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α-minimal prime ideals in almost distributive lattices

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Abstract

The concept of α-minimal prime ideal of an ADL is introduced and its characterizations are established. The set of all α-minimal prime ideals of an ADL is topologized and resulting space is studied.

Keywords: ADL, minimal prime ideal, relative α-annihilator, α-minimal prime ideal, α-maximal filter, α-pseudo complementation, hull-kernel topology.

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1. Introduction

The notion of Almost Distributive Lattice (ADL) was introduced by Swamy and Rao [7] as a common abstraction of several lattice theoretic and ring theoretic generalizations of a Boolean algebra. An ADL is an algebra $(A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ which satisfies all the axioms of a distributive lattice except possibly the commutativity of the operations $\vee$ and $\wedge$. It is known that, in an ADL, the commutativity of $\vee$ is equivalent to that of $\wedge$ and also to the right distributivity of $\vee$ over $\wedge$. The class of ADLs with pseudo-complementation was introduced in [8] and proved it is equationally definable. In [5], we introduced the notion of α-pseudo-complementation on an ADL A by fixing an arbitrary element α in A as the natural generalization of the notion of pseudo-complementation on an ADL. In [4], we introduced the concepts of α-dense element and α-maximal filter in an ADL A and studied these in connection with α-pseudo-complementation on A. Here, we introduced the concept of α-minimal prime ideal of an ADL A and characterized these in terms of α-maximal filter, relative α-annihilator, α-dense element, and α-pseudo complementation. Mainly, we considered the spaces $h_{\mathbb{P}}(\alpha)$ and $\min h_{\mathbb{P}}(\alpha)$ of prime ideals containing the element α in A and α-minimal prime ideals respectively, together with the hull-kernel topologies, and proved certain properties of these.

2. Preliminaries

In this section, we recall certain definitions, results and notations which will be needed later on are presented.

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Definition 2.1 ([7]). An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (ADL) if, for any $x, y, z \in A$,

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
(2) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$;
(3) $(x \vee y) \wedge x = x$;
(4) $(x \vee y) \wedge y = y$;
(5) $x \vee (x \wedge y) = x$;
(6) $0 \wedge x = 0$.

Example 2.2 ([7]). Let X be a non-empty set and $a_0 \in X$. For any $a, b \in X$, define,

$$a \wedge b = \begin{cases} b, & \text{if } a \neq a_0, \\ a_0, & \text{if } a = a_0, \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} a, & \text{if } a \neq a_0, \\ b, & \text{if } a = a_0. \end{cases}$$

Then $(X, \wedge, \vee, a_0)$ is an ADL and this is called discrete ADL.

Definition 2.3 ([7]). Let A be an ADL. For any $x, y \in A$, define $x \leqslant y$ iff $x = x \wedge y$ or, equivalently $x \vee y = y$, then $\leqslant$ is a partial ordering on A.

Proposition 2.4 ([7]). *Let A be an ADL. For any $x, y, z \in A$, we have the following:*

(1) $x \wedge y = x \Leftrightarrow x \vee y = y$;
(2) $x \wedge y = y \Leftrightarrow x \vee y = x$;
(3) $x \wedge y = y \wedge x$ *whenever* $x \leqslant y$;
(4) $\wedge$ *is associative in* A;
(5) $x \wedge y \wedge z = y \wedge x \wedge z$;
(6) $(x \vee y) \wedge z = (y \vee x) \wedge z$;
(7) $x \wedge y = 0 \Leftrightarrow y \wedge x = 0$;
(8) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;
(9) $x \wedge (x \vee y) = x$, $(x \wedge y) \vee y = y$ *and* $x \vee (y \wedge x) = x$;
(10) $x \leqslant x \vee y$ *and* $x \wedge y \leqslant y$;
(11) $x \wedge x = x$ *and* $x \vee x = x$;
(12) 0 *is the identity for the operation* $\vee$ *(that is;* $x \vee 0 = x = 0 \vee x$*)*;
(13) 0 *is the zero element for the operation* $\wedge$ *(that is;* $x \wedge 0 = 0$*)*;
(14) *if* $x \leqslant z$, $y \leqslant z$, *then* $x \wedge y = y \wedge x$ *and* $x \vee y = y \vee x$.

An element m of an ADL A is called maximal, if m is a maximal element in the poset $(A, \leqslant)$. It is known that, m is maximal $\Leftrightarrow m \wedge x = x \Leftrightarrow m \vee x = m$ for all $x \in A$. In any discrete ADL, every non-zero element is maximal.

An ADL A is said to be associative ADL, if the operation $\vee$ on A is associative. Throughout this paper A denotes an ADL with a maximal element in which $\vee$ is associative; that is $(x \vee y) \vee z = x \vee (y \vee z)$ for all $x, y, z \in A$.

For any $x, y \in A$, with $x \leqslant y$, the set $[x, y] = \{z \in A : x \leqslant z \leqslant y\}$ is a bounded distributive lattice with respect to the operations induced by those on A. If in addition $[x, y]$ is a Boolian algebra then A is called a relatively complemented ADL and, in this case, the operation $\vee$ is associative. Every discrete ADL is relatively complemented.

A non-empty subset I of A is said to be an ideal (filter) of A, if $x \vee y \in I (x \wedge y \in I)$ and $x \wedge a \in I (a \vee x \in I)$ whenever $x, y \in I$ and $a \in A$. If I is an ideal (filter) of A and $x, y \in A$, then $x \wedge y \in I \Leftrightarrow y \wedge x \in I (x \vee y \in I \Leftrightarrow y \vee x \in I)$. For any $S \subseteq A$, the smallest ideal of A containing S is called the ideal generated by S in A and is denoted by $(S]$. If $S = \{x\}$, we simply write $(x]$ for $(\{x\}]$. We have that for any $S \subseteq A$ and

$x \in A$, $(S] = \{(\bigvee_{i=1}^{n} s_i) \wedge a : n \geqslant 0,\ s_i \in S \text{ and } a \in A\}$, and $(x] = \{x \wedge a : a \in A\}$=$\{y \in A : x \wedge y = y\}$, $(x]$ is called the principal ideal generated by x. For any $x \in A$, $[x) = \{a \vee x : a \in A\}$=$\{y \in A : y \vee x = y\}$ is called the principal filter generated by x. For any $S \subseteq A$, the set $S^* = \{x \in A : x \wedge s = 0 \text{ for all } s \in S\}$ is always an ideal of A and is called an annihilator of S in A. Note that $S^* = (S]^*$. For any $x \in A$, we have $(x]^* = \{x\}^* = \{y \in A : x \wedge y = 0\}$. A proper ideal (filter) P of A is said to be prime if, for any $x, y \in A$, $x \wedge y \in P (x \vee y \in P)$ implies either $x \in P$ or $y \in P$. A prime ideal (filter) of an ADL is called a minimal prime ideal (filter) if there is no other prime ideal (filter) properly contained in it. A proper ideal (filter) M of A is said to be maximal if, there is no proper ideal (filter) N of A such that $M \subset N$.

Proposition 2.5 ([7]). *For any subset P of A, P is a prime filter of A iff A-P is a prime ideal of A.*

Proposition 2.6 ([7]). *Let A be an ADL, I an ideal (filter) of A and $x \in A$-I. Then there exists a prime ideal (filter) P of A such that $I \subseteq P$ and $x \notin P$.*

Proposition 2.7 ([7]). *Every prime ideal (filter) of A contains a minimal prime ideal (filter) .*

Proposition 2.8 ([7]). *Every maximal ideal (filter) is prime ideal (filter).*

Proposition 2.9 ([7]). *An ideal P of A is a minimal prime ideal iff A-P is a maximal filter, and a filter Q of A is a minimal prime filter iff A-Q is a maximal ideal.*

Definition 2.10 ([7]). An equivalence relation θ on an ADL $A = (A, \wedge, \vee, 0)$ is said to be a congruence if θ is compatible with $\wedge$ and $\vee$ on A; that is, for any $a, b, c, d \in A$, (a, b) and $(c, d) \in \theta \Rightarrow (a \wedge c,\ b \wedge d) \in \theta$ and $(a \vee c,\ b \vee d) \in \theta$. If θ is a congruence on A, then the set $x/\theta = \{y \in A : (x, y) \in \theta\}$ is called the congruence class of x in A corresponding to θ.

Proposition 2.11 ([7]). *For any $a \in A$, $\theta_a = \{(x, y) \in A \times A : a \vee x = a \vee y\}$ is a congruence relation on A.*

Proposition 2.12 ([7]). *For any ideal I of A, the relation $\theta_I = \{(x, y) \in A \times A : a \vee x = a \vee y \text{ for some } a \in I\}$ is a congruence on A and is the smallest congruence on A containing $I \times I$. Moreover, for any $a \in A$, $\theta_{(a]} = \theta_a$. Also, $0/\theta_I$=I and this is the only congruence class of θ_I which is an ideal of A. This congruence is called ideal congruence.*

Definition 2.13 ([8]). A unary operation $*$ on A is called a pseudo complementation on A if, for any $x, y \in A$,

(1) $x \wedge y = 0 \Rightarrow x^* \wedge y = y$;
(2) $x \wedge x^* = 0$;
(3) $(x \vee y)^* = x^* \wedge y^*$.

Definition 2.14 ([6]). For any elements x and a in A, the relative a-annihilator of x is defined by $\langle x, a \rangle = \{y \in A : x \wedge y \in (a]\}$. Note that $\langle x, a \rangle$ is an ideal of A.

Definition 2.15 ([5]). Let a be an arbitrary fixed element in A. Then a unary operation $x \mapsto x * a$ on A is called an a-pseudo-complementation on A, if for any $x, y \in A$;

(1) $\langle x, a \rangle = (x * a]$;
(2) $(x \vee y) * a = (x * a) \wedge (y * a)$.

Definition 2.16 ([4]). Let a be a fixed arbitrary element in A. Then an element $x \in A$ is said to be a-dense, if $\langle x, a \rangle \subseteq (a]$ (and hence $\langle x, a \rangle = (a]$). D_a denotes the set of all a-dense elements in A.

Definition 2.17 ([4]). Let a be an arbitrary fixed element in A. Then a filter F of A is said to be a-maximal, if F is maximal with respect to the property of not containing a.

Proposition 2.18 ([4]). *For any filter F of A and $a \in$A-F, there exists a-maximal filter containing F.*

Proposition 2.19 ([4]). *A filter F of A is a-maximal filter iff $a \notin F$ and for every $x \notin F$, $\langle x, a\rangle \cap F \neq \phi$.*

Proposition 2.20 ([4]). *Every a-maximal filter of A is a prime filter.*

Proposition 2.21 ([4]). *The following are equivalent:*

(1) $x \in D_a$;
(2) $x * a \sim a$;
(3) $(x * a) * a = a * a$.

Proposition 2.22 ([4]). *For every $x \in A$, $x \vee (x * a) \in D_a$.*

3. a-minimal prime ideals

Definition 3.1. Let A be an ADL and $a \in A$. Then a prime ideal P of A containing a is called a-minimal prime ideal if there is no prime ideal of A containing a and properly contained in P.

The aim of this article is to study some characterizations and properties of a-minimal prime ideals. First we have the following as an application of Zorn's lemma which allow us to denote the existence of a-minimal prime ideals.

Theorem 3.2. *Let Q be a prime ideal of A containing an element a in A. Then there exists an a-minimal prime ideal M of A such that $M \subseteq Q$.*

Corollary 3.3. *For $a \in A$, the intersection of all a-minimal prime ideals of A is $(a]$.*

Proposition 2.6 and Theorem 3.2 yield the following theorem.

Theorem 3.4. *For any x and $a \in A$, $\langle x, a\rangle = \bigcap\{M : M$ is a-minimal prime ideal and $x \notin M\}$.*

Theorem 3.5. *Following statements are equivalent for any prime ideal P of A containing an element a in A.*

(1) *P is a-minimal prime ideal;*
(2) *A-P is a-maximal filter;*
(3) *$\langle x, a\rangle \nsubseteq P$, for each $x \in P$.*

Proof.

(1)⇒(2) Suppose P is a-minimal prime ideal. Then, clearly A-P is a filter of A and $a \notin A$-P. Let F be a filter of A such that A-$P \subseteq F$ and $a \notin F$. Then, by Proposition 2.6, there exists a prime filter Q of A such that $F \subseteq Q$ and $a \notin Q$. By Proposition 2.5, A-Q is a prime ideal of A and $a \in A$-Q. Since A-$P \subseteq F \subseteq Q$, we get A-$Q \subseteq P$. Since P is a-minimal prime ideal, we get A-$Q = P$ and hence A-$P = Q$. This implies that A-$P = F$. Thus A-P is a-maximal filter.

(2)⇒(3) Let $x \in P$. Then $x \notin A$-P. By (2) and Proposition 2.19, $\langle x, a\rangle \cap (A\text{-}P) \neq \phi$ which implies that $\langle x, a\rangle \nsubseteq P$.

(3)⇒(1) Let Q be a prime ideal of A such that $a \in Q \subseteq P$. Now,

$$\begin{aligned} x \in P \Rightarrow \langle x, a\rangle \nsubseteq P \quad & \text{(by (3))} \\ \Rightarrow\ & \text{there exists } y \in A \text{ such that } x \wedge y \in (a] \text{ and } y \notin P \\ \Rightarrow\ & y \notin Q \text{ and } x \wedge y \in Q \quad (\text{since } a \in Q \Leftrightarrow (a] \subseteq Q) \\ \Rightarrow\ & x \in Q \quad (\text{since } Q \text{ is prime}). \end{aligned}$$

Therefore $P \subseteq Q$ and hence $Q = P$. Thus P is a-minimal prime ideal. □

Another characterization of a-minimal prime ideals in connection with a-pseudo complementations is given below.

Theorem 3.6. *Let $x \mapsto x * a$ be an a-pseudo complementation on A and P a prime ideal of A containing $a \in A$. Then P is a-minimal prime ideal iff $x * a \notin P$ for each $x \in P$.*

Proof. For any $x \in A$, we have that $x * a \in P \Leftrightarrow \langle x, a \rangle \subseteq P$. Now, the theorem follows from Theorem 3.5. □

Definition 3.7. Let P be a prime ideal of A and $a \in P$. Define, $a(P) = \{y \in A : y \wedge x \in (a] \text{ for some } x \in A\text{-}P\}$.

Some basic properties of $a(P)$ are stated below.

Lemma 3.8. *For any prime ideal P of A and $a, b \in A$, we have the following.*

(1) $a(P) = \bigcup_{x \in A-P} \langle x, a \rangle$.
(2) $a(P)$ *is an ideal of* A *and* $a \in a(P) \subseteq P$.
(3) $a \wedge b(P) = b \wedge a(P) = a(P) \cap b(P)$.
(4) $a \vee b(P) = b \vee a(P)$ *and* $a(P) \vee b(P) \subseteq a \vee b(P)$.
(5) $a \leqslant b \Rightarrow a(P) \subseteq b(P)$.
(6) $a \sim b \Rightarrow a(P) = b(P)$.
(7) a *is maximal iff* $a(P) = A$.

Theorem 3.9. *Let P be a prime ideal of A and $a \in P$. Then P is a-minimal prime ideal iff $a(P) = P$.*

Proof. Suppose P is a-minimal prime ideal and $x \in P$. Then $\langle x, a \rangle \nsubseteq P$ and hence there exists $y \notin P$ such that $x \wedge y \in (a]$; that is $x \in a(P)$. Therefore $P \subseteq a(P)$. By Lemma 3.8 (2), $a(P) \subseteq P$. Hence $a(P) = P$.

Conversely, suppose that $a(P) = P$. Let $x \in P$. Then $x \in a(P)$ and hence $x \wedge y \in (a]$ for some $y \notin P$. This implies $y \in \langle x, a \rangle$ and $y \notin P$. Hence $\langle x, a \rangle \nsubseteq P$. Thus, by Theorem 3.5, P is a-minimal prime ideal. □

Theorem 3.10. *Let $x \mapsto x * a$ be an a-pseudo complementation on A and $a \in M \subset A$. Then the following statements are equivalent:*

(1) A-M *is a-maximal filter;*
(2) A-M *is a prime filter and* $x \vee (x * a) \in$ A-M *for each* $x \in A$;
(3) M *is a-minimal prime ideal;*
(4) M *is a prime ideal, and* $x \in M \Rightarrow (x * a) * a \in M$;
(5) M *is a prime ideal and* $M \cap D_a = \phi$.

Proof.

(1)⇒(2) Assume (1). Then A-M is a prime filter of A (by Proposition 2.20). To prove the second part : if $x \notin$ A-M, then $\langle x, a \rangle \cap (\text{A-M}) \neq \phi$ (by (1)). Choose $y \in \langle x, a \rangle \cap (\text{A-M})$. Then $y \in \langle x, a \rangle = (x * a]$ so that $(x * a) \wedge y = y \in$ A-M and so $x * a \in$ A-M. Therefore $x \vee (x * a) \in$ A-M.

(2)⇒(3) Suppose the condition (2) is satisfied. Then M is a prime ideal. Let P be a prime ideal of A such that $a \in P \subset M$. Then select $x \in M$ such that $x \notin P$. Now $x \wedge (x * a) \in (a] \subseteq P$. Since P is prime and $x \notin P$, we get $x * a \in P$. So that $x \vee (x * a) \in M$; a contradiction to our supposition. Thus M is a-minimal prime ideal of A.

(3)⇒(4) Assume(3). Then M is a prime ideal. If $x \in M$, then $x * a \notin M$ (by Theorem 3.6). Now $(x * a) \wedge ((x * a) * a) \in (a] \subseteq M$. Since M is prime, we get $(x * a) * a \in M$.

(4)⇒(5) Assume the condition (4). If $M \cap D_a \neq \phi$ and choose $x \in M \cap D_a$, then $(x * a) * a \in M$ (by (4)), and $(x * a) * a = a * a$ (by Proposition 2.21). Hence $(x * a) * a$ is maximal since $a * a$ is maximal. So that $a * a \in M$ and hence M=A; a contradiction. Thus $M \cap D_a = \phi$.

(5)⇒(1) Assume (5). Then A-M is a prime filter and $a \notin$A-M. To prove A-M is a-maximal filter: let $x \notin$ A-M. Then $x \vee (x * a)$ is a-dense element in A (by Proposition 2.22); that is, $x \vee (x * a) \in D_a$ and hence $x \vee (x * a) \in$ A-M. As A-M is prime filter and $x \notin$A-M, we get $x * a \in$ A-M. Therefore, $x * a \in \langle x, a \rangle \cap (\text{A-M})$ so that $\langle x, a \rangle \cap (\text{A-M}) \neq \phi$. By Proposition 2.19, A-M is a-maximal filter. □

4. Hull space

Let H be a non-empty set of prime ideals of A. For any $S \subseteq A$, let $H(S) = \{P \in H : S \nsubseteq P\}$. Then it can be easily proved that the class $\{H(S) : S \subseteq A\}$ is a topology on H. This topology is called the hull-kernel topology. For any $S \subseteq A$, we have $H(S) = \bigcup\limits_{s \in S} H(s)$, where $H(s) = H(\{s\})$ and hence the class $\{H(s) : s \in A\}$ is a base for the hull-kernel topology on H. The closed set H-$H(S)$ is called the hull of S in H and is denoted by $h_H(S)$. Note that $h_H(S) = \{P \in H : S \subseteq P\}$. Also, for any $U \subseteq H$, the kernel of U is defined by $k(U) = \bigcap\{P \in H : P \in U\}$. The hull of any $S \subseteq A$ is closed in H, and for any $U \subseteq H$ the kernel $k(U)$ of U is an ideal of A. The name hull-kernel topology is justified by the reason being that for any $U \subseteq H$, $\overline{U} = h_H(k(U))$, where $\overline{U}$ is the closure of U with respect to the hull-kernel topology on H. It can be easily seen that, if $S \subseteq A$ and $I = (S]$, then $H(S) = H(I)$ and hence every open set in H is of the form $H(I)$ for some ideal I of A and every closed set in H is of the form $h_H(I)$ for some ideal I of A.

Let $\mathbb{P}$ denote the set of all prime ideals of A. Then the set $\mathbb{P}$ together with the hull-kernel topology is called the prime spectrum of A and it is denoted by $\mathrm{Spec}(A)$. Let $\mathbb{P}_m$ denote the set of all minimal prime ideals of A. This set together with the subspace topology relative to the hull-kernel topology on $\mathrm{Spec}(A)$ is called the minimal prime ideal space of A and is denoted by $\min\ \mathrm{Spec}(A)$. Throughout this paper whenever we talk about the topology on $\mathbb{P}$ or any $H \subseteq \mathbb{P}$ we mean the hull-kernel topology. For an arbitrary fixed element $a \in A$, we define $h_{\mathbb{P}}(a) = \{P \in \mathbb{P} : a \in P\}$. Note that $h_{\mathbb{P}}(a)$ is the hull of $\{a\}$ in $\mathrm{Spec}(A)$. In this section we study some basic properties of the set $h_{\mathbb{P}}(a)$. The following is a straight forward verification.

Lemma 4.1. *Let $H = h_{\mathbb{P}}(a)$. Then the following hold for any x, y in A.*

(1) $H(x) = H(a \vee x) = H(x \vee a)$.
(2) $H(x) \cap H(y) = H(x \wedge y) = H(y \wedge x)$.
(3) $H(x) \cup H(y) = H(x \vee y) = H(y \vee x)$.
(4) $(x] \subseteq (y] \Rightarrow H(x) \subseteq H(y)$.
(5) $H(x) \subseteq H(y) \Leftrightarrow (a \vee x] \subseteq (a \vee y]$.
(6) $H(x) = H(y) \Leftrightarrow (a \vee x] = (a \vee y]$.
(7) $H(x) = \phi \Leftrightarrow x \in (a] \Leftrightarrow a \wedge x = x$.
(8) $H(a) = \phi = H(0)$.
(9) $H(x) = H \Leftrightarrow a \vee x$ *is maximal.*
(10) $H(x) \subseteq H(y) \Rightarrow (x] \subseteq (y]$ *whenever* $a \in (y]$.
(11) $H(x) = H(y) \Leftrightarrow x \sim y$ *whenever* $a \in (x] \cap (y]$.

Theorem 4.2. *Let $H = h_{\mathbb{P}}(a)$ and $Y \subseteq H$. Then Y is compact open iff* $Y{=}H(x)$ *for some $x \in A$.*

Proof. Suppose that $Y{=}H(x)$ for some $x \in A$. Then, Y is open. Let $\{H(s) : s \in S\}$ be a basic open cover of Y, where $S \subseteq A$. Then,

$$Y \subseteq \bigcup_{s \in S} H(s) = \bigcup_{s \in S} H(a \vee s) = H(T) = H((T]),$$

where $T = \{a \vee s : s \in S\}$. If $x \notin (T]$, then, by Proposition 2.6, there exists a prime ideal P of A such that $(T] \subseteq P$ and $x \notin P$ so that $P \in H(x)$ and $P \notin H((T])$ a contradiction. Therefore $x \in (T]$ and hence, $x = (\bigvee\limits_{i=1}^{n} (a \vee s_i)) \wedge y$ for some $y \in A$ and $s_1, s_2, \ldots, s_n \in S$. Now, $P \in H(x) \Rightarrow x \notin P \Rightarrow \bigvee\limits_{i=1}^{n} (a \vee s_i) \notin P$ $\Rightarrow P \in H(\bigvee\limits_{i=1}^{n} (a \vee s_i)) = \bigcup\limits_{i=1}^{n} H(a \vee s_i) = \bigcup\limits_{i=1}^{n} H(s_i)$. Therefore $Y \subseteq \bigcup\limits_{i=1}^{n} H(s_i)$. Thus Y is compact.

Conversely, suppose that Y is compact open. Since Y is open, $Y{=}\bigcup\limits_{s \in S} H(S), S \subseteq A$. By compactness of Y, $Y{=}\bigcup\limits_{i=1}^{n} H(s_i) = H(\bigvee\limits_{i=1}^{n} s_i) = H(s)$, where $s = \bigvee\limits_{i=1}^{n} s_i$. □

Note that, for any $x \in A$, the interval $[x, \infty) = \{y \in A : x \leqslant y < \infty\}$ is an ADL under the induced operations $\wedge$ and $\vee$ with x as its smallest element.

Theorem 4.3. *Let* $H = h_{\mathbb{P}}(a)$. *Then the following statements are equivalent:*

(1) H *is a Haussdorff space;*
(2) H *is a* T_1*-space;*
(3) P *is maximal ideal for each* $P \in H$*;*
(4) P *is* a*-minimal prime ideal for each* $P \in H$*;*
(5) $a(P) = P$ *for all* $P \in H$*;*
(6) $[a, \infty)$ *is a relatively complemented ADL.*

Proof.

(1)⇒(2) It is clear.

(2)⇒(3) Let $P \in H$. By(2), $\{P\}$ is closed in H and hence $\overline{\{P\}} = \{P\}$; that is $h_H(P) = h_H(k(\{P\})) = P$ which implies that there is no prime ideal containing a and P, other than P it self. Thus P is maximal ideal.

(3)⇒(4) It is trivial.

(4)⇒(5) It follows by Theorem 3.9.

(5)⇒(6) Let b and $x \in A$ such that $a \leqslant x \leqslant b$. Put $J = \langle x, a\rangle$. If $b \notin (x] \vee J$, then by Proposition 2.6, there exists a prime ideal P of A such that $b \notin P$ and $(x] \vee J \subseteq P$ so that $P \in H$. By (5), $a(P) = P$. Since $x \in P$, $x \in \langle y, a\rangle$ for some $y \in$A-P. Therefore $y \wedge x \in (a]$ and hence $x \wedge y \in (a]$, where $y \in$ A-P. This implies that $y \in \langle x, a\rangle$ so that $y \in P$; a contradiction. Therefore $b \in (x] \vee J$. Then $b = (x \wedge f) \vee g$ for some $f \in A$ and $g \in J$. As $g \in \langle x, a\rangle$, $x \wedge g \in (a]$ so that $a \wedge x \wedge g = x \wedge g$. Clearly $a \leqslant a \vee g \leqslant b$. Now,

$$x \wedge (a \vee g) = (x \wedge a) \vee (x \wedge g) = a \vee (a \wedge x \wedge g) = a \vee (a \wedge g) = a$$

and

$$x \vee b = x \vee \big((x \wedge f) \vee g\big) = (x \vee (x \wedge f)) \vee g = x \vee g.$$

Then, $x \vee (a \vee g) = (x \vee a) \vee g = x \vee g = x \vee b = b$. Therefore $a \vee g$ is the complement of x in $[a, b]$ and hence $[a, b]$ is a Boolean algebra. Thus $[a, \infty)$ is a relatively complemented ADL.

(6)⇒(1) Let $P \in H$ and Q is an ideal of A such that $P \subset Q$. Choose $x \in Q$ such that $x \notin P$. Let $y \in A$. Then $a \vee x$ and $a \vee y \in [a, \infty)$. Since the ADL $[a, \infty)$ is relatively complemented, there exists $z \in [a, \infty)$ such that $(a \vee x) \wedge z = a$ and $(a \vee x) \vee z = (a \vee x) \vee (a \vee y)$. Since $x \notin P$, $a \vee x \notin P$. Also $(a \vee x) \wedge z \in P$. Since P is prime we get $z \in P$. Then $(a \vee x) \vee z \in Q$ which implies $a \vee y \in Q$. Now $y = (a \vee y) \wedge y \in Q$. Therefore $Q = A$. Hence P is a maximal ideal of A. So every element in H is a maximal ideal and hence a-minimal prime ideal of A. Let $P, Q \in H$ such that $P \neq Q$. Then $P \nsubseteq Q$ and $Q \nsubseteq P$ since P and Q are maximal. Choose an element $x \in P$ such that $x \notin Q$. As $x \in P$, $\langle x, a\rangle \nsubseteq P$ (by Theorem 3.5) and hence there exists $y \notin P$ such that $x \wedge y \in (a]$; that is, $a \wedge x \wedge y = x \wedge y$. Now $H(y)$ and $H(x)$ are open sets in H containing P and Q, respectively and

$$\begin{aligned} H(x) \cap H(y) &= H(x \wedge y) \quad \text{(by Lemma 4.1 (2))} \\ &= H(a \wedge x \wedge y) = H(a) \cap H(x \wedge y) = \phi \cap H(x \wedge y) = \phi \quad \text{(by Lemma 4.1 (8))}. \end{aligned}$$

Thus H is a Haussdorff space. □

5. The space of a-minimal prime ideals

In this section, we consider the space of all a-minimal prime ideals of A, which will be denoted by min $h_{\mathbb{P}}(a)$. We fix the following notations.

$$H = h_{\mathbb{P}}(a);$$
$$H_m = \{M : M \text{ is } a\text{-minimal prime ideal of } A\};$$

$$\begin{aligned} H_m(S) &= H(S) \cap \min\ h_{\mathbb{P}}(a) = \{M \in H_m :\ S \nsubseteq M\};\\ H_m(x) &= H(x) \cap \min\ h_{\mathbb{P}}(a) = \{M \in H_m :\ x \notin M\};\\ \text{also, } h_{H_m}(S) &= \{M \in H_m :\ S \subseteq M\};\\ \text{and } h_{H_m}(x) &= \{M \in H_m :\ x \in M\}. \end{aligned}$$

Theorem 5.1. *For each* $x \in A$, $H_m(x)$ *is closed and open in* $\min\ h_{\mathbb{P}}(a)$.

Proof. By Theorem 3.5, it follows that $H_m(x) \cap H_m(\langle x, a\rangle) = \phi$ and $H_m(x) \cup H_m(\langle x, a\rangle) = H_m$. Hence $H_m(x) = H_m$ - $H_m(\langle x, a\rangle)$; so that $H_m(x)$ is closed. □

Corollary 5.2. $\min\ h_{\mathbb{P}}(a)$ *is a Haussdorff space.*

Note that, for any $a \in A$, the relation defined by $\eta_a = \{(x, y) \in A \times A : \langle x, a\rangle = \langle y, a\rangle\}$ is a congruence relation on A.

The following is a straight forward verification and it gives us a set of relations between relative a-annihilators of A and the basic open (closed) sets of $\min\ h_{\mathbb{P}}(a)$.

Lemma 5.3. *For any* $x, y, z \in A$, *the following hold.*

(1) $H_m(x) = h_{H_m}(\langle x, a\rangle)$.
(2) $H_m(\langle x, a\rangle) = h_{H_m}(x)$.
(3) $h_{H_m}(x) = h_{H_m}(\ \langle x, a\rangle^*)$.
(4) $H_m(x) \subseteq H_m(y) \Leftrightarrow \langle y, a\rangle \subseteq \langle x, a\rangle$.
(5) $h_{H_m}(x) \subseteq h_{H_m}(y) \Leftrightarrow \langle x, a\rangle \subseteq \langle y, a\rangle$.
(6) $H_m(x) = H_m(y) \Leftrightarrow (x, y) \in \eta_a$.
(7) $\langle z, a\rangle = \langle x, a\rangle \cap \langle y, a\rangle \Leftrightarrow h_{H_m}(z) = h_{H_m}(x) \cap h_{H_m}(y)$.
(8) $\langle x, a\rangle^* = \langle y, a\rangle \Leftrightarrow h_{H_m}(x) = h_{H_m}(\langle y, a\rangle)$.

Now we give a characterization of a-dense elements.

Theorem 5.4. $D_a = \{x \in A : h_{H_m}(x) = \phi\}$.

Proof. Let $x \in D_a$. Then $h_{H_m}(x) = H_m(\langle x, a\rangle) = H_m((a]) = H_m(a) = \phi$. Now,

$$\begin{aligned} h_{H_m}(x) = \phi &\Rightarrow H_m(\langle x, a\rangle) = \phi\\ &\Rightarrow \langle x, a\rangle \subseteq M \text{ for all } M \in H_m\\ &\Rightarrow \langle x, a\rangle \subseteq \bigcap_{M \in H_m} M = (a] \qquad \text{(by Corollary 3.3)}\\ &\Rightarrow x \text{ is } a\text{-dense; so that } x \in D_a. \end{aligned}$$

□

Now, for any congruence θ on A, the quotient $A/\theta = \{x/\theta : x \in A\}$ is an ADL under the operations $\wedge$ and $\vee$ on A/θ defined by $x/\theta \wedge y/\theta = (x \wedge y)/\theta$ and $x/\theta \vee y/\theta = (x \vee y)/\theta$; and its zero element is $0/\theta$.

Theorem 5.5. *Let* J *be an ideal of* A *and* A/θ_J *be the quotient of* A *by an ideal congruence* θ_J. *Then the map*

$$f(I) = I/\theta_J = \{x/\theta_J : x \in I\}$$

is an isomorphism of the lattice of all ideals of A *containing* J *onto the lattice of all ideals of* A/θ_J *and this induces a homeomorphism from* $h_{\mathbb{P}}(J)$ *onto* $\mathrm{Spec}(A/\theta_J)$.

Proof. Since the natural map $x \mapsto x/\theta_J$ of A onto A/θ_J is a homomorphism (in fact, it is an epimorphism), one can easily verify that for any ideal I of A containing J, I/θ_J is an ideal of the ADL A/θ_J. Hence f is well defined. For any ideals P and Q containing J of A, it is clear that

$$P \subseteq Q \Rightarrow P/\theta_J \subseteq Q/\theta_J \Rightarrow f(P) \subseteq f(Q).$$

On the other hand, suppose that $f(P) \subseteq f(Q)$. Then

$$\begin{aligned} x \in P &\Rightarrow x/\theta_J \in P/\theta_J \subseteq Q/\theta_J \\ &\Rightarrow x/\theta_J = y/\theta_J \text{ for some } y \in Q \\ &\Rightarrow (x, y) \in \theta_J \text{ for some } y \in Q \\ &\Rightarrow a \vee x = a \vee y \text{ for some } a \in J \text{ and } y \in Q \\ &\Rightarrow x = (a \vee x) \wedge x = (a \vee y) \wedge x \in Q \end{aligned}$$

and therefore $P \subseteq Q$. Hence, $P \subseteq Q \Leftrightarrow f(P) \subseteq f(Q)$; that is f and f^{-1} are order homomorphism. To prove f is onto, let K be an ideal of A/θ_J. Put

$$I = \{x \in A : x/\theta_J \in K\}.$$

Clearly I is an ideal of A and $f(I) = I/\theta_J = K$ and hence f is onto. Therefore $f : \mathcal{I}_J(A) \mapsto \mathcal{I}(A/\theta_J)$ is an order isomorphism and thus f is an isomorphism.

To prove the second part; we note that, if K is a prime ideal in A/θ_J, then there exists a prime ideal P of A containing J such that $f(P) = P/\theta_J = K$. Now, let $\mathbb{P}_{A/\theta_J}(K)$ be an open subset of $\mathrm{Spec}(A/\theta_J)$, where K is an ideal of A/θ_J. Then

$$\begin{aligned} f^{-1}(\mathbb{P}_{A/\theta_J}(K)) &= \{P \in h_{\mathbb{P}}(J) : f(P) \in \mathbb{P}_{A/\theta_J}(K)\} \\ &= \{P \in h_{\mathbb{P}}(J) : K \nsubseteq f(P)\} \\ &= \{P \in h_{\mathbb{P}}(J) : f^{-1}(K) \nsubseteq P\} \\ &= \{P \in h_{\mathbb{P}}(J) : P \in \mathbb{P}(f^{-1}(K))\} \\ &= h_{\mathbb{P}}(J) \cap \mathbb{P}(f^{-1}(K)), \end{aligned}$$

which is an open subset of $h_{\mathbb{P}}(J)$ and hence f is continuous.

Again, let $\mathbb{P}(I) \cap h_{\mathbb{P}}(J)$ be an open subset of $h_{\mathbb{P}}(J)$ where I is an ideal of A. Then,

$$\begin{aligned} f(\mathbb{P}(I) \cap h_{\mathbb{P}}(J)) &= \{f(P) : f^{-1}(f(P)) \in \mathbb{P}(I) \cap h_{\mathbb{P}}(J)\} \\ &= \{f(P) : P \in \mathbb{P}(I) \cap h_{\mathbb{P}}(J)\} \\ &= \{f(P) : I \nsubseteq P \text{ and } J \subseteq P\} \\ &= \{f(P) : f(I) \nsubseteq f(P) \text{ and } 0/\theta_J = f(J) \subseteq f(P)\} \\ &= \mathbb{P}_{A/\theta_J}(f(I)), \end{aligned}$$

and this is an open subset of $\mathrm{Spec}(A/\theta_J)$; so that f^{-1} is continuous. Hence $h_{\mathbb{P}}(J)$ and $\mathrm{Spec}(A/\theta_J)$ are homeomorphic; that is, $h_{\mathbb{P}}(J) \cong \mathrm{Spec}(A/\theta_J)$. □

Now, if J is an ideal of A, then by a minimal prime ideal belonging to J, we mean a minimal element in the set of all prime ideals of A containing J. In [3], it was proved that, a prime ideal P of A containing an ideal J is a minimal prime ideal belonging to J iff for each $x \in P$ there exists $y \notin P$ such that $x \wedge y \in J$.

For any ideal J of A, let m_A^J denote the set of all minimal prime ideals belonging to J, and $\min p^J A$ denote the space of minimal prime ideals belonging to J equipped with the hull-kernel topology. Then we have the following Corollaries as a direct consequence of the above theorem.

Corollary 5.6. $\min p^J A \cong \min\ \mathrm{Spec}(A/\theta_J)$.

Corollary 5.7. $\min\ h_{\mathbb{P}}(a) \cong\ \min\ \mathrm{Spec}(A/\theta_a)$.

Theorem 5.8. *For any* $x \in A$, $H_m(x) \cong \min\ \mathrm{Spec}(A/\theta_{\langle x,a\rangle})$.

Proof. By Corollary 5.6, $\min\ \mathrm{Spec}(A/\theta_{\langle x,a\rangle}) \cong \min p^{\langle x,a\rangle} A = m_A^{\langle x,a\rangle}$. So it sufficies to prove that $H_m(x) = m_A^{\langle x,a\rangle}$. Let $M \in H_m(x)$. Then M is a-minimal prime ideal and $x \notin M$, which implies that $\langle x, a\rangle \subseteq M$, so that M is a minimal prime ideal belonging to $\langle x, a\rangle$; that is $M \in m_A^{\langle x,a\rangle}$. On the other hand let $M \in m_A^{\langle x,a\rangle}$, then, for any $s \in M$ there exists $t \notin M$ such that $s \wedge t \in \langle x, a\rangle$. Therefore $x \notin M$; for if $x \in M$, then $\langle x, \langle x, a\rangle\rangle = \langle x, a\rangle \nsubseteq M$, a contradiction. Thus, for $s \in M$, we have $x \wedge t \notin M$ and $s \wedge x \wedge t \in (a]$, so that $\langle s, a\rangle \nsubseteq M$. Therefore M is a-minimal prime ideal and $x \notin M$; that is $M \in H_m(x)$. Hence $H_m(x) = m_A^{\langle x,a\rangle}$. □

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