Fixed point results for $(\beta, \alpha)$-implicit contractions in two generalized b-metric spaces

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Abstract
The aim of this paper is to introduce $(\beta, \alpha)$-implicit contractive of two mappings on two generalized b-metric spaces and derive some new fixed point theorems for $(\beta, \alpha)$-implicit contractive in two complete and compact generalized b-Metric spaces.

Keywords: Fixed points, $(\beta, \alpha)$-implicit contractions, generalized b-metric spaces.

1. Introduction
Mustafa and Sims [11] introduced the concept of G-metric spaces as a generalization of a metric space, since then, several interesting results for existence of fixed point in G-metric spaces have been obtained (see [2–4, 7, 10, 11]).

In the other hand, Papa et al. are proved some fixed point theorems for functions satisfying an implicit relations in metric spaces and G-metric spaces (see [9, 13–18]).

The notion of b-metric space was introduced by Czerwik [8], and many authors studied fixed point theorems in b-metric spaces (see[1, 6, 9, 12, 19]). Recently, Aghajani et al. [5] generalized the concept of b-metric spaces by using the notions of b -metric spaces and G-metric spaces, which is called $G_b$-metric spaces and they discussed some properties and common fixed point results of $G_b$-metric.

The aim of this paper is to introduce $(\beta,\alpha)$-implicit contractive of two mappings on two generalized b-metric spaces and derive some new fixed point theorems for $(\beta, \alpha)$-implicit contractive in two complete and compact generalized b-Metric spaces.

We recall some basic definitions and results, for details on the following notions (see [5]). Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}_+$ denotes the set of nonnegative reals and $\mathbb{N}$ denotes the set of natural numbers.

Definition 1.1. Let $X$ be a non empty set and $\delta \geq 1$ be a given real number. A mapping $G: X^3 \to \mathbb{R}_+$ is said to be a generalized b-metric ($G_b$-metric on $X$) if for all $x, y, z, \alpha \in X$, the following axioms are satisfied:
A pair \((X,G)\) is called a generalized b-metric.

**Example 1.2.** Let \((X,G)\) be a G-metric space. Consider \(G_b(x, y, z) = (G(x, y, z))^p\), where \(p > 1\), is a real number. Then \(G_b\) is a \(G_b\)-metric space with \(\delta = 2^{p-1}\).

Each G-metric space is a \(G_b\)-metric space with \(\delta = 1\). The following example shows that a \(G_b\)-metric space on \(X\) does not need to be a G-metric space on \(X\).

**Example 1.3.** Let \(X = \mathbb{R}\) and \(G_b = \frac{1}{6}(|x - y| + |z - y| + |z - x|)^2\). Then \(G_b\) is a \(G_b\)-metric space on \(\mathbb{R}\) but not a G-metric space on \(\mathbb{R}\). Indeed, let \(x = 3, y = 5, z = 7\) and \(a = \frac{7}{2}\), we have \(G_b(3, 5, 7) = \frac{64}{9}, G_b(3, \frac{7}{2}, \frac{7}{2}) = \frac{1}{9}\) and \(G_b(\frac{7}{2}, 5, 7) = \frac{49}{9}\) so \(G_b(3, 5, 7) > G_b(3, \frac{7}{2}, \frac{7}{2}) + G_b(\frac{7}{2}, 5, 7)\).

We present some definitions and propositions in \(G_b\)-metric space.

**Proposition 1.4.** Let \((X, G)\) be a \(G_b\)-metric space. Then for any \(x, y, z\) and \(a \in X\), it follows that:

1. if \(G(x, y, z) = 0\) then \(x = y = z\);
2. \(G(x, y, z) \leq G(x, x, y) + G(x, x, z)\);
3. \(G(x, y, y) \leq 2G(y, x, x)\);
4. \(G(x, y, z) \leq G(a, a, z) + G(a, a, y)\).

**Definition 1.5.** Let \((X, G)\) be a \(G_b\)-metric space, and \((x_n)\) be a sequence of points of \(X\), we say that \((x_n)\) is \(G_b\)-convergent to \(x\) if for any \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(G(x_n, x_m, x_l) < \varepsilon\), for all \(n, m, l \geq n_0\).

**Proposition 1.6.** Let \((X, G)\) be a \(G_b\)-metric space. Then the following are equivalent:

1. \((x_n)\) is \(G\)-convergent to \(x\);
2. \(G(x_n, x_n, x) \to 0\), as \(n \to \infty\);
3. \(G(x_n, x, x) \to 0\), as \(n \to \infty\).

**Definition 1.7.** Let \((X, G)\) be a \(G_b\)-metric space, a sequence \((x_n)\) is called \(G_b\)-Cauchy if given \(\varepsilon > 0\), there is \(n_0 \in \mathbb{N}\) such that \(G(x_n, x_m, x_l) < \varepsilon\), for all \(n, m, l \geq n_0\).

**Proposition 1.8.** Let \(X\) be a \(G_b\)-metric space, then the following are equivalent:

1. the sequence \(\{x_n\}\) is \(G_b\)-Cauchy;
2. for any \(\varepsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(G(x_n, x_m, x_l) < \varepsilon\), for all \(m, n \geq n_0\).

**Definition 1.9.** Let \((X, G)\) and \((X^*, G^*)\) be \(G_b\)-metric space and let \(h : (X, G) \to (X^*, G^*)\) be a function, then \(h\) is said to be \(G_b\)-continuous at a point \(a \in X\), if given \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(x, y \in X, G(a, x, y) < \delta\) implies \(G^*(h(a), h(x), h(y)) < \varepsilon\).

**Definition 1.10.** A \(G_b\)-metric space \((X, G)\) is said to be \(G_b\)-complete if every \(G_b\)-Cauchy sequence in \((X, G)\) is \(G_b\)-convergent in \((X, G)\).

**Definition 1.11.** A \(G_b\)-metric space \((X, G)\) is said to be a compact \(G_b\)-metric space if it is \(G_b\)-complete and \(G_b\)-totally bounded.
2. Main results

Now, in this paper, we introduce \((\beta, \alpha)\)-implicit contractive and prove our first new results in two complete \(G_b\)-metric spaces.

**Definition 2.1.** Let \(\pi\) be the set of all upper semi-continuous function in each variable \(h(r_1, r_2, r_3, r_4, r_5) : \mathbb{R}^5 \to \mathbb{R}\) satisfying:

\((\pi_1)\) \(h\) is non-decreasing in variable \(r_1\) and non-increasing in variables \(r_3, r_4, r_5\);

\((\pi_2)\) if either \(h(\mu, \nu, 0, \mu, 2\delta \nu) \leq 0\) or \(h(\mu, \nu, \mu, 0) \leq 0\) for all \(\mu, \nu \geq 0\), then \(\mu \leq \frac{1}{8\delta^3}\nu\).

**Example 2.2.** \(h(r_1, r_2, r_3, r_4, r_5) = r_1 - \frac{1}{4\delta^3} \max(r_2, r_3, r_4, \frac{1}{25} r_5)\).

**Example 2.3.** \(h(r_1, r_2, r_3, r_4, r_5) = r_1 - \frac{1}{25} r_2 + \frac{1}{25^3} \max(2\delta^3 r_3, 2\delta^3 r_4, \frac{1}{25} r_5)\).

**Example 2.4.** \(h(r_1, r_2, r_3, r_4, r_5) = r_1 - \frac{1}{4\delta^3} r_2 + r_3 + r_4 - \frac{1}{8\delta^3} r_5\).

**Definition 2.5.** Let \(\varphi : X \to Y, \psi : Y \to X\) be mappings and \(\alpha : X \times X \to [0, \infty), \beta : Y \times Y \to [0, \infty)\). We say that \(\varphi \psi : X \to X\) is an \(\alpha\)-admissible if \(x_1, x_2 \in X, \alpha(x_1, x_2) \geq 1 \implies \alpha(\psi(\varphi(x_1)), \psi(\varphi(x_2))) \geq 1\) and \(\varphi \psi : Y \to Y\) is a \(\beta\)-admissible if \(y_1, y_2 \in Y, \beta(y_1, y_2) \geq 1 \implies \beta(\varphi(\psi(y_1)), \varphi(\psi(y_2))) \geq 1\).

**Definition 2.6.** Let \((X, G_1)\) and \((Y, G_2)\) be complete \(G_b\)-metric spaces, \(\varphi : X \to Y, \psi : Y \to X\) be two given mappings and \(\alpha : X \times X \to [0, \infty), \beta : Y \times Y \to [0, \infty)\). The pair \((\varphi, \psi)\) is said to be an \((\beta, \alpha)\)-implicit contractive pair of mappings whenever there exists \(h, g \in \pi\) such that

\[
\begin{align*}
\h(\beta(s_1, \varphi \psi s_1) G_2(\varphi t_1, \varphi \psi s_1, \varphi \psi s_2), G_1(t, \varphi \psi s_1, s_2, \varphi t_1), G_2(s_1, \varphi \psi s_1, \varphi \psi s_2), G_1(t, t, \varphi \psi t_1)) \leq 0, \\
g(\alpha(t, \varphi \psi t_1) G_1(\varphi \psi s_1, \varphi \psi s_2, \varphi \psi t_1), G_2(s_1, \varphi \psi s_1, \varphi \psi s_2), G_1(t, t, \varphi \psi t_1), G_2(s_1, \varphi \psi s_1, \varphi \psi s_2)) \leq 0.
\end{align*}
\]

for all \(t \in X, s_1, s_2 \in Y\).

**Theorem 2.7.** Let \((X, G_1)\) and \((Y, G_2)\) be complete \(G_b\)-metric spaces, and \(\varphi : X \to Y, \psi : Y \to X\) be an \((\beta, \alpha)\)-implicit contractive mappings. Suppose that:

(i) \(\varphi \psi\) is an \(\beta\)-admissible and \(\psi \varphi\) is an \(\alpha\)-admissible;

(ii) there exists \(x_0 \in X\) and \(y_0 \in Y\) such that \(\alpha(x_0, \varphi \psi(x_0)) \geq 1, \beta(y_0, \varphi \psi(y_0)) \geq 1\);

(iii) \(\alpha(x, x) \geq 1, \beta(y, y) \geq 1, x \in X, y \in Y\).

Then \(\psi \varphi, \varphi \psi\) have a unique fixed points \(\xi, q\) in \(X\) and \(q\) in \(Y\), respectively. Further \(\varphi \xi = q\) and \(\psi q = \xi\).

**Proof.** Define two sequences \((t_n)\) in \(X\), and \((s_n)\) in \(Y\), by \(t_n = (\varphi \psi)^{n-1}t, s_n = \varphi(\psi)^{n-1}t\), for \(n = 1, 2, ..., t \in X\). Suppose that \(t_n \neq t_{n+1}\) and \(s_n \neq s_{n+1}\), for all \(n\). Applying (2.1), we have

\[
\begin{align*}
\h(\beta(s_n, \varphi \psi s_n) G_2(\varphi t_{n-1}, \varphi \psi s_n, \varphi \psi s_n), G_1(t_{n-1}, \varphi \psi s_n, \varphi \psi s_n), G_2(s_n, s_n, \varphi \psi s_n), G_1(t_{n-1}, t_{n-1}, t_{n-1})) \leq 0, \\
\h(\beta(s_n, s_{n+1}) G_2(s_n, s_{n+1}, s_{n+1}), G_1(t_{n-1}, t_{n-1}, t_{n-1}), 0, G_2(s_n, s_{n+1}, s_{n+1}), G_1(t_{n-1}, t_{n-1}, t_{n-1})) \leq 0.
\end{align*}
\]

By using the conditions (i) and (ii), we deduce that \(\beta(s_0, s_1) = \beta(s_0, \varphi \psi(s_0)) \geq 1 \implies \beta(\varphi \psi(s_0), \varphi \psi(s_1)) = \beta(s_1, s_2) \geq 1\). By iterating the process, we get \(\beta(s_n, s_{n+1}) \geq 1, \alpha(t_0, t_1) = \alpha(t_0, \varphi \psi(t_0)) \geq 1 \implies \alpha(\varphi \psi(t_0), \varphi \psi(t_1)) = \alpha(t_1, t_2) \geq 1\). By iterating the process, we get \(\alpha(t_n, t_{n+1}) \geq 1\).

Applying \((\pi_1), (\pi 2)\) we have

\[
\begin{align*}
\h(G_2(s_n, s_{n+1}, s_{n+1}), G_1(t_{n-1}, t_{n-1}, t_{n-1}), 0, G_2(s_n, s_{n+1}, s_{n+1}), 2\delta G_1(t_{n-1}, t_{n-1}, t_{n-1})) \leq 0, \\
G_2(s_n, s_{n+1}, s_{n+1}) \leq \frac{1}{4\delta^3} G_1(t_{n-1}, t_{n-1}, t_{n-1}).
\end{align*}
\]
Similarly, applying the inequality (2.2),
\[ g(\alpha(t_n, \psi \varphi t_n)G_1(\psi s_n, \psi s_n, \psi \varphi t_n), G_2(s_n, s_n, \varphi t_n), G_1(t_n, \psi s_n, \psi s_n), \\
G_1(t_n, t_n, t_{n+1}), G_2(s_n, \psi \varphi s_n, \varphi \psi s_n)) \leq 0, \]
\[ g(\alpha(t_n, t_{n+1})G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_n, s_{n+1}), G_1(t_n, t_n, t_n), G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_{n+1}, s_{n+1})) \leq 0, \]
\[ g(\alpha(t_n, t_{n+1})G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_n, s_{n+1}), G_1(t_n, t_n, t_n), G_1(t_n, t_n, t_{n+1}), 2\delta G_2(s_n, s_{n+1}, s_{n+1})) \leq 0 \]
and using the property (π1), (π2), we have
\[ g(G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_n, s_{n+1}), 0, G_1(t_n, t_n, t_{n+1}), 2\delta G_2(s_n, s_{n+1}, s_{n+1})) \leq 0, \]
\[ G_1(t_n, t_n, t_{n+1}) \leq \frac{1}{4\delta^3} G_2(s_n, s_n, s_{n+1}). \]  

Using Proposition 1.4, we get,
\[ \frac{1}{2\delta} G_1(t_n, t_{n+1}, t_{n+1}) \leq G_1(t_{n+1}, t_{n}, t_{n}) = G_1(t_n, t_n, t_{n+1}), \]  
\[ G_2(s_n, s_n, s_{n+1}) = G_2(s_{n+1}, s_n, s_n) \leq 2\delta G_2(s_n, s_{n+1}, s_{n+1}). \]

From (2.5), (2.6), and (2.4), we obtain
\[ G_1(t_n, t_{n+1}, t_{n+1}) \leq \frac{1}{\delta} G_2(s_n, s_{n+1}, s_{n+1}). \]  

Now it follows from the inequalities (2.3) and (2.7) that
\[ G_1(t_n, t_{n+1}, t_{n+1}) \leq \frac{1}{4\delta^4} G_1(t_{n-1}, t_t, t_n). \]

Hence, by induction we get
\[ G_1(t_n, t_{n+1}, t_{n+1}) \leq \left(\frac{1}{4\delta^4}\right)^n G_1(t, t, t_1), n = 1, 2, \ldots. \]

Hence, \( \lim_{n \to \infty} G_1(t_n, t_{n+1}, t_{n+1}) = 0. \) Put \( \kappa = \frac{1}{4\delta^7} \), for any \( p \in \mathbb{N}^+ \), we get
\[ G_1(t_n, t_{n+p}, t_{n+p}) \leq \delta G_1(t_{n+t}, t_{n_t+1}, t_{n+1}) + \delta^2 G_1(t_{n+t+2}, t_{n+2}) + \cdots + \delta^{p-1} G_1(t_{n+p-1}, t_{n+p}, t_{n+p}) \]
\[ \leq \delta \kappa^n G_1(t, t, t_1) + \delta^2 \kappa^{n+1} G_1(t, t, t_1) + \cdots + \delta^p \kappa^{n+p-1} G_1(t, t, t_1) \]
\[ = \delta \kappa^n \left(1 + \frac{(\delta \kappa)^p}{1 - \delta \kappa}\right) G_1(t, t, t_1). \]

Since \( \delta \kappa^n < 1 \), we have \( \lim_{n \to \infty} G_1(t_n, t_{n+p}, t_{n+p}) = 0 \), this due to \( \lim_{n \to \infty} G_1(t_n, t_{n}, t_{n}) = 0 \), follows that \( t_n \) and \( s_n \) are \( G_1 \)-Cauchy sequences with limits \( x \) in \( X \) and \( q \) in \( Y \). Using the inequality (2.1), we have
\[ h(\beta(s_n, \varphi \psi s_n)G_2(\varphi t_n, \varphi \psi s_n, \varphi \psi s_n), G_1(t_n, \psi s_n, \psi s_n), G_2(s_n, s_n, s_n), \varphi t_n), \\
G_1(t_n, t_n, t_{n-1}), H_2(t_n, t_{n-1}, \varphi \psi t_n, \varphi \psi t_n)) \leq 0, \\
h(\beta(s_n, s_n)G_2(\varphi t_n, s_n, s_n), G_1(t_n, t_n, t_{n-1}), G_2(s_n, s_n, s_n), \varphi t_n), G_2(s_n, s_n, s_n), G_1(t_n, t_n, t_{n-1})) \leq 0 \]
Taking \( n \) tend to \( \infty \), we have
\[ h(\beta(q, q)G_2(\varphi \xi, q, q), 0, G_2(q, q, \varphi \xi), 0, 0) \leq 0, \]
by the condition (iii), we get \( h(G_2(\varphi \xi, q, q), 0, G_2(q, q, \varphi \xi), 0, 0) \leq 0 \), hence \( q = \varphi \xi \). Using the inequality (2.2), we have

\[
\begin{align*}
g(\alpha(t_n, \psi \varphi t_n)G_1(\psi s_n, \psi s_n, \psi \varphi t_{n-1}), G_2(s_n, s_n, \varphi t_{n-1}), G_1(t_{n-1}, \psi s_n, \psi s_n), \\
g_1(t_{n-1}, t_{n-1}, \psi \varphi t_{n-1}), G_2(s_{n-1}, \varphi \psi s_{n-1}, \varphi \psi s_{n-1})) \leq 0,
\end{align*}
\]

and so by the condition (iii), we get \( q \) by the condition (iii), we get \( \psi \).

Thus \( \psi \varphi \xi = \psi q = \xi \), \( \varphi \psi q = \varphi \xi = q \), and so \( \psi \varphi \) has a fixed point \( \xi \), and \( \varphi \psi \) has a fixed point \( q \).

Suppose that \( \psi \varphi \) has a another fixed point \( \xi_1 \) and \( \varphi \psi \) has a another fixed point \( q_1 \). We apply (2.1), (ii) and using the property (iii), we get

\[
\begin{align*}
h(\beta(q_1, \varphi \psi q_1)G_2(\varphi \xi_1, \varphi \psi q_1, \varphi \psi q_1), G_1(\xi, \psi q_1, \psi q_1), G_2(q_1, q_1, \varphi \xi_1), \\
h(\beta(q_1, q_1)G_2(\varphi \psi q_1, \varphi \psi q_1, \varphi \psi q_1), G_1(\xi, q_1, q_1), G_2(q_1, q_1, \varphi \xi_1), \\
h(\psi, q_1, q_1), G_1(\psi, q_1, q_1), G_2(q_1, q_1, q_1), 0, 0) \leq 0,
\end{align*}
\]

which implies that

\[
G_2(q_1, q_1, q_1) \leq \frac{1}{45^3}G_1(\psi, q_1, q_1). \tag{2.8}
\]

We apply the inequality (2.2) and (iii), we obtain

\[
\begin{align*}
g(\alpha(\psi q_1, \psi \varphi q_1)G_1(\psi \psi q_1, \psi \psi q_1, \psi \varphi q_1), G_2(\varphi q, \varphi q, \varphi q), \\
g_1(\psi q_1, \psi \varphi q_1, \psi \varphi q_1), G_1(\psi q_1, \psi \varphi q_1, \psi \varphi q_1)) \leq 0,
\end{align*}
\]

and so \( q = q_1 \) since \( \frac{1}{45^3} < 1 \). Now \( \varphi \psi \xi_1 = \xi_1 \) implies \( \varphi \psi \varphi \xi_1 = \varphi \xi_1 \) and so \( \varphi \xi_1 = q \). Thus \( \xi = \psi \varphi \xi = q = \psi q \xi_1 = \xi_1 \).
Corollary 2.8. Let \((X, G_1)\) and \((Y, G_2)\) be complete \(G_b\)-metric spaces, and \(\varphi: X \to Y, \psi: Y \to X\) be a mappings. Suppose that there exists a functions \(\alpha: X \times X \to [0, \infty), \beta: Y \times Y \to [0, \infty),\) such that

\[
\beta(s_1, \varphi\psi s_1)G_2(\varphi t, \varphi\psi s_1, \varphi\psi s_2) \leq \frac{1}{4\delta^2} \max\{G_1(t, \psi s_1, \psi s_2), G_2(s_1, s_2, \varphi t)\},
\]

\[
G_2(s_1, \varphi\psi s_1, \varphi\psi s_2), \frac{1}{2\delta} G_1(t, t, \psi\varphi t))
\]

\[
\alpha(t, \psi\varphi t)G_1(\psi s_1, \psi s_2, \psi\varphi t) \leq \frac{1}{4\delta^2} \max G_2(s_1, s_2, \varphi t),
\]

\[
G_1(t, \psi s_1, \psi s_2), G_1(t, t, \psi\varphi t), \frac{1}{2\delta} G_2(s_1, \varphi\psi s_1, \varphi\psi s_2))
\]

for all \(t \in X, s_1, s_2 \in Y.\) Suppose also that:

(i) \(\varphi\psi\) is an \(\beta\)-admissible and \(\psi\varphi\) is an \(\alpha\)-admissible;
(ii) there exists \(x_0 \in X\) and \(y_0 \in Y\) such that \(\alpha(x_0, \varphi\psi(x_0)) \geq 1, \beta(y_0, \psi\varphi(y_0)) \geq 1;\)
(iii) \(\alpha(x, x) \geq 1, \beta(y, y) \geq 1, x \in X, y \in Y.\)

Then \(\psi\varphi\) and \(\varphi\psi\) have a unique fixed points \(\xi, q\) in \(X\) and \(q\) in \(Y\), respectively. Further \(\varphi\xi = q\) and \(\psi q = \xi.\)

We prove an analogous results for compact \(G_b\)-metric spaces.

Definition 2.9. Let \(\pi^*\) be the set of all upper semi-continuous function in each variable \(h(r_1, r_2, r_3, r_4, r_5): \mathbb{R}^5_+ \to \mathbb{R}\) satisfying:

\((\pi^*1)\) \(h\) is non-decreasing in variable \(r_1\) and non-increasing in variables \(r_3, r_4, r_5;\)

\((\pi^*2)\) if either \(h(\mu, \nu, 0, \mu, 2\nu) < 0\) or \(h(\mu, \nu, \nu, 0) < 0\) for all \(\mu, \nu \geq 0\), then \(\mu < \frac{1}{2\nu}.\)

Definition 2.10. Let \((X, G_1)\) and \((Y, G_2)\) be compact \(G_b\)-metric spaces, \(\varphi: X \to Y, \psi: Y \to X\) be two given mappings and \(\alpha: X \times X \to [0, \infty), \beta: Y \times Y \to [0, \infty).\) The pair \((\varphi, \psi)\) is said to be an \((\beta, \alpha)\)-implicit contractive pair of mappings whenever there exists \(h, g \in \pi^*\) such that

\[
h(\beta(s_1, \varphi\psi s_1)G_2(\varphi t, \varphi\psi s_1, \varphi\psi s_2), G_1(t, \psi s_1, \psi s_2), G_2(s_1, s_2, \varphi t), G_2(s_1, \varphi\psi s_1, \varphi\psi s_2), G_1(t, t, \psi\varphi t)) < 0, \tag{2.10}
\]

\[
g(\alpha(t, \psi\varphi t)G_1(\psi s_1, \psi s_2, \psi\varphi t), G_2(s_1, s_2, \varphi t), G_1(t, \psi s_1, \psi s_2), G_1(t, t, \psi\varphi t), G_2(s_1, \varphi\psi s_1, \varphi\psi s_2)) < 0 \tag{2.11}
\]

for all \(t \in X, s_1, s_2 \in Y.\)

Theorem 2.11. Let \((X, G_1)\) and \((Y, G_2)\) be compact \(G_b\)-metric spaces, and \(\varphi: X \to Y, \psi: Y \to X\) be an \((\beta, \alpha)\)-implicit contractive continuous mappings satisfying the conditions:

(i) \(\psi\varphi\) is an \(\beta\)-admissible and \(\psi\varphi\) is an \(\alpha\)-admissible;
(ii) there exists \(x_0 \in X\) and \(y_0 \in Y\) such that \(\alpha(x_0, \varphi\psi(x_0)) \geq 1, \beta(y_0, \psi\varphi(y_0)) \geq 1;\)
(iii) \(\alpha(x, x) \geq 1, \beta(y, y) \geq 1, x \in X, y \in Y.\)

Then \(\psi\varphi\) has a unique fixed point \(\xi_1\) in \(X\) and \(\varphi\psi\) has a unique fixed point \(q\) in \(Y.\) Further, \(\varphi\xi_1 = q\) and \(\psi q = \xi_1.\)

Proof. Let \(\omega: X \to \mathbb{R}^+\) defined by \(\omega(t) = G_1(t, \psi\varphi t, \psi\varphi t)\) is \(G_b\)-continuous on \(X.\) Since \(X\) is compact, there exists a point \(\xi\) in \(X\) such that

\[
\omega(\xi) = G_1(\xi, \psi\varphi\xi, \psi\varphi\xi) = \min\{G_1(t, \psi\varphi t, \psi\varphi t); t \in X\}.
\]

Suppose that \(\varphi\xi \not= \psi\varphi\xi.\) Then \(\xi \not= \psi\varphi\xi.\) Put \(s_1 = s_2 = \varphi\xi, t = \psi s = \psi\varphi\xi,\) in \((2.11),\) we get

\[
g(\alpha(\psi\varphi\xi, \psi\varphi\psi\varphi\xi)G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi)) < 0,
\]

\[
g(\alpha(\psi\varphi\xi, \psi\varphi\psi\varphi\xi)G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi)) < 0,
\]

\[
g(\alpha(\psi\varphi\xi, \psi\varphi\psi\varphi\xi)G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi)) < 0,
\]

\[
g(\alpha(\psi\varphi\xi, \psi\varphi\psi\varphi\xi)G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\psi\varphi\xi, \psi\varphi\psi\varphi\xi, \psi\varphi\psi\varphi\xi)) < 0.
\]
Using Proposition 1.4 and (2.12), we get
\[ g(\alpha(\psi \varphi \xi, \psi \varphi \psi \xi)G_1(\psi \varphi \xi, \psi \varphi \xi, \psi \varphi \psi \xi), G_2(\varphi \xi, \varphi \xi, \varphi \psi \xi), 0), \]
\[ G_1(\psi \varphi \xi, \psi \varphi \xi, \psi \varphi \psi \xi), 2\delta G_2(\varphi \xi, \varphi \xi, \psi \varphi \xi) < 0. \]

By using the condition (i) and (ii), we obtain \( \alpha(\xi, \psi \varphi \xi) \geq 1 \implies \alpha(\psi \varphi \xi, \psi \varphi \psi \xi) \geq 1 \), and using \((\pi^*1), (\pi^*2)\), we get
\[ G_1(\psi \varphi \xi, \psi \varphi \xi, \psi \varphi \psi \xi) < \frac{1}{2\delta} G_2(\varphi \xi, \varphi \xi, \psi \varphi \xi). \]
(2.12)

Using Proposition 1.4 and (2.12), we get
\[ G_2(\varphi \psi \varphi \xi, \psi \varphi \xi) \leq 2\delta G_2(\varphi \xi, \varphi \psi \xi, \varphi \psi \xi), G_1(\psi \varphi \xi, \psi \varphi \xi, \psi \varphi \psi \xi) < G_2(\varphi \xi, \varphi \psi \xi, \varphi \psi \xi). \]
(2.13)

Putting \( s_1 = s_2 = \varphi \xi \), \( t = \xi \) in (2.10), we have
\[
\begin{align*}
&h(\beta(\psi \xi, \psi \varphi \xi)G_2(\varphi \xi, \psi \varphi \xi, \psi \varphi \xi), G_1(\xi, \psi \varphi \xi, \psi \varphi \xi), G_2(\varphi \xi, \varphi \xi, \xi), \\
&G_2(\varphi \xi, \psi \varphi \xi, \xi, \varphi \psi \xi), G_1(\xi, \xi, \varphi \psi \xi) < 0, \\
&h(\beta(\psi \xi, \psi \varphi \xi)G_2(\varphi \xi, \psi \varphi \xi, \psi \varphi \xi), G_1(\xi, \psi \varphi \xi, \psi \varphi \xi), 0, G_2(\varphi \xi, \xi, \varphi \psi \xi, \xi, \varphi \xi), \\
&2\delta G_1(\xi, \varphi \xi, \psi \varphi \xi) < 0.
\end{align*}
\]

Applying conditions (i) and (ii), put \( x_0 = \varphi \xi, \beta(\varphi \xi, \varphi \psi \xi) \geq 1 \), and using \((\pi^*1), (\pi^*2)\), we obtain
\[ G_2(\varphi \xi, \varphi \psi \xi, \psi \varphi \xi) < \frac{1}{2\delta} G_1(\xi, \psi \varphi \xi, \psi \varphi \xi). \]
(2.14)

From (2.13) and (2.14), we obtain
\[ G_1(\psi \varphi \xi, \psi \varphi \xi, \psi \varphi \psi \xi) < \frac{1}{2\delta} G_1(\xi, \varphi \xi, \psi \varphi \xi), \]
\[ \frac{1}{2\delta} G_1(\psi \varphi \xi, \psi \varphi \psi \xi, \psi \varphi \psi \xi) \leq G_1(\psi \varphi \psi \xi, \psi \varphi \xi, \psi \xi), \]
\[ \frac{1}{2\delta} G_1(\psi \varphi \xi, \psi \varphi \psi \xi, \psi \varphi \psi \xi) < \frac{1}{2\delta} G_1(\xi, \varphi \xi, \varphi \xi). \]

Hence \( \psi \varphi \xi < \omega(\xi) \), and we have a contradiction. So \( \varphi \psi \xi = \varphi \xi \). If \( \varphi \xi = q \) and \( \psi q = \xi_1 \), then we have \( \varphi \psi \xi = \psi \xi = \psi q = \xi_1 \), and \( q = \varphi q = \varphi \psi \xi = \varphi \psi \xi = \varphi \xi_1 \). Then \( \psi q = \xi_1 \) is a fixed point of \( \psi \) and \( \varphi \xi_1 = q \) is a fixed point of \( \varphi \psi \xi \).

Suppose that \( \psi \) has a another fixed point \( \xi_2 \). Then applying (2.11), we have,
\[ g(\alpha(\xi_2, \varphi \xi_2)G_1(\psi \varphi \xi_1, \varphi \xi_1, \varphi \xi_2), G_2(\varphi \xi_1, \varphi \xi_1, \varphi \xi_2), G_1(\xi_2, \xi_1, \xi_1), \\
G_1(\xi_2, \varphi \xi_2, \psi \xi_2), G_2(\varphi \xi_2, \varphi \psi \xi_2, \varphi \psi \xi_2)) < 0, \]
\[ g(\alpha(\xi_2, \varphi \xi_2)G_2(\xi_1, \xi_1, \xi_2), G_2(\varphi \xi_1, \varphi \xi_1, \varphi \xi_2), G_1(\xi_2, \xi_1, \xi_1), \\
G_1(\xi_2, \varphi \xi_2, \psi \xi_2), G_2(\varphi \xi_2, \varphi \xi_1, \varphi \xi_1)) < 0, \]
\[ g(\alpha(\xi_2, \xi_2)G_1(\xi_1, \xi_1, \xi_2), G_2(\varphi \xi_1, \varphi \xi_1, \varphi \xi_2), G_1(\xi_2, \xi_1, \xi_1), 0, 2\delta G_2(\varphi \xi_1, \varphi \xi_2, \varphi \xi_2)) < 0. \]

Applying condition (iii) and using \((\pi^*1), (\pi^*2)\), we obtain
\[ G_1(\xi_1, \xi_1, \xi_2) < \frac{1}{2\delta} G_2(\varphi \xi_1, \varphi \xi_1, \varphi \xi_2). \]
(2.15)

Using (2.10) we have,
\[
\begin{align*}
&h(\beta(\varphi \xi_2, \varphi \psi \xi_2)G_2(\varphi \xi_1, \varphi \psi \xi_2, \varphi \psi \xi_2), G_1(\xi_1, \varphi \xi_2, \varphi \psi \xi_2), \\
&G_2(\varphi \xi_2, \varphi \xi_2, \varphi \xi_1), G_2(\varphi \xi_2, \varphi \psi \xi_2, \varphi \psi \xi_2), G_1(\xi_2, \varphi \xi_2, \varphi \psi \xi_2)) < 0, \\
&h(\beta(\varphi \xi_2, \varphi \psi \xi_2)G_2(\varphi \xi_1, \varphi \xi_2, \varphi \xi_2), G_1(\xi_1, \xi_2, \xi_2), G_2(\varphi \xi_2, \varphi \xi_2, \varphi \xi_1), 0, 0) < 0,
\end{align*}
\]
and it follows that
\[ G_2(\varphi \xi_1, \varphi \xi_2, \varphi \xi_2) < \frac{1}{2\delta} G_1(\xi_1, \xi_2, \xi_2), \]
\[ \frac{1}{2\delta} G_2(\varphi \xi_2, \varphi \xi_1, \varphi \xi_1) \leq G_2(\varphi \xi_1, \varphi \xi_2, \varphi \xi_2) < \frac{1}{2\delta} G_1(\xi_1, \xi_2, \xi_2). \]  \tag{2.16}
From (2.15) and (2.16), it follows that
\[ G_1(\xi_1, \xi_1, \xi_2) < \frac{1}{2\delta} G_1(\xi_1, \xi_2, \xi_2) \leq \frac{1}{2\delta} 2\delta G_1(\xi_1, \xi_1, \xi_2), \]
which gives a contradiction and so the fixed point \( \xi_1 \) must be unique. Similarly, \( q \) is the unique fixed point of \( \varphi \psi \).

**Corollary 2.12.** Let \((X, G_1)\) and \((Y, G_2)\) be compact \(G_\delta\)-metric spaces, and \( \varphi : X \to Y, \psi : Y \to X \) be a continuous mappings. Suppose that there exists a functions \( \alpha : X \times X \to [0, \infty), \beta : Y \times Y \to [0, \infty), \) such that
\[ \beta(s_1, \varphi \psi s_1) G_2(\varphi t, \varphi \psi s_1, \varphi \psi s_2) < \frac{1}{2\delta} \max(G_1(t, \psi s_1, \psi s_2), G_2(s_1, s_2, \varphi t), G_2(s_1, \varphi \psi s_1, \varphi \psi s_2), \frac{1}{2\delta} G_1(t, t, \psi \varphi t)), \]
\[ \alpha(t, \psi \varphi t) G_1(\psi s_1, \psi s_2, \psi \varphi t) < \frac{1}{2\delta} \max G_2(s_1, s_2, \varphi t), G_1(t, \psi s_1, \psi s_2), G_1(t, t, \psi \varphi t), \frac{1}{2\delta} G_2(s_1, \varphi \psi s_1, \varphi \psi s_2)) \]
for all \( t \in X, s_1, s_2 \in Y \). Suppose also that:

(i) \( \varphi \psi \) is an \( \beta \)-admissible and \( \psi \varphi \) is an \( \alpha \)-admissible;
(ii) there exists \( x_0 \in X \) and \( y_0 \in Y \) such that \( \alpha(\psi \varphi(x_0)) \geq 1, \beta(y_0, \psi \varphi(y_0)) \geq 1; \)
(iii) \( \alpha(x, x) \geq 1, \beta(y, y) \geq 1, x \in X, y \in Y. \)

Then \( \psi \varphi \) has a unique fixed point \( \xi \) in \( X \) and \( \varphi \psi \) has a unique fixed point \( q \) in \( Y \), furthermore \( \varphi \xi = q \) and \( \psi q = \xi \).

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**References**


