A coincidence continuation theory between multi-valued maps with continuous selections and compact admissible maps

Donal O'Regan

School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland.

Abstract

We establish a topological transversality theorem and a Leray-Schauder alternative for coincidences between multi-valued maps with continuous selections and compact admissible maps.

Keywords: Continuous selections, admissible maps, essential maps, coincidence theory.


1. Introduction

This paper discusses coincidences between multi-valued maps with continuous selections and compact admissible maps. In particular we present a general Granas type topological transversality theorem [5, 6, 9], a general Leray-Schauder type alternatives [6, 9] and also a general Furi-Pera type result [3] for coincidences. Even though some of the results presented here could be modified from the results of O’Regan [9] (Φ replaced by Φ−1 there) however we feel it is more natural to construct this theory from the well known fixed point result of Gorniewicz [4, 8]. To motivate our theory we present below a very simple coincidence result in a general setting.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \rightarrow X$, $H(f)$ is the induced linear map $f_* = \{f_*q\}$ where $f_*q : H_q(X) \rightarrow H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \rightarrow X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

Email address: donal.oregan@nuigalway.ie (Donal O’Regan)
doi: 10.22436/jnsa.014.03.01
Received: 2020-06-14 Revised: 2020-07-25 Accepted: 2020-07-30
(ii) $p$ is a perfect map, i.e., $p$ is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multi-valued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of $Y$). A pair $(p, q)$ of single valued continuous maps of the form $X \xrightarrow{p} \Gamma \xrightarrow{q} Y$ is called a selected pair of $\phi$ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) $p$ is a Vietoris map;
(ii) $q[p^{-1}(x)] \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [4]. A upper semi-continuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in \Ad(X, Y)$) provided there exists a selected pair $(p, q) \subset \phi$.

Let $Z$ and $W$ be subsets of Hausdorff topological vector spaces $Y_1$ and $Y_2$ and $G$ a multi-function. We say $G \in \DTK(Z, W)$ [1, 7] if $W$ is convex and there exists a map $S : Z \to W$ with $\co(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in $Z$) for each $w \in W$.

By a space we mean a Hausdorff topological space. Let $Q$ be a class of topological spaces. A space $Y$ is an extension space for $Q$ (written $Y \in ES(Q)$) if for all $X \in Q$ and all $K \subseteq X$ closed in $X$, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$.

Now we recall the following fixed point result from the literature [4, 8].

**Theorem 1.1.** Let $X \in ES(\text{compact})$ and $\Psi \in \Ad(X, X)$ a compact map. Then there exists a $x \in X$ with $x \in \Psi(x)$.

We note that one can use Theorem 1.1 to generate coincidence results. For convenience we present one simple result to illustrate the strategy.

**Theorem 1.2.** Let $X$ and $Y$ be subsets of a Hausdorff topological vector space $E$ with $X$ convex and $Y$ paracompact. Suppose $F \in \Ad(X, Y)$ is a compact map and $G \in \DTK(Y, X)$. In addition suppose $Y \in ES(\text{compact})$ (respectively, $X \in ES(\text{compact})$). Then there exists a $y \in Y$ with $G(y) \cap F^{-1}(y) \neq \emptyset$ (respectively, there exists a $x \in X$ with $G^{-1}(x) \cap F(x) \neq \emptyset$).

**Proof.** Since $Y$ is paracompact, then from [1, 7] there exists a selection $g \in C(Y, X)$ (note $\theta \in C(Y, X)$ if $\theta : Y \to X$ is a continuous (single valued) map) of $G$. Now $Fg \in \Ad(Y, Y)$ (respectively, $gF \in \Ad(X, X)$) is a compact map. Now Theorem 1.1 guarantees that there exists a $y \in Y$ with $y \in Fg(y)$ (respectively, there exists a $x \in X$ with $x \in gF(x)$).

**Remark 1.3.** In Theorem 1.2 one could replace $F$ is a compact map with $G$ is a compact map.

2. Continuation theory

Let $E$ be a completely regular topological space and $U$ an open subset of $E$.

**Definition 2.1.** We say $\Phi \in B(E, E)$ if $\Phi \in \Ad(E, E)$ and $\Phi$ is a compact map.

**Remark 2.2.** An example of a map $\Phi \in \Ad(E, E)$ is if $\Phi : E \to K(E)$; here $K(E)$ denotes the family of nonempty, compact, acyclic subsets of $E$. In this paper we consider $\Phi \in B(E, E)$ but we note if we wish one could consider $\Phi \in B(E, \overline{U})$ throughout the paper; here $\overline{U}$ denotes the closure of $U$ in $E$.

**Definition 2.3.** We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to 2^E$ and there exists a continuous (single valued) selection $f : \overline{U} \to E$ (we write $f \in C(\overline{U}, E)$) of $F$.

**Remark 2.4.**

(i) Suppose $E$ is a topological vector space and $\overline{U}$ is paracompact. An example of a map $F \in A(\overline{U}, E)$ is $F \in \DTK(\overline{U}, E)$. As an aside, note metrizable spaces are paracompact and closed subsets of paracompact spaces are paracompact.
(ii). In this paper we always assume \( \Phi \in B(E, E) \) is a compact map and \( F \in A(\overline{U}, E) \) if there exists a continuous selection of \( F \). However it is easy to adjust the theory if we assume \( \Phi \in B(E, E) \) means \( \Phi \in \mathcal{A}(\overline{U}, E) \) and \( F \in A(\overline{U}, E) \) means there exists a continuous compact selection of \( F \).

In this paper we fix a \( \Phi \in B(E, E) \).

**Definition 2.5.** We say \( F \in A_{\partial U}(\overline{U}, E) \) (respectively, \( f \in C_{\partial U}(\overline{U}, E) \)) if \( F \in A(U, E) \) (respectively, \( f \in C(U, E) \)) with \( F(x) \cap \Phi^{-1}(x) = \emptyset \) (respectively, \( x \notin \Phi(f(x)) \)) for \( x \in \partial U \); where \( \partial U \) denotes the boundary of \( U \) in \( E \) and \( \Phi^{-1}(x) = \{ z \in E : x \in \Phi(z) \} \).

**Definition 2.6.** Let \( f, g \in C_{\partial U}(\overline{U}, E) \). We say \( f \equiv g \) in \( C_{\partial U}(\overline{U}, E) \) if there exists a continuous map \( h : \overline{U} \times [0, 1] \to E \) with \( x \notin \Phi(h_t(x)) \) for \( x \in \partial U \) and \( t \in (0, 1) \) (here \( h_t(x) = h(x, t) \)), \( h_0 = f \) and \( h_1 = g \).

**Remark 2.7.** Note \( \equiv \) in \( C_{\partial U}(\overline{U}, E) \) is an equivalence relation.

**Definition 2.8.** Let \( F, G \in A_{\partial U}(\overline{U}, E) \). We say \( F \equiv G \) in \( A_{\partial U}(\overline{U}, E) \) if for any selection \( f \in C_{\partial U}(\overline{U}, E) \) (respectively, \( g \in C_{\partial U}(\overline{U}, E) \)) of \( F \) (respectively, \( G \)) we have \( f \equiv g \) in \( C_{\partial U}(\overline{U}, E) \).

**Definition 2.9.** We say \( F \in A_{\partial U}(\overline{U}, E) \) is essential in \( A_{\partial U}(\overline{U}, E) \) if for any selection \( f \in C_{\partial U}(\overline{U}, E) \) of \( F \) and any map \( j \in C_{\partial U}(\overline{U}, E) \) with \( j|_{\partial U} = f|_{\partial U} \) there exists a \( x \in U \) with \( x \in \Phi(j(x)) \).

**Remark 2.10.** If \( F \in A_{\partial U}(\overline{U}, E) \) is essential in \( A_{\partial U}(\overline{U}, E) \) and if \( F \in C_{\partial U}(\overline{U}, E) \) is any selection of \( F \), then there exists a \( x \in U \) with \( x \in \Phi(f(x)) \) (take \( j = f \) in Definition 2.9) and so \( F(x) \cap \Phi^{-1}(x) \neq \emptyset \).

**Theorem 2.11.** Let \( E \) be a completely regular topological space, \( U \) an open subset of \( E \), \( F \in A_{\partial U}(\overline{U}, E) \) and \( G \in A_{\partial U}(\overline{U}, E) \) is essential in \( A_{\partial U}(\overline{U}, E) \). Also suppose

\[
\left\{ \begin{array}{ll}
\text{for any selection } f \in C_{\partial U}(\overline{U}, E) \ (\text{respectively, } g \in C_{\partial U}(\overline{U}, E)) \\
\text{of } F \ (\text{respectively, of } G) \text{ and any map } j \in C_{\partial U}(\overline{U}, E) \text{ with } j|_{\partial U} = f|_{\partial U} \text{ we have } g \equiv j \text{ in } C_{\partial U}(\overline{U}, E).
\end{array} \right.
\]

Then \( F \) is essential in \( A_{\partial U}(\overline{U}, E) \).

**Proof.** Let \( f \in C_{\partial U}(\overline{U}, E) \) be any selection of \( F \) and consider any map \( j \in C_{\partial U}(\overline{U}, E) \) with \( j|_{\partial U} = f|_{\partial U} \). We must show there exists an \( x \in U \) with \( x \in \Phi(j(x)) \). Let \( g \in C_{\partial U}(\overline{U}, E) \) be any selection of \( G \). Now (2.1) guarantees that there is a continuous map \( h : \overline{U} \times [0, 1] \to E \) with \( x \notin \Phi(h_t(x)) \) for \( x \in \partial U \) and \( t \in (0, 1) \), \( h_0 = g \) and \( h_1 = j \). Let

\[ K = \{ x \in \overline{U} : x \in \Phi(h_t(x)) \text{ for some } t \in [0, 1] \} \quad \text{and} \quad D = \{ (x, t) \in \overline{U} \times [0, 1] : x \notin \Phi(h_t(x)) \}. \]

Note \( D \neq \emptyset \) (take \( t = 0 \) and note \( G \in A_{\partial U}(\overline{U}, E) \) is essential in \( A_{\partial U}(\overline{U}, E) \)) and \( D \) is closed (note \( \Phi \) is upper semi-continuous and \( h \) is continuous) and so compact (note \( \Phi \) is a compact map). Let \( \pi : \overline{U} \times [0, 1] \to \overline{U} \) be a projection. Now \( K = \pi(D) \) is closed (see Kuratowski’s theorem [2]) and so in fact compact (recall projections are continuous). Also note \( K \cap \partial U = \emptyset \) (since \( x \notin \Phi(h_t(x)) \) for \( x \in \partial U \) and \( t \in (0, 1) \)) so since \( E \) is Tychonoff there exists a continuous map \( \mu : \overline{U} \to [0, 1] \) with \( \mu(\partial U) = 0 \) and \( \mu(K) = 1 \). Let \( r(x) = h(x, \mu(x)) = h_{\mu(x)}(x) \) for \( x \in \overline{U} \). Note \( r \in C_{\partial U}(\overline{U}, E) \) with \( r|_{\partial U} = h_{\mu|_{\partial U}} \) if \( g_{\partial U} \). Now since \( G \) is essential in \( A_{\partial U}(\overline{U}, E) \) there exists a \( x \in U \) with \( x \in \Phi(r(x)) \), i.e., \( x \in \Phi(h_{\mu(x)}(x)) \). Thus \( x \in K \) so \( \mu(x) = 1 \) and \( x \in \Phi(h_1(x)) = \Phi(j(x)) \), as required.

Now we present the topological transversality theorem for \( A_{\partial U}(\overline{U}, E) \) maps. Let \( E \) be a topological vector space (recall topological vector spaces are completely regular). Next note

\[
\text{if } \phi, \psi \in C_{\partial U}(\overline{U}, E) \text{ with } \phi|_{\partial U} = \psi|_{\partial U}, \text{ then } \phi \equiv \psi \text{ in } C_{\partial U}(\overline{U}, E); \tag{2.2}
\]

to see this let \( h(x, t) = (1 - t)\phi(x) + t\psi(x) \) and note \( x \notin \Phi(h_t(x)) \) for \( x \in \partial U \) and \( t \in (0, 1) \) (since if \( x \in \partial U \) and \( t \in (0, 1) \), then since \( \phi|_{\partial U} = \psi|_{\partial U} \) we have \( \Phi(h_t(x)) = \Phi((1 - t)\psi(x) + t\psi(x)) = \Phi(\psi(x)) \)).
Theorem 2.12. Let $E$ be a topological vector space and $U$ an open subset of $E$. Suppose $F$ and $G$ are two maps in $A_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $A_{\partial U}(\overline{U}, E)$. Now $F$ is essential in $A_{\partial U}(\overline{U}, E)$ if and only if $G$ is $\Phi$-essential in $A_{\partial U}(\overline{U}, E)$. (In Theorem 2.12 if $E$ a topological vector space is replaced by $E$ a completely regular topological space, then the result in Theorem 2.12 again holds provided we assume (2.2).)

Proof. Assume $F$ is essential in $A_{\partial U}(\overline{U}, E)$. We will apply Theorem 2.11 here. Let $f \in C_{\partial U}(\overline{U}, E)$ be any selection of $F$ and let $g \in C_{\partial U}(\overline{U}, E)$ be any selection of $G$ and consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = f|_{\partial U}$. Now since $F \cong G$ in $A_{\partial U}(\overline{U}, E)$ we have $f \cong g$ in $C_{\partial U}(\overline{U}, E)$. Also from (2.2) (here $\psi = j$ and $\Phi = f$) we have $j \cong f$ in $C_{\partial U}(\overline{U}, E)$. Combining gives $g \cong j$ in $C_{\partial U}(\overline{U}, E)$, i.e., (2.1). Thus Theorem 2.11 guarantees that $F$ is essential in $A_{\partial U}(\overline{U}, E)$. A similar argument shows if $F$ is essential in $A_{\partial U}(\overline{U}, E)$, then $G$ is essential in $A_{\partial U}(\overline{U}, E)$.

Next we present an example of an essential in $A_{\partial U}(\overline{U}, E)$ map which will enable us to present a Leray-Schauder type alternative.

Theorem 2.13. Let $E$ be a locally convex metrizable topological vector space, $U$ an open subset of $E$ and $\Phi(0) \subseteq U$. Then the zero map is essential in $A_{\partial U}(\overline{U}, E)$.

Proof. Consider any selection $g \in C_{\partial U}(\overline{U}, E)$ of the zero map (note $g = 0$). Now consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = 0|_{\partial U}$. We must show there exists a $x \in U$ with $x \in \Phi(j(x))$. Let

$$\psi(x) = \begin{cases} j(x), & x \in \overline{U} \\ 0, & x \in E \setminus \overline{U} \end{cases}$$

Now $\psi \in C(E, E)$ (a map $\theta \in C(E, E)$ if $\theta : E \to E$ is a continuous map) so $\Phi \psi$ is an admissible compact map. Then Theorem 1.1 (note from Dugundji extension theorem every locally convex metrizable topological vector space is an AR) guarantees that there exists a $x \in E$ with $x \in \Phi(\psi(x))$. If $x \in E \setminus U$, then $x \in \Phi(0)$, a contradiction since $\Phi(0) \subseteq U$. Thus $x \in U$ so $x \in \Phi(j(x))$.

Theorem 2.14. Let $E$ be a locally convex metrizable topological vector space, $U$ an open subset of $E$, $F \in A_{\partial U}(\overline{U}, E)$, $\Phi(0) \subseteq U$ and $tF(x) \cap \Phi^{-1}(x) = \emptyset$ for $x \in \partial U$ and $t \in (0, 1)$. Then $F$ is essential in $A_{\partial U}(\overline{U}, E)$ (so in particular there exists a $x \in U$ with $F(x) \cap \Phi^{-1}(x) \neq \emptyset$).

Proof. From Theorem 2.13 we know that the zero map is essential in $A_{\partial U}(\overline{U}, E)$. We will apply Theorem 2.11 to show $F$ is essential in $A_{\partial U}(\overline{U}, E)$. Note that topological vector spaces are completely regular so we need only to show (2.1) with $G = 0$ (so automatically $g = 0$). Let $f \in C_{\partial U}(\overline{U}, E)$ be any selection of $F$ and consider any map $j \in C_{\partial U}(\overline{U}, E)$ with $j|_{\partial U} = f|_{\partial U}$. Now let $h(x, t) = tf(x)$ and note $j \preceq 0$ in $C_{\partial U}(\overline{U}, E)$ (note if $x \in \partial U$ and $t \in (0, 1)$, then $x \not\in \Phi(h_t(x))$ since $j|_{\partial U} = f|_{\partial U}$ gives $\Phi(h_t(x)) = \Phi(tf(x))$. Thus (2.1) holds).

Remark 2.15. Theorem 2.14 gives a stronger conclusion, namely $F$ is essential in $A_{\partial U}(\overline{U}, E)$. The usual conclusion in a Leray-Schauder type alternative is that there exists a $x \in U$ with $F(x) \cap \Phi^{-1}(x) \neq \emptyset$. We note that this can be proved directly without any reference to essential maps. Let $F \in C(\overline{U}, E)$ be any selection of $F$ and let

$$K = \{x \in U : x \in \Phi(tf(x)) \text{ for some } t \in [0, 1]\}$$

Note $K \neq \emptyset$ (take $t = 0$ and note $\Phi(0) \subseteq U$ is compact and $K \cap \partial U = \emptyset$ (since $tF(x) \cap \Phi^{-1}(x) = \emptyset$ for $x \in \partial U$ and $t \in (0, 1)$) so there exists a continuous map $\mu : \overline{U} \to [0, 1]$ with $\mu(\partial U) = 0$ and $\mu(K) = 1$. Let $\theta : E \to E$ be given by

$$\theta(x) = \begin{cases} \mu(x)f(x), & x \in \overline{U} \\ 0, & x \in E \setminus \overline{U} \end{cases}$$

Now $\theta \in C(E, E)$ so $\Phi \theta$ is an admissible compact map. Then Theorem 1.1 guarantees that there exists a $x \in E$ with $x \in \Phi(\theta(x))$. If $x \in E \setminus U$, then $x \in \Phi(0)$, a contradiction since $\Phi(0) \subseteq U$. Thus $x \in U$ so $x \in \Phi(\mu(x)f(x))$ and as a result $x \in K$. Thus $\mu(x) = 1$ and so $x \in \Phi(f(x))$ so $F(x) \cap \Phi^{-1}(x) \neq \emptyset$. 
A special case of Remark 2.15 (i.e., when $A = C$) is the following.

**Theorem 2.16.** Let $E$ be a locally convex metrizable topological vector space, $U$ an open subset of $E$, $f \in C_{\partial U}(\overline{U}, E)$, $\Phi(0) \subseteq U$ and $x \notin \Phi(f(t(x)))$ for $x \in \partial U$ and $t \in (0, 1)$. Then there exists a $x \in U$ with $x \in \Phi(f(x))$.

**Remark 2.17.** There is an obvious analogue of Theorem 2.14, when $A = C$ also.

Now we prove a Furi-Pera type result. Here $E$ will be a locally convex metrizable topological vector space and $Q$ a closed convex subset of $E$. In our next result we assume $\partial Q = Q$ (the case when $\text{int}(Q) \neq \emptyset$ is also easily handled; see Remark 2.19).

**Theorem 2.18.** Let $E$ be a locally convex metrizable topological vector space, $Q$ a closed convex subset of $E$, $\partial Q = Q$, $F \in A(Q, E)$ and $\Phi \in B(E, E)$ with $\Phi(0) \subseteq Q$. In addition assume

$$
\begin{cases}
\text{if } \{(x_i, \lambda_j)\}_{i=1}^{\infty} \text{ is a sequence in } \partial Q \times [0, 1] \text{ converging to } (x, \lambda) \text{ with } \lambda \Phi(f(x)) \neq 0 \\
\text{to } (x, \lambda) \text{ with } \lambda \Phi^{-1}(x) \neq 0 \text{ and } 0 < \lambda < 1, \text{ then } \{\Phi(\lambda_j f(x_i))\} \subseteq Q \text{ for } j \text{ sufficiently large.}
\end{cases}
$$

(2.3)

Then there exists a $x \in Q$ with $F(x) \cap \Phi^{-1}(x) \neq \emptyset$.

**Proof.** From Dugundji’s theorem we know there exists a retraction $r : E \to Q$. Let $f \in C(Q, E)$ be a selection of $F$ and let

$$
\Omega = \{x \in E : x \in \Phi(f(r(x)))\}.
$$

Note $\Omega \neq \emptyset$ from Theorem 1.1 (note $Fr$ is a compact admissible map) and $\Omega$ is compact. We claim $\Omega \cap Q \neq \emptyset$. To show this we argue by contradiction. Suppose $\Omega \cap Q = \emptyset$. Then since $\Omega$ is compact and $Q$ is closed, there exists a $\delta > 0$ with $\text{dist}(Q, \Omega) > \delta$. Choose $m \in \{1, 2, \ldots\}$ with $1 < \delta m$ and let

$$
U_i = \left\{ x \in E : d(x, Q) < \frac{1}{i} \right\} \text{ for } i \in \{m, m+1, \ldots\};
$$

where $d$ is the metric associated with $E$. Fix $i \in \{m, m+1, \ldots\}$. Since $\text{dist}(Q, \Omega) > \delta$ we see that $\Omega \cap U_i = \emptyset$. Now Theorem 2.16 (note $fr \in C(E, E)$ and $\Phi(0) \subseteq Q \subseteq U_i$) guarantees that there exists $\lambda_i \in (0, 1)$ and $y_i \in \partial U_i$ with $y_i \in \Phi(\lambda_i fr(y_i))$. Since $y_i \in \partial U_i$ we have $\{\Phi(\lambda_i fr(y_i))\} \subseteq Q$ for $i \in \{m, m+1, \ldots\}$ and so

$$
\{\Phi(\lambda_i Fr(y_i))\} \subseteq Q \text{ for } i \in \{m, m+1, \ldots\}.
$$

(2.4)

Let

$$
D = \{x \in E : x \in \Phi(\lambda fr(x)) \text{ for some } \lambda \in [0, 1]\}.
$$

Now $D \neq \emptyset$ (see Theorem 1.1 and take $\lambda = 1$) and $D$ is compact. This together with

$$
d(y_j, Q) = \frac{1}{j} \text{ and } |y_j| \leq 1 \text{ for } j \in \{m, m+1, \ldots\}
$$

implies that we may assume without loss of generality that $\lambda_j \to \lambda^* \in [0, 1]$ and $y_j \to y^* \in \partial Q$. In addition since $f$ and $r$ are continuous, $\Phi$ is upper semi-continuous and $y_j \in \Phi(\lambda_j fr(y_j))$ we have $y^* \in \Phi(\lambda^* fr(y^*))$. Thus since $r(y^*) = y^*$ we have $y^* \in \Phi(\lambda^* f(y^*))$. If $\lambda^* = 1$, then $y^* \in \Phi(\lambda^* f(y^*)) = \Phi(fr(y^*))$, which contradicts $\Omega \cap Q = \emptyset$. Thus $0 \leq \lambda^* < 1$. Now (2.3) with $x_i = r(y_j)$ (note $y_j \in \partial U_i$ and $r(y_j) \in \partial Q$) and $x = y^* = r(y^*)$ and $y^* \in \Phi(\lambda^* f(y^*))$ (so $\lambda^* F(y^*) \cap \Phi^{-1}(y^*) \neq \emptyset$) implies

$$
\{\Phi(\lambda_j f(x_j))\} \subseteq Q \text{ for } j \text{ sufficiently large.}
$$

This contradicts (2.4). Thus $\Omega \cap Q \neq \emptyset$ so there exists a $x \in Q$ with $x \in \Phi(fr(x)) = \Phi(f(x))$, so $F(x) \cap \Phi^{-1}(x) \neq \emptyset$. □
Remark 2.19. In Theorem 2.18 we assumed $\partial Q = Q$. However this is easily removed since if $\text{int}(Q) \neq \emptyset$ (assume without loss of generality that $0 \in \text{int}(Q)$), then one can take the retraction $r : E \to Q$ as

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \text{ for } x \in E,$$

where $\mu$ is the Minkowski functional on $Q$ (i.e., $\mu(x) = \inf\{\alpha > 0 : x \in \alpha Q\}$). Note $r(z) \in \partial Q$ if $z \in E \setminus Q$. The argument in Theorem 2.18 now remains the same (once one notes that $r(y_j)$ in the proof is in $\partial Q$).

A special case of Theorem 2.18 and Remark 2.19 (i.e., when $\Lambda = C$) is the following.

**Theorem 2.20.** Let $E$ be a locally convex metrizable topological vector space, $Q$ a closed convex subset of $E$, $f \in C(Q, E)$ and $\Phi \in B(E, E)$ with $\Phi(0) \subseteq Q$. In addition assume

$$\left\{ \begin{array}{l}
\text{if } \{(x_j, \lambda_j)\}_{j=1}^\infty \text{ is a sequence in } \partial Q \times [0, 1] \\
\text{converging}
\end{array} \right.
$$

$$\text{to } (x, \lambda) \text{ with } x \in \Phi(\Lambda f(x)) \text{ and } 0 \leq \lambda < 1,$$

then $\{(\lambda_j f(x_j))\} \subseteq Q$ for $j$ sufficiently large.

Then there exists $x \in Q$ with $x \in \Phi(f(x))$.

References


