + u_{3}. \end{cases}$$

$$(4.4)$$

We design the active functions in system (4.4) as follows:

$$\begin{cases} u_{1} = -e_{1}y_{1} - e_{2}(e_{1} + x_{1}) + v_{1}, \\ u_{2} = me_{2}^{2} + 2me_{2}y_{1} + ne_{1}(e_{1} + 2x_{1}) + v_{2}, \\ u_{3} = v_{3}. \end{cases}$$

$$(4.5)$$

With the choice of the active functions in system (4.5) error dynamical system (4.4) becomes:

$$\begin{cases} \dot{e}_{1} = ee_{3} - de_{1} + v_{1}, \\ \dot{e}_{2} = v_{2}, \\ \dot{e}_{3} = -\alpha e_{3} - \gamma e_{1} - \delta e_{2} + v_{3}. \end{cases}$$
(4.6)

For the control inputs  $v_1$ ,  $v_2$  and  $v_3$ , let's choose this particular form:

$$\begin{cases} v_1 = -ee_3, \\ v_2 = -e_2, \\ v_3 = \gamma e_1 + \delta e_2. \end{cases}$$

For this particular choice, the closed-loop system (4.6) has eigenvalues which are -d, -1 and  $-\alpha$ . So, the error system is asymptotically stable. When the parameters' values are taken as d = 0.3, m = 0.02,  $\alpha = 1$ , l = 1, n = 0.1,  $\delta = 0.05$ , e = 1.2, and  $\gamma = 1$ , the new chaotic financial system shows chaotic behavior (see Figure 1). For the drive system, initial conditions are taken as (1.2, 1.5, 1.6) and for the response system (0.2, 0.5, 0.6) are taken. We use step size h = 0.001 to solve both systems with the fourth order Runge-Kutta integration method.



Figure 5: Time histories of synchronized drive system (blue line), and response system (red line) and time evolution of the error functions for identical systems. The control inputs are activated at t = 100. (a) Time histories of the states  $x_1$ ,  $x_2$ , (b) time histories of the states  $y_1$ ,  $y_2$ , (c) time histories of the states  $z_1$ ,  $z_2$ , and (d) time histories of the error functions  $e_1$  (blue line),  $e_2$  (red line), and  $e_3$  (green line).

Figures 5 (a)-5 (d) show the simulation results of the chaos synchronization between two identical new chaotic financial systems. Figures 5 (a)-5 (c) display the time evolutions of synchronized two identical systems with the controller applied at a time  $t \ge 100$ . Figure 5 (d) shows that the time response of the error signals between the states of both systems are converged to zero as time evolves and this shows that the synchronization has been achieved.

#### 4.1.2. Chaos synchronization between non-identical systems

In this part, synchronization between two different chaotic systems is achieved via active control. The financial system reported by the authors [27, 28] is considered as the response system. The new chaotic financial system

$$\begin{cases} \dot{x}_{1} = ez_{1} + (y_{1} - d_{1})x_{1}, \\ \dot{y}_{1} = -m_{1}y_{1}^{2} - nx_{1}^{2} + l, \\ \dot{z}_{1} = -\alpha z_{1} - \gamma x_{1} - \delta y_{1}, \end{cases}$$
(4.7)

is chosen as the drive system, in which  $d_1 = 0.3$ ,  $m_1 = 0.02$ ,  $\alpha = 1$ , l = 1, n = 0.1,  $\delta = 0.05$ , e = 1.2 and  $\gamma = 1$ . And the response system, the financial system with control input  $u_i$  (i = 1, 2, 3) defined as follows:

$$\begin{cases} \dot{x}_2 = z_2 + (y_2 - d_2)x_2 + u_1, \\ \dot{y}_2 = 1 - m_2 y_2 - x_2^2 + u_2, \\ \dot{z}_2 = -x_2 - \alpha z_2 + u_3, \end{cases}$$
(4.8)

where  $d_2 = 3$ ,  $m_2 = 0.1$  and  $\alpha = 1$ . The last system (response system) shows chaotic behavior without control inputs, i.e., when  $u_i(i = 1, 2, 3) = 0$  and initial conditions (2.0, 3.0, 2.0) are taken (see Figures 6 (a)-6 (b)).



Figure 6: Phase diagrams of system (4.8) with the control inputs  $u_i(i = 1, 2, 3) = 0$  and initial conditions (2.0, 3.0, 2.0) projected onto: (a) x-y phase plane and (b) x-z phase plane.

Eq. (4.3) with systems (4.7) and (4.8) give the error dynamical system as follows:

$$\begin{cases} \dot{e}_{1} = e_{3} + (1 - e)z_{1} + e_{2}(e_{1} + x_{1}) + e_{1}y_{1} - d_{2}e_{1} + (d_{1} - d_{2})x_{1} + u_{1}, \\ \dot{e}_{2} = 1 - m_{2}e_{2} + y_{1}(m_{1}y_{1} - m_{2}) - e_{1}(e_{1} + 2x_{1}) + x_{1}^{2}(n - 1) - l + u_{2}, \\ \dot{e}_{3} = -\alpha e_{3} - e_{1} + (\gamma - 1)x_{1} + \delta y_{1} + u_{3}, \end{cases}$$

$$(4.9)$$

where  $u_i(i = 1, 2, 3)$  have the same sense as the above case. We design the active functions in system (4.9) as follows:

$$\begin{cases} u_1 = -(1-e)z_1 - e_2(e_1 + x_1) - e_1y_1 - (d_1 - d_2)x_1 + v_1, \\ u_2 = -1 - y_1(m_1y_1 - m_2) + e_1(e_1 + 2x_1) - x_1^2(n-1) + l + v_2, \\ u_3 = -(\gamma - 1)x_1 - \delta y_1 + v_3. \end{cases}$$

Thus, the error dynamical system (4.9) becomes:

$$\begin{cases} \dot{e}_1 = e_3 - d_2 e_1 + v_1, \\ \dot{e}_2 = -m_2 e_2 + v_2, \\ \dot{e}_3 = -\alpha e_3 - e_1 + v_3. \end{cases}$$
(4.10)

For a choice of the control inputs  $v_1$ ,  $v_2$ , and  $v_3$ , let's take:

$$\begin{cases} v_1 = -e_3, \\ v_2 = 0, \\ v_3 = e_1. \end{cases}$$



Figure 7: Time histories of synchronized drive system (blue line), and response system (red line) and time evolution of the error functions for non-identical systems. The control inputs are activated at t = 100. (a) Time histories of the states  $x_1$ ,  $x_2$ , (b) time histories of the states  $y_1$ ,  $y_2$ , (c) time histories of the states  $z_1$ ,  $z_2$ , and (d) time histories of the errors  $e_1$  (blue line),  $e_2$  (red line), and  $e_3$  (green line).

For this particular choice, the closed-loop system (4.10) has eigenvalues which are  $-d_2$ ,  $-m_2$  and  $-\alpha$ . Thus, the error dynamical system is asymptotically stable. So, this choice will lead to the synchronization of both systems. The initial conditions for the financial system (response system) (4.8), and the new chaotic financial system (drive system) (4.7) are (2.0, 3.0, 2.0) and (1.2, 1.5, 1.6), respectively. We use step size h = 0.001 to solve both systems. Figures 7 (a)-7 (d) show the simulation results of the synchronization between the financial system and the new chaotic financial system. Figures 7 (a)-7 (c) display the time evolutions of synchronized two non-identical systems with the controller applied at a time  $t \ge 100$ . Figure 7 (d) shows that the time response of the error signals between the states of both systems are converged to zero and this shows that the synchronization has been achieved.

#### 4.2. Fractional order case

In this section, we study the synchronization between two identical fractional order new financial systems, and between the new fractional order financial system and the fractional order version of the other financial system proposed by Chen [8]. Caputo definition for fractional order derivatives will be adopted in this part. For more details, see reference [34].

#### 4.2.1. Chaos synchronization between identical systems

For the fractional order case, the standard derivatives are replaced by fractional order derivatives. Thus, for the synchronization between identical systems, we assume that the drive system (master system) denoted by index 1 drives the response system (slave system) denoted by index 2. The systems are:

$$\begin{cases} D^{q}x_{1} = ez_{1} + (y_{1} - d)x_{1}, \\ D^{q}y_{1} = -my_{1}^{2} - nx_{1}^{2} + l, \\ D^{q}z_{1} = -\alpha z_{1} - \gamma x_{1} - \delta y_{1}, \end{cases}$$
(4.11)

and

$$\begin{cases} D^{q}x_{2} = ez_{2} + (y_{2} - d)x_{2} + u_{1}, \\ D^{q}y_{2} = -my_{2}^{2} - nx_{2}^{2} + l + u_{2}, \\ D^{q}z_{2} = -\alpha z_{2} - \gamma x_{2} - \delta y_{2} + u_{3}, \end{cases}$$

$$(4.12)$$

where  $0 < q \leq 1$  and  $u_1$ ,  $u_2$ ,  $u_3$  are active functions to be designed. Let us define the error functions between the response system (4.12) and the drive system (4.11) as follows:

$$e_1 = x_2 - x_1, \qquad e_2 = y_2 - y_1, \qquad \text{and} \qquad e_3 = z_2 - z_1.$$
 (4.13)

Eq. (4.13) with systems (4.11) and (4.12) give the error dynamical system as follows:

$$\begin{cases} D^{q}e_{1} = ee_{3} - de_{1} + e_{1}y_{1} + e_{2}(e_{1} + x_{1}) + u_{1}, \\ D^{q}e_{2} = -me_{2}^{2} - 2me_{2}y_{1} - ne_{1}(e_{1} + 2x_{1}) + u_{2}, \\ D^{q}e_{3} = -\alpha e_{3} - \gamma e_{1} - \delta e_{2} + u_{3}. \end{cases}$$

$$(4.14)$$

We design the active control in system (4.14) as follows:

$$\begin{cases} u_1 = -e_1y_1 - e_2(e_1 + x_1) + v_1, \\ u_2 = me_2^2 + 2me_2y_1 + ne_1(e_1 + 2x_1) + v_2, \\ u_3 = v_3. \end{cases}$$

Thus, the error dynamical system (4.14) becomes:

$$\begin{cases} D^{q}e_{1} = ee_{3} - de_{1} + v_{1}, \\ D^{q}e_{2} = v_{2}, \\ D^{q}e_{3} = -\alpha e_{3} - \gamma e_{1} - \delta e_{2} + v_{3}. \end{cases}$$
(4.15)

The choice of the control inputs  $v_i$  (i = 1, 2, 3) has to lead all eigenvalues  $\lambda_i$  of system (4.15) to verify the condition [30, 38]:

$$|\arg(\lambda_i)| > q\pi/2. \tag{4.16}$$

Thus, let's choose the control inputs  $v_1$ ,  $v_2$ , and  $v_3$  as follows:

$$\begin{cases} v_1 = -ee_3, \\ v_2 = -e_2, \\ v_3 = \gamma e_1 + \delta e_2. \end{cases}$$

For this particular choice, the linear system (4.15) has eigenvalues which are -d, -1 and  $-\alpha$ . Hence, for  $q \leq 1$  the condition (4.16) is satisfied. Thus, the error system (4.15) is stable and the synchronization is achieved. The parameters' values are taken as d = 0.3, m = 0.02,  $\alpha = 1$ , l = 1, n = 0.1,  $\delta = 0.05$ , e = 1.2 and  $\gamma = 1$ . The fractional order is chosen as q = 0.99 for which the system is chaotic as shows in Figure 8. The initial conditions for the drive, the response, and the error systems are  $(x_1(0), y_1(0), z_1(0)) = (1.2, 1.5, 1.6)$ ,  $(x_2(0), y_2(0), z_2(0)) = (0.2, 0.5, 0.6)$  and  $(e_1(0), e_2(0), e_3(0)) = (-1, -1, -1)$ , respectively. Both systems are solved with step size h = 0.01 using the Adams-Bashforth-Moulton predictor-corrector method proposed by Diethelm et al. [12]. Figures 9 (a)-9 (d) show the simulation results of the synchronization between two identical fractional order new chaotic financial systems. Figures 9 (a)-9 (c) display the time evolutions of synchronized two identical fractional order systems with the controller applied at a time t  $\geq 30$ . Figure 9 (d) shows that the time response of the error signals between the states of both systems are converged to zero and this shows that the synchronization has been achieved.



Figure 8: Phase diagrams and time histories of system (4.11) with q = 0.99: (a) projected onto x-y phase plane, (b) projected onto x-z phase plane, (c) time history of x and (d) time history of y.



Figure 9: Time histories of synchronized drive system (blue line), and response system (red line) and time evolution of the error functions for identical systems with order q = 0.99. The control inputs are activated at t = 30. (a) Time histories of the states  $x_1$ ,  $x_2$ , (b) time histories of the states  $y_1$ ,  $y_2$ , (c) time histories of the states  $z_1$ ,  $z_2$ , and (d) time histories of the errors  $e_1$  (blue line),  $e_2$  (red line), and  $e_3$  (green line).



Figure 10: Time histories of synchronized drive system (blue line), and response system (red line) and time evolution of the error functions for non-identical systems with order q = 0.99. The control inputs are activated at t = 30. (a) Time histories of the states  $x_1$ ,  $x_2$ , (b) time histories of the states  $y_1$ ,  $y_2$ , (c) time histories of the states  $z_1$ ,  $z_2$ , and (d) time histories of the errors  $e_1$  (blue line),  $e_2$  (red line), and  $e_3$  (green line).

## 4.2.2. Chaos synchronization between non-identical systems

In this section, the synchronization of two different chaotic fractional order systems is realized. Assuming that the new chaotic financial is the drive system given by:

$$\begin{cases} D^{q}x_{1} = ez_{1} + (y_{1} - d_{1})x_{1}, \\ D^{q}y_{1} = -m_{1}y_{1}^{2} - nx_{1}^{2} + l, \\ D^{q}z_{1} = -\alpha z_{1} - \gamma x_{1} - \delta y_{1}, \end{cases}$$
(4.17)

and the other financial system is the response system given by:

$$\begin{cases} D^{q}x_{2} = z_{2} + (y_{2} - d_{2})x_{2} + u_{1}, \\ D^{q}y_{2} = 1 - m_{2}y_{2} - x_{2}^{2} + u_{2}, \\ D^{q}z_{2} = -x_{2} - \alpha z_{2} + u_{3}. \end{cases}$$

$$(4.18)$$

The unknown terms  $u_1$ ,  $u_2$ ,  $u_3$  in system (4.18) have the same sense as above case. By taking into account Eq. (4.13), the error dynamical system between the drive system (4.17) and the response system (4.18) can be written as follows:

$$\begin{cases} D^{q}e_{1} = e_{3} + (1 - e)z_{1} + e_{2}(e_{1} + x_{1}) + e_{1}y_{1} - d_{2}e_{1} + (d_{1} - d_{2})x_{1} + u_{1}, \\ D^{q}e_{2} = 1 - m_{2}e_{2} + y_{1}(m_{1}y_{1} - m_{2}) - e_{1}(e_{1} + 2x_{1}) + (n - 1)x_{1}^{2} - l + u_{2}, \\ D^{q}e_{3} = -e_{1} - \alpha e_{3} + (\gamma - 1)x_{1} + \delta y_{1} + u_{3}. \end{cases}$$

$$(4.19)$$

The active control can be designed as follows:

$$\begin{cases} u_{1} = -(1-e)z_{1} - e_{2}(e_{1} + x_{1}) - e_{1}y_{1} - (d_{1} - d_{2})x_{1} + v_{1}, \\ u_{2} = -1 - y_{1}(m_{1}y_{1} - m_{2}) + e_{1}(e_{1} + 2x_{1}) - (n-1)x_{1}^{2} + l + v_{2}, \\ u_{3} = -(\gamma - 1)x_{1} - \delta y_{1} + v_{3}. \end{cases}$$
(4.20)

With the choice of active functions in system (4.20), the error dynamical system (4.19) becomes:

$$\begin{cases} D^{q}e_{1} = e_{3} - d_{2}e_{1} + v_{1}, \\ D^{q}e_{2} = -m_{2}e_{2} + v_{2}, \\ D^{q}e_{3} = -e_{1} - \alpha e_{3} + v_{3}. \end{cases}$$
(4.21)

Let the control inputs  $v_1$ ,  $v_2$  and  $v_3$  be

$$\begin{cases} v_1 = -e_3, \\ v_2 = 0, \\ v_3 = e_1. \end{cases}$$



Figure 11: Phase diagrams of system (4.17): (a-b) for order q = 0.93 projected onto x-y, and x-z phase planes, (c-d) for order q = 0.87 projected onto x-y and x-z phase planes.



Figure 12: Time histories of synchronized drive system (blue line), and response system (red line) and time evolution of the error functions for non-identical systems with order q = 0.93. The control inputs are activated at t = 30. (a) Time histories of the states  $x_1$ ,  $x_2$ , (b) time histories of the states  $y_1$ ,  $y_2$ , (c) time histories of the states  $z_1$ ,  $z_2$ , and (d) time histories of the errors  $e_1$  (blue line),  $e_2$  (red line), and  $e_3$  (green line).



Figure 13: Time histories of synchronized drive system (blue line), and response system (red line) and time evolution of the error functions for non-identical systems with order q = 0.87. The control inputs are activated at t = 30. (a) Time histories of the states  $x_1$ ,  $x_2$ , (b) time histories of the states  $y_1$ ,  $y_2$ , (c) time histories of the states  $z_1$ ,  $z_2$ , and (d) time histories of the errors  $e_1$  (blue line),  $e_2$  (red line), and  $e_3$  (green line).

Then, the eigenvalues of the linear system (4.21) are  $-d_2$ ,  $-m_2$  and  $-\alpha$ . Hence, for  $q \le 1$  the condition (4.16) is satisfied. Thus, the error dynamical system is asymptotically stable. So, this choice will lead to the synchronization of both systems. The parameters values are taken as  $d_1 = 0.3$ ,  $m_1 = 0.02$ ,  $\alpha = 1$ , l = 1, n = 0.1,  $\delta = 0.05$ , e = 1.2,  $\gamma = 1$  for the drive system, and for the response system, the parameters are taken as  $d_2 = 3$ ,  $m_2 = 0.1$  and  $\alpha = 1$ . The fractional order is chosen as q = 0.99 for both systems and for which the other fractional order financial system presented by Chen [8] is also chaotic. The initial conditions for the drive, the response, and the error systems are  $(x_1(0), y_1(0), z_1(0)) = (1.2, 1.5, 1.6)$ ,  $(x_2(0), y_2(0), z_2(0)) = (2.0, 3.0, 2.0)$  and  $(e_1(0), e_2(0), e_3(0)) = (0.8, 1.5, 0.4)$ , respectively. Both systems are solved with step size h = 0.01 using the Adams-Bashforth-Moulton predictor-corrector method proposed by Diethelm et al. [12]. Figure 10 shows the simulation results of the synchronization between the financial systems with the controller applied at a time t  $\geq 30$ . Figure 10 (d) shows that the time response of the error signals between the states of both systems are converged to zero and this shows that the synchronization has been achieved.

To show the influence of fractional order q on synchronization, values of q = 0.87 and q = 0.93 have been also taken for both systems and for which the fractional version of the other financial system proposed by Chen [8] is chaotic. Chaotic attractors for the new chaotic financial system are shown in Figure 11 for q = 0.87 and q = 0.93. When the parameters for both systems and all initial conditions are the same as the above case, the dynamics of the synchronization between the financial system and the new chaotic financial system for q = 0.93 and q = 0.87 are shown in Figures 12 and 13, respectively. It can be seen from Figures 13, 12, and 10, especially for  $y_1, y_2$ , versus t, we can observe that when the order q increases, the synchronization starts early as is observed in reference [6].

### 5. Conclusion

In this paper, we showed that the chaotic behavior in a new financial system with integer, and fractional order can be stabilized by using feedback control and synchronized by using active control. The linear feedback controllers are designed based on the Routh-Hurwitz criterion. Also, synchronization between two identical new chaotic financial systems and two non-identical financial systems with integer and fractional order is realized by using active control. Numerical simulations have been done to show the effectiveness of the proposed control and synchronization techniques. The proposed feedback control method is very effective for chaos control and the active control is also suitable for the synchronization of the new chaotic financial system. Moreover, it is shown that the synchronization is achieved quickly when fractional order q increases. Linear feedback control and active control are used when the system's parameters are known, so for future works, the adaptive control and synchronization of this new financial system can be considered.

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