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A coincidence continuation theory between multi-valued maps with continuous selections and compact admissible maps

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Abstract

We establish a topological transversality theorem and a Leray-Schauder alternative for coincidences between multi-valued maps with continuous selections and compact admissible maps.

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1. Introduction

This paper discusses coincidences between multi-valued maps with continuous selections and compact admissible maps. In particular we present a general Granas type topological transversality theorem [5, 6, 9], a general Leray-Schauder type alternatives [6, 9] and also a general Furi-Pera type result [3] for coincidences. Even though some of the results presented here could be modified from the results of O'Regan [9] (Φ replaced by Φ^{-1} there) however we feel it is more natural to construct this theory from the well known fixed point result of Gorniewicz [4, 8]. To motivate our theory we present below a very simple coincidence result in a general setting.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geqslant 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

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(ii) p is a perfect map, i.e., p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multi-valued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map;
(ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [4]. A upper semi-continuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ.

Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multi-function. We say $G \in DKT(Z, W)$ [1, 7] if W is convex and there exists a map $S : Z \to W$ with $co(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$.

By a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$ if for all $X \in Q$ and all $K \subseteq X$ closed in X, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

Now we recall the following fixed point result from the literature [4, 8].

Theorem 1.1. *Let $X \in ES(compact)$ and $\Psi \in Ad(X, X)$ a compact map. Then there exists a $x \in X$ with $x \in \Psi(x)$.*

We note that one can use Theorem 1.1 to generate coincidence results. For convenience we present one simple result to illustrate the strategy.

Theorem 1.2. *Let X and Y be subsets of a Hausdorff topological vector space E with X convex and Y paracompact. Suppose $F \in Ad(X, Y)$ is a compact map and $G \in DTK(Y, X)$. In addition suppose $Y \in ES(compact)$ (respectively, $X \in ES(compact)$). Then there exists a $y \in Y$ with $G(y) \cap F^{-1}(y) \neq \emptyset$ (respectively, there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$).*

Proof. Since Y is paracompact, then from [1, 7] there exists a selection $g \in C(Y, X)$ (note $\theta \in C(Y, X)$ if $\theta : Y \to X$ is a continuous (single valued) map) of G. Now $Fg \in Ad(Y, Y)$ (respectively, $gF \in Ad(X, X)$) is a compact map. Now Theorem 1.1 guarantees that there exists a $y \in Y$ with $y \in Fg(y)$ (respectively, there exists a $x \in X$ with $x \in gF(x)$). □

Remark 1.3. In Theorem 1.2 one could replace F is a compact map with G is a compact map.

2. Continuation theory

Let E be a completely regular topological space and U an open subset of E.

Definition 2.1. We say $\Phi \in B(E, E)$ if $\Phi \in Ad(E, E)$ and Φ is a compact map.

Remark 2.2. An example of a map $\Phi \in Ad(E, E)$ is if $\Phi : E \to K(E)$; here $K(E)$ denotes the family of nonempty, compact, acyclic subsets of E. In this paper we consider $\Phi \in B(E, E)$ but we note if we wish one could consider $\Phi \in B(E, \overline{U})$ throughout the paper; here $\overline{U}$ denotes the closure of U in E.

Definition 2.3. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and there exists a continuous (single valued) selection $f : \overline{U} \to E$ (we write $f \in C(\overline{U}, E)$) of F.

Remark 2.4.

(i). Suppose E is a topological vector space and $\overline{U}$ is paracompact. An example of a map $F \in A(\overline{U}, E)$ is $F \in DKT(\overline{U}, E)$. As an aside, note metrizable spaces are paracompact and closed subsets of paracompact spaces are paracompact.

(ii). In this paper we always assume $\Phi \in B(E, E)$ is a compact map and $F \in A(\overline{U}, E)$ if there exists a continuous selection of F. However it is easy to adjust the thory if we assume $\Phi \in B(E, E)$ means $\Phi \in Ad(\overline{U}, E)$ and $F \in A(\overline{U}, E)$ means there exists a continuous compact selection of F.

In this paper we fix a $\Phi \in B(E, E)$.

Definition 2.5. We say $F \in A_{\partial U}(\overline{U}, E)$ (respectively, $f \in C_{\partial U}(\overline{U}, E)$) if $F \in A(\overline{U}, E)$ (respectively, $f \in C(\overline{U}, E)$) with $F(x) \cap \Phi^{-1}(x) = \emptyset$ (respectively, $x \notin \Phi(f(x))$) for $x \in \partial U$; here ∂U denotes the boundary of U in E and $\Phi^{-1}(x) = \{z \in E : x \in \Phi(z)\}$.

Definition 2.6. Let $f, g \in C_{\partial U}(\overline{U}, E)$. We say $f \cong g$ in $C_{\partial U}(\overline{U}, E)$ if there exists a continuous map $h : \overline{U} \times [0, 1] \to E$ with $x \notin \Phi(h_t(x))$ for $x \in \partial U$ and $t \in (0, 1)$ (here $h_t(x) = h(x, t)$), $h_0 = f$ and $h_1 = g$.

Remark 2.7. Note $\cong$ in $C_{\partial U}(\overline{U}, E)$ is an equivalence relation.

Definition 2.8. Let $F, G \in A_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ if for any selection $f \in C_{\partial U}(\overline{U}, E)$ (respectively, $g \in C_{\partial U}(\overline{U}, E)$) of F (respectively, G) we have $f \cong g$ in $C_{\partial U}(\overline{U}, E)$.

Definition 2.9. We say $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ if for any selection $f \in C_{\partial U}(\overline{U}, E)$ of F and any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = f|_{\partial U}$ there exists a $x \in U$ with $x \in \Phi(j(x))$.

Remark 2.10. If $F \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$ and if $f \in C_{\partial U}(\overline{U}, E)$ is any selection of F, then there exists a $x \in U$ with $x \in \Phi(f(x))$ (take $j = f$ in Definition 2.9) and so $F(x) \cap \Phi^{-1}(x) \neq \emptyset$.

Theorem 2.11. *Let E be a completely regular topological space, U an open subset of E,* $F \in A_{\partial U}(\overline{U}, E)$ *and* $G \in A_{\partial U}(\overline{U}, E)$ *is essential in* $A_{\partial U}(\overline{U}, E)$. *Also suppose*

$$\begin{cases} \text{for any selection } f \in C_{\partial U}(\overline{U}, E) \text{ (respectively, } g \in C_{\partial U}(\overline{U}, E)) \\ \text{of } F \text{ (respectively, of } G) \text{ and any map } j \in C_{\partial U}(\overline{U}, E) \text{ with } j|_{\partial U} = f|_{\partial U} \text{ we have } g \cong j \text{ in } C_{\partial U}(\overline{U}, E). \end{cases} \tag{2.1}$$

Then F is essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Let $f \in C_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = f|_{\partial U}$. We must show there exists an $x \in U$ with $x \in \Phi(j(x))$. Let $g \in C_{\partial U}(\overline{U}, E)$ be any selection of G. Now (2.1) guarantees that there is a continuous map $h : \overline{U} \times [0, 1] \to E$ with $x \notin \Phi(h_t(x))$ for $x \in \partial U$ and $t \in (0, 1)$, $h_0 = g$ and $h_1 = j$. Let

$$K = \left\{x \in \overline{U} : x \in \Phi(h_t(x)) \text{ for some } t \in [0, 1]\right\} \quad \text{and} \quad D = \left\{(x, t) \in \overline{U} \times [0, 1] : x \in \Phi(h_t(x))\right\}.$$

Note $D \neq \emptyset$ (take $t = 0$ and note $G \in A_{\partial U}(\overline{U}, E)$ is essential in $A_{\partial U}(\overline{U}, E)$) and D is closed (note Φ is upper semi-continuous and h is continuous) and so compact (note Φ is a compact map). Let $\pi : \overline{U} \times [0, 1] \to \overline{U}$ be a projection. Now $K = \pi(D)$ is closed (see Kuratowski's theorem [2]) and so in fact compact (recall projections are continuous). Also note $K \cap \partial U = \emptyset$ (since $x \notin \Phi(h_t(x))$ for $x \in \partial U$ and $t \in (0, 1)$) so since E is Tychonoff there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $r(x) = h(x, \mu(x)) = h_{\mu(x)}(x)$ for $x \in \overline{U}$. Note $r \in C_{\partial U}(\overline{U}, E)$ with $r|_{\partial U} = h_0|_{\partial U} = g|_{\partial U}$. Now since G is essential in $A_{\partial U}(\overline{U}, E)$ there exists a $x \in U$ with $x \in \Phi(r(x))$, i.e., $x \in \Phi(h_{\mu(x)}(x))$. Thus $x \in K$ so $\mu(x) = 1$ and $x \in \Phi(h_1(x)) = \Phi(j(x))$, as required. □

Now we present the topological transversality theorem for $A_{\partial U}(\overline{U}, E)$ maps. Let E be a topological vector space (recall topological vector sapces are completely regular). Next note

$$\text{if } \phi, \psi \in C_{\partial U}(\overline{U}, E) \text{ with } \phi|_{\partial U} = \psi|_{\partial U}, \text{then } \phi \cong \psi \text{ in } C_{\partial U}(\overline{U}, E); \tag{2.2}$$

to see this let $h(x, t) = (1 - t)\phi(x) + t\psi(x)$ and note $x \notin \Phi(h_t(x))$ for $x \in \partial U$ and $t \in (0, 1)$ (since if $x \in \partial U$ and $t \in (0, 1)$, then since $\phi|_{\partial U} = \psi|_{\partial U}$ we have $\Phi(h_t(x)) = \Phi((1 - t)\psi(x) + t\psi(x)) = \Phi(\psi(x))$).

Theorem 2.12. *Let E be a topological vector space and U an open subset of E. Suppose F and G are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Now F is essential in $A_{\partial U}(\overline{U}, E)$ if and only if G is Φ-essential in $A_{\partial U}(\overline{U}, E)$. (In Theorem 2.12 if E a topological vector space is replaced by E a completely regular topological space, then the result in Theorem 2.12 again holds provided we assume (2.2).)*

Proof. Assume G is essential in $A_{\partial U}(\overline{U}, E)$. We will apply Theorem 2.11 here. Let $f \in C_{\partial U}(\overline{U}, E)$ be any selection of F and let $g \in C_{\partial U}(\overline{U}, E)$ be any selection of G and consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = f|_{\partial U}$. Now since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ we have $f \cong g$ in $C_{\partial U}(\overline{U}, E)$. Also from (2.2) (here $\phi = j$ and $\psi = f$) we have $j \cong f$ in $C_{\partial U}(\overline{U}, E)$. Combining gives $g \cong j$ in $C_{\partial U}(\overline{U}, E)$, i.e., (2.1). Thus Theorem 2.11 guarantees that F is essential in $A_{\partial U}(\overline{U}, E)$. A similar argument shows if F is essential in $A_{\partial U}(\overline{U}, E)$, then G is essential in $A_{\partial U}(\overline{U}, E)$. □

Next we present an example of an essential in $A_{\partial U}(\overline{U}, E)$ map which will enable us to present a Leray-Schauder type alternative.

Theorem 2.13. *Let E be a locally convex metrizable topological vector space, U an open subset of E and $\Phi(0) \subseteq U$. Then the zero map is essential in $A_{\partial U}(\overline{U}, E)$.*

Proof. Consider any selection $g \in C_{\partial U}(\overline{U}, E)$ of the zero map (note $g = 0$). Now consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = 0|_{\partial U}$. We must show there exists a $x \in U$ with $x \in \Phi(j(x))$. Let

$$\psi(x) = \begin{cases} j(x), & x \in \overline{U}, \\ 0, & x \in E \backslash \overline{U}. \end{cases}$$

Now $\psi \in C(E, E)$ (a map $\theta \in C(E, E)$ if $\theta : E \to E$ is a continuous map) so $\Phi\psi$ is an admissible compact map. Then Theorem 1.1 (note from Dugundji extension theorem every locally convex metrizable topological vector space is an AR) guarantees that there exists a $x \in E$ with $x \in \Phi(\psi(x))$. If $x \in E \backslash U$, then $x \in \Phi(0)$, a contradiction since $\Phi(0) \subseteq U$. Thus $x \in U$ so $x \in \Phi(j(x))$. □

Theorem 2.14. *Let E be a locally convex metrizable topological vector space, U an open subset of E, $F \in A_{\partial U}(\overline{U}, E)$, $\Phi(0) \subseteq U$ and $tF(x) \cap \Phi^{-1}(x) = \emptyset$ for $x \in \partial U$ and $t \in (0, 1)$. Then F is essential in $A_{\partial U}(\overline{U}, E)$ (so in particular there exists a $x \in U$ with $F(x) \cap \Phi^{-1}(x) \neq \emptyset$).*

Proof. From Theorem 2.13 we know that the zero map is essential in $A_{\partial U}(\overline{U}, E)$. We will apply Theorem 2.11 to show F is essential in $A_{\partial U}(\overline{U}, E)$. Note that topological vector spaces are completely regular so we need only to show (2.1) with $G = 0$ (so automatically $g = 0$). Let $f \in C_{\partial U}(\overline{U}, E)$ be any selection of F and consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = f|_{\partial U}$. Now let $h(x, t) = tj(x)$ and note $j \cong 0$ in $C_{\partial U}(\overline{U}, E)$ (note if $x \in \partial U$ and $t \in (0, 1)$, then $x \notin \Phi(h_t(x))$ since $j|_{\partial U} = f|_{\partial U}$ gives $\Phi(h_t(x)) = \Phi(tj(x)) = \Phi(tf(x))$). Thus (2.1) holds. □

Remark 2.15. Theorem 2.14 gives a strong conclusion, namely F is essential in $A_{\partial U}(\overline{U}, E)$. The usual conclusion in a Leray-Schauder type alternative is that there exists a $x \in U$ with $F(x) \cap \Phi^{-1}(x) \neq \emptyset$. We note that this can be proved directly without any reference to essential maps. Let $f \in C(\overline{U}, E)$ be any selection of F and let

$$K = \{x \in \overline{U} : x \in \Phi(tf(x)) \text{ for some } t \in [0, 1]\}.$$

Note $K \neq \emptyset$ (take $t = 0$ and note $\Phi(0) \subseteq U$) is compact and $K \cap \partial U = \emptyset$ (since $tF(x) \cap \Phi^{-1}(x) = \emptyset$ for $x \in \partial U$ and $t \in (0, 1)$) so there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $\theta : E \to E$ be given by

$$\theta(x) = \begin{cases} \mu(x)f(x), & x \in \overline{U}, \\ 0, & x \in E \backslash \overline{U}. \end{cases}$$

Now $\theta \in C(E, E)$ so $\Phi\theta$ is an admissible compact map. Then Theorem 1.1 guarantees that there exists a $x \in E$ with $x \in \Phi(\theta(x))$. If $x \in E \backslash U$, then $x \in \Phi(0)$, a contradiction since $\Phi(0) \subseteq U$. Thus $x \in U$ so $x \in \Phi(\mu(x)f(x))$ and as a result $x \in K$. Thus $\mu(x) = 1$ and so $x \in \Phi(f(x))$ so $F(x) \cap \Phi^{-1}(x) \neq \emptyset$.

A special case of Remark 2.15 (i.e., when $A = C$) is the following.

Theorem 2.16. *Let E be a locally convex metrizable topological vector space, U an open subset of E, $f \in C_{\partial U}(\overline{U}, E)$, $\Phi(0) \subseteq U$ and $x \notin \Phi(tf(x))$ for $x \in \partial U$ and $t \in (0,1)$. Then there exists a $x \in U$ with $x \in \Phi(f(x))$.*

Remark 2.17. There is an obvious analogue of Theorem 2.14, when $A = C$ also.

Now we prove a Furi-Pera type result. Here E will be a locally convex metrizable topological vector space and Q a closed convex subset of E. In our next result we assume $\partial Q = Q$ (the case when $\text{int}(Q) \neq \emptyset$ is also easily handled; see Remark 2.19).

Theorem 2.18. *Let E be a locally convex metrizable topological vector space, Q a closed convex subset of E, $\partial Q = Q$, $F \in A(Q, E)$ and $\Phi \in B(E, E)$ with $\Phi(0) \subseteq Q$. In addition assume*

$$\begin{cases} \text{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \text{ is a sequence in } \partial Q \times [0,1] \text{ converging} \\ \text{to } (x, \lambda) \text{ with } \lambda F(x) \cap \Phi^{-1}(x) \neq \emptyset \text{ and } 0 \leqslant \lambda < 1, \text{ then } \{\Phi(\lambda_j F(x_j))\} \subseteq Q \text{ for } j \text{ sufficiently large.} \end{cases} \tag{2.3}$$

Then there exists a $x \in Q$ with $F(x) \cap \Phi^{-1}(x) \neq \emptyset$.

Proof. From Dugundji's theorem we know there exists a retraction $r : E \to Q$. Let $f \in C(Q, E)$ be a selection of F and let

$$\Omega = \{x \in E : x \in \Phi(f(r(x))\}.$$

Note $\Omega \neq \emptyset$ from Theorem 1.1 (note Φfr is a compact admissibe map) and Ω is compact. We claim $\Omega \cap Q \neq \emptyset$. To show this we argue by contradiction. Suppose $\Omega \cap Q = \emptyset$. Then since Ω is compact and Q is closed, there exists a $\delta > 0$ with $\text{dist}(Q, \Omega) > \delta$. Choose $m \in \{1, 2, \ldots\}$ with $1 < \delta m$ and let

$$U_i = \left\{ x \in E : d(x, Q) < \frac{1}{i} \right\} \text{ for } i \in \{m, m+1, \ldots, \};$$

here d is the metric associated with E. Fix $i \in \{m, m+1, \ldots\}$. Since $\text{dist}(Q, \Omega) > \delta$ we see that $\Omega \cap \overline{U_i} = \emptyset$. Now Theorem 2.16 (note $fr \in C(E, E)$ and $\Phi(0) \subseteq Q \subseteq U_i$) guarantees that there exists $\lambda_i \in (0,1)$ and $y_i \in \partial U_i$ with $y_i \in \Phi(\lambda_i fr(y_i))$. Since $y_i \in \partial U_i$ we have $\{\Phi(\lambda_i fr(y_i))\} \not\subseteq Q$ for $i \in \{m, m+1, \ldots\}$ and so

$$\{\Phi(\lambda_i Fr(y_i))\} \not\subseteq Q \text{ for } i \in \{m, m+1, \ldots\}. \tag{2.4}$$

Let

$$D = \{x \in E : x \in \Phi(\lambda fr(x)) \text{ for some } \lambda \in [0,1]\}.$$

Now $D \neq \emptyset$ (see Theorem 1.1 and take $\lambda = 1$) and D is compact. This together with

$$d(y_j, Q) = \frac{1}{j} \text{ and } |j_j| \leqslant 1 \text{ for } j \in \{m, m+1, \ldots\}$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^\star \in [0,1]$ and $y_j \to y^\star \in \partial Q$. In addition since f and r are continuous, Φ is upper semi-continuous and $y_j \in \Phi(\lambda_j fr(y_j))$ we have $y^\star \in \Phi(\lambda^\star fr(y^\star))$. Thus since $r(y^\star) = y^\star$ we have $y^\star \in \Phi(\lambda^\star fy^\star)$. If $\lambda^\star = 1$, then $y^\star \in \Phi(fy^\star)(= \Phi(fr(y^\star))$, which contradicts $\Omega \cap Q = \emptyset$. Thus $0 \leqslant \lambda^\star < 1$. Now (2.3) with $x_j = r(y_j)$ (note $y_j \in \partial U_j$ and $r(y_j) \in \partial Q$) and $x = y^\star = r(y^\star)$ and $y^\star \in \Phi(\lambda^\star f(y^\star))$ (so $\lambda^\star F(y^\star) \cap \Phi^{-1}(y^\star) \neq \emptyset$) implies

$$\{\Phi(\lambda_j Fx_j)\} \subseteq Q \text{ for } j \text{ sufficiently large.}$$

This contradicts (2.4). Thus $\Omega \cap Q \neq \emptyset$ so there exists a $x \in Q$ with $x \in \Phi(fr(x)) = \Phi(f(x))$, so $F(x) \cap \Phi^{-1}(x) \neq \emptyset$. □

Remark 2.19. In Theorem 2.18 we assumed $\partial Q = Q$. However this is easily removed since if $\text{int}(Q) \neq \emptyset$ (assume without loss of generality that $0 \in \text{int}(Q)$), then one can take the retraction $r : E \to Q$ as

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E,$$

where μ is the Minkowski functional on Q (i.e., $\mu(x) = \inf\{\alpha > 0 : x \in \alpha Q\}$). Note $r(z) \in \partial Q$ if $z \in E \backslash Q$. The argument in Theorem 2.18 now remains the same (once one notes that $r(y_j)$ in the proof is in ∂Q).

A special case of Theorem 2.18 and Remark 2.19 (i.e., when $A = C$) is the following.

Theorem 2.20. *Let E be a locally convex metrizable topological vector space, Q a closed convex subset of E, $f \in C(Q, E)$ and $\Phi \in B(E, E)$ with $\Phi(0) \subseteq Q$. In addition assume*

$$\begin{cases} \textit{if } \{(x_j, \lambda_j)\}_{j=1}^{\infty} \textit{ is a sequence in } \partial Q \times [0,1] \textit{ converging} \\ \textit{to } (x, \lambda) \textit{ with } x \in \Phi(\lambda f(x)) \textit{ and } 0 \leqslant \lambda < 1, \textit{ then } \{\Phi(\lambda_j f(x_j))\} \subseteq Q \textit{ for } j \textit{ sufficiently large.} \end{cases}$$

Then there exists a $x \in Q$ with $x \in \Phi(f(x))$.

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