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Fixed point results for (β, α)-implicit contractions in two generalized b-metric spaces

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Abstract

The aim of this paper is to introduce (β,α)-implicit contractive of two mappings on two generalized b-metric spaces and derive some new fixed point theorems for (β,α)-implicit contractive in two complete and compact generalized b-Metric spaces.

Keywords: Fixed points, (β,α)-implicit contractions, generalized b-metric spaces.

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1. Introduction

Mustafa and Sims [11] introduced the concept of G-metric spaces as a generalization of a metric space, since then, several interesting results for existence of fixed point in G-metric spaces have been obtained (see [2–4, 7, 10, 11]).

In the other hand, Papa et al. are proved some fixed point theorems for functions satisfying an implicit relations in metric spaces and G-metric spaces (see [9, 13–18]).

The notion of b-metric space was introduced by Czerwik [8], and many authors studied fixed point theorems in b-metic spaces (see[1, 6, 9, 12, 19]). Recently, Aghajani et al. [5] generalized the concept of b-metic spaces by using the notions of b -metric spaces and G-metric spaces, which is called G_b-metric spaces and they discussed some properties and common fixed point results of G_b-metric.

The aim of this paper is to introduce (β,α)-implicit contractive of two mappings on two generalized b-metric spaces and derive some new fixed point theorems for (β,α)-implicit contractive in two complete and compact generalized b-Metric spaces.

We recall some basic definitions and results, for details on the following notions (see [5]). Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}_+$ denotes the set of nonnegative reals and N denotes the set of natural numbers.

Definition 1.1. Let X be a non empty set and $\delta \geqslant 1$ be a given real number. A mapping $G : X^3 \to \mathbb{R}^+$ is said to be a generalized b-metric (G_b-metric on X) if for all x, y, z, $a \in X$, the following axioms are satisfied:

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(G_b1) $G(x,y,z) = 0$ if $x = y = z$;
(G_b2) $0 < G(x,x,y)$, with $x \neq y$;
(G_b3) $G(x,x,y) \leqslant G(x,y,z)$, with $z \neq y$;
(G_b4) $G(x,y,z) = G(p(x,z,y))$ where p is a permutation of x,y,z (symmetry);
(G_b5) $G(x,y,z) \leqslant \delta(G(x,a,a) + G(a,y,z))$, (rectangle inequality).

A pair (X,G) is called a generalized b-metric.

Example 1.2. Let (X,G) be a G-metric space. Consider $G_b(x,y,z) = (G(x,y,z))^p$, where $p > 1$, is a real number. Then G_b is a G_b-metric space with $\delta = 2^{p-1}$.

Each G-metric space is a G_b-metric space with $\delta = 1$. The following example shows that a G_b-metric space on X does not need to be a G-metric space on X.

Example 1.3. Let $X = \mathbb{R}$ and $G_b = \frac{1}{9}(\mid x-y\mid + \mid y-z\mid + \mid z-x\mid)^2$. Then G_b is a G_b-metric space on $\mathbb{R}$ but not a G-metric space on $\mathbb{R}$. Indeed, let $x = 3, y = 5, z = 7$ and $a = \frac{7}{2}$, we have $G_b(3,5,7) = \frac{64}{9}$, $G_b(3,\frac{7}{2},\frac{7}{2}) = \frac{1}{9}$ and $G_b(\frac{7}{2},5,7) = \frac{49}{9}$ so $G_b(3,5,7) > G_b(3,\frac{7}{2},\frac{7}{2}) + G_b(\frac{7}{2},5,7)$.

We present some definitions and propositions in G_b-metric space.

Proposition 1.4. *Let* (X,G)*be a* G_b*-metric space. Then for any* x,y,z *and* $a \in X$, *it follows that:*

(1) *if* $G(x,y,z) = 0$ *then* $x = y = z$;
(2) $G(x,y,z) \leqslant \delta(G(x,x,y) + G(x,x,z))$;
(3) $G(x,y,y) \leqslant 2\delta G(y,x,x)$;
(4) $G(x,y,z) \leqslant \delta(G(x,a,z) + G(a,y,z))$.

Definition 1.5. Let (X,G) be a G_b-metric space, and (x_n) be a sequence of points of X, we say that (x_n) is G_b-convergent to x if for any $\varepsilon > 0$, there exists $n_0 \in N$ such that $G(x,x_n,x_m) < \varepsilon$, for all $n,m \geqslant n_0$.

Proposition 1.6. *Let* (X,G) *be a* G_b*-metric space. Then the following are equivalent:*

(1) (x_n) *is G-convergent to* x;
(2) $G(x_n,x_n,x) \longrightarrow 0$, *as* $n \longrightarrow \infty$;
(3) $G(x_n,x,x) \longrightarrow 0$, *as* $n \longrightarrow \infty$.

Definition 1.7. Let (X,G) be a G_b-metric space, a sequence (x_n) is called G_b-Cauchy if given $\varepsilon > 0$, there is $n_o \in N$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geqslant n_o$.

Proposition 1.8. *Let* X *be a* G_b*-metric space, then the following are equivalent:*

1. *the sequence* $\{x_n\}$ *is* G_b*-Cauchy;*
2. *for any* $\varepsilon > 0$, *there exists* $n_0 \in N$ *such that* $G(x_n,x_m,x_m) < \varepsilon$, *for all* $m,n \geqslant n_0$.

Definition 1.9. Let (X,G) and (X^*,G^*) be G_b-metric space and let $h : (X,G) \longrightarrow (X^*,G^*)$ be a function, then h is said to be G_b-continuous at a point $a \in X$, if given $\varepsilon > 0$, there exists $\delta > 0$ such that $x,y \in X$, $G(a,x,y) < \delta$ implies $G^*(h(a), h(x), h(y)) < \varepsilon$.

Definition 1.10. A G_b-metric space (X,G) is said to be G_b-complete if every G_b-Cauchy sequence in (X,G) is G_b-convergent in (X,G).

Definition 1.11. A G_b-metric space (X,G) is said to be a compact G_b-metric space if it is G_b-complete and G_b-totally bounded.

2. Main results

Now, in this paper, we introduce (β,α)-implicit contractive and prove our first new results in two complete G_b-metric spaces.

Definition 2.1. Let π be the set of all upper semi-continuous function in each variable $h(r_1, r_2, r_3, r_4, r_5) : \mathbb{R}^5_+ \to \mathbb{R}$ satisfying:

(π1) h is non-decreasing in variable r_1 and non-increasing in variables r_3, r_4, r_5;
(π2) if either $h(\mu, \nu, 0, \mu, 2\delta\nu) \leqslant 0$ or $h(\mu, \nu, \mu, 0, 0) \leqslant 0$ for all $\mu, \nu \geqslant 0$, then $\mu \leqslant \frac{1}{4\delta^3}\nu$.

Example 2.2. $h(r_1, r_2, r_3, r_4, r_5) = r_1 - \frac{1}{4\delta^3} \max(r_2, r_3, r_4, \frac{1}{2\delta} r_5)$.

Example 2.3. $h(r_1, r_2, r_3, r_4, r_5) = r_1 - \frac{1}{2\delta^3} r_2 + \frac{1}{2\delta^3} \max(2\delta^3 r_3, 2\delta^3 r_4, \frac{1}{2\delta} r_5)$.

Example 2.4. $h(r_1, r_2, r_3, r_4, r_5) = r_1 - \frac{1}{4\delta^3} r_2 + r_3 + r_4 - \frac{1}{8\delta^4} r_5$.

Definition 2.5. Let $\varphi : X \longrightarrow Y$, $\psi : Y \longrightarrow X$ be a mappings and α: $X \times X \to [0, \infty)$, $\beta : Y \times Y \to [0, \infty)$.

We say that $\psi\varphi : X \to X$ is an α-admissible if $x_1, x_2 \in X$, $\alpha(x_1, x_2) \geqslant 1 \Longrightarrow \alpha(\psi\varphi(x_1), \psi\varphi(x_2)) \geqslant 1$ and $\varphi\psi : Y \to Y$ is an β-admissible if $y_1, y_2 \in Y$, $\beta(y_1, y_2) \geqslant 1 \Longrightarrow \beta(\varphi\psi(y_1), \varphi\psi(y_2)) \geqslant 1$.

Definition 2.6. Let (X, G_1) and (Y, G_2) be complete G_b-metric spaces, φ: $X \longrightarrow Y$, ψ :$Y \longrightarrow X$ be two given mappings and α: $X \times X \to [0, \infty)$, $\beta : Y \times Y \to [0, \infty)$. The pair (φ, ψ) is said to be an (β, α)-implicit contractive pair of mappings whenever there exists $h, g \in \pi$ such that

$$h(\beta(s_1,\varphi\psi s_1)G_2(\varphi t,\varphi\psi s_1,\varphi\psi s_2),G_1(t,\psi s_1,\psi s_2), G_2(s_1, s_2,\varphi t), G_2(s_1,\varphi\psi s_1,\varphi\psi s_2), G_1(t, t,\psi\varphi t)) \leqslant 0, \tag{2.1}$$

$$g(\alpha(t,\psi\varphi t)G_1(\psi s_1,\psi s_2,\psi\varphi t),G_2(s_1, s_2,\varphi t), G_1(t,\psi s_1,\psi s_2), G_1(t, t,\psi\varphi t), G_2(s_1,\varphi\psi s_1,\varphi\psi s_2)) \leqslant 0, \tag{2.2}$$

for all $t \in X$, $s_1, s_2 \in Y$.

Theorem 2.7. *Let (X, G_1) and (Y, G_2) be complete G_b- metric spaces, and φ: $X \longrightarrow Y$, ψ :$Y \longrightarrow X$ be an (β, α)-implicit contractive mappings. Suppose that:*

(i) *$\varphi\psi$ is an β-admissible and $\psi\varphi$ is an α-admissible;*
(ii) *there exists $x_0 \in X$ and $y_0 \in Y$ such that $\alpha(x_0, \psi\varphi(x_0)) \geqslant 1, \beta(y_0, \varphi\psi(y_0)) \geqslant 1$;*
(iii) *$\alpha(x, x)) \geqslant 1$, $\beta(y, y) \geqslant 1, x \in X, y \in Y$.*

Then $\psi\varphi$, $\varphi\psi$ have a unique fixed points ξ in X and q in Y, respectively. Further $\varphi\xi = q$ and $\psi q = \xi$.

Proof. Define two sequences (t_n) in X, and (s_n) in Y, by $t_n = (\psi\varphi)^n t$, $s_n = \varphi(\psi\varphi)^{n-1} t$, for $n = 1, 2, \ldots, t \in X$. Suppose that $t_n \neq t_{n+1}$ and $s_n \neq s_{n+1}$, for all n. Applying (2.1), we have

$$\begin{aligned}
&h(\beta(s_n, \varphi\psi s_n)G_2(\varphi t_{n-1}, \varphi\psi s_n, \varphi\psi s_n), G_1(t_{n-1}, \psi s_n, \psi s_n), G_2(s_n, s_n, \varphi t_{n-1}),\\
&\quad G_2(s_n, \varphi\psi s_n, \varphi\psi s_n), G_1(t_{n-1}, t_{n-1}, t_n)) \leqslant 0,\\
&h(\beta(s_n, s_{n+1})G_2(s_n, s_{n+1}, s_{n+1}), G_1(t_{n-1}, t_n, t_n), 0, G_2(s_n, s_{n+1}, s_{n+1}),\\
&\quad G_1(t_n, t_{n-1}, t_{n-1})) \leqslant 0.
\end{aligned}$$

By using the conditions (i) and (ii), we deduce that $\beta(s_0, s_1) = \beta(s_0, \varphi\psi(s_0)) \geqslant 1 \Longrightarrow \beta(\varphi\psi(s_0), \varphi\psi(s_1)) = \beta(s_1, s_2) \geqslant 1$. By iterating the process, we get $\beta(s_n,s_{n+1}) \geqslant 1$, and $\alpha(t_0, t_1) = \alpha(t_0, \psi\varphi(t_0)) \geqslant 1 \Longrightarrow \alpha(\psi\varphi(t_0), \psi\varphi(t_1)) = \alpha(t_1, t_2) \geqslant 1$. By iterating the process, we get $\alpha(t_n,t_{n+1}) \geqslant 1$.

Applying (π1), (π2) we have

$$\begin{aligned}
&h(G_2(s_n, s_{n+1}, s_{n+1}), G_1(t_{n-1}, t_n, t_n), 0, G_2(s_n, s_{n+1}, s_{n+1}), 2\delta G_1(t_{n-1}, t_n, t_n)) \leqslant 0,\\
&G_2(s_n, s_{n+1}, s_{n+1}) \leqslant \frac{1}{4\delta^3} G_1(t_{n-1}, t_n, t_n).
\end{aligned} \tag{2.3}$$

Similarly, applying the inequality (2.2),

$$\begin{aligned}
&g(\alpha(t_n, \psi\varphi t_n)G_1(\psi s_n, \psi s_n, \psi\varphi t_n), G_2(s_n, s_n, \varphi t_n), G_1(t_n, \psi s_n, \psi s_n),\\
&G_1(t_n, t_n, t_{n+1}), G_2(s_n, \varphi\psi s_n, \varphi\psi s_n)) \leqslant 0,\\
&g(\alpha(t_n, t_{n+1})G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_n, s_{n+1}), G_1(t_n, t_n, t_n), G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_{n+1}, s_{n+1})) \leqslant 0,\\
&g(\alpha(t_n, t_{n+1})G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_n, s_{n+1}), G_1(t_n, t_n, t_n), G_1(t_n, t_n, t_{n+1}), 2\delta G_2(s_n, s_n, s_{n+1})) \leqslant 0
\end{aligned}$$

and using the property $(\pi 1), (\pi 2)$, we have

$$\begin{aligned}
&g(G_1(t_n, t_n, t_{n+1}), G_2(s_n, s_n, s_{n+1}), 0, G_1(t_n, t_n, t_{n+1}), 2\delta G_2(s_n, s_n, s_{n+1})) \leqslant 0,\\
&G_1(t_n, t_n, t_{n+1}) \leqslant \frac{1}{4\delta^3} G_2(s_n, s_n, s_{n+1}).
\end{aligned} \tag{2.4}$$

Using Proposition 1.4, we get,

$$\frac{1}{2\delta} G_1(t_n, t_{n+1}, t_{n+1}) \leqslant G_1(t_{n+1}, t_n, t_n) = G_1(t_n, t_n, t_{n+1}), \tag{2.5}$$

and

$$G_2(s_n, s_n, s_{n+1}) = G_2(s_{n+1}, s_n, s_n) \leqslant 2\delta G_2(s_n, s_{n+1}, s_{n+1}). \tag{2.6}$$

From (2.5), (2.6), and (2.4), we obtain

$$G_1(t_n, t_{n+1}, t_{n+1}) \leqslant \frac{1}{\delta} G_2(s_n, s_{n+1}, s_{n+1}). \tag{2.7}$$

Now it follows from the inequalities (2.3) and (2.7) that

$$G_1(t_n, t_{n+1}, t_{n+1}) \leqslant \frac{1}{4\delta^4} G_1(t_{n-1}, t_n, t_n).$$

Hence, by induction we get

$$G_1(t_n, t_{n+1}, t_{n+1}) \leqslant (\frac{1}{4\delta^4})^n G_1(t, t_1, t_1), n = 1, 2, \ldots.$$

Hence, $\lim_{n \longrightarrow \infty} G_1(t_n, t_{n+1}, t_{n+1}) = 0$. Put $\kappa = \frac{1}{4\delta^4}$, for any $p \in \mathbb{N}^+$, we get

$$\begin{aligned}
G_1(t_n, t_{n+p}, t_{n+p}) &\leqslant \delta G_1(t_n, t_{n+1}, t_{n+1}) + \delta^2 G_1(t_{n+1}, t_{n+2}, t_{n+2}) + \cdots + \delta^{p-1} G_1(t_{n+p-1}, t_{n+p}, t_{n+p})\\
&\leqslant \delta\kappa^n G_1(t, t_1, t_1) + \delta^2\kappa^{n+1} G_1(t, t_1, t_1) + \cdots + \delta^p \kappa^{n+p-1} G_1(t, t_1, t_1)\\
&= \delta\kappa^n (1 + \delta\kappa + \cdots + \delta^{p-1}\kappa^{p-1}) G_1(t, t_1, t_1)\\
&= \delta\kappa^n (\frac{1 - (\delta\kappa)^p}{1 - \delta\kappa}) G_1(t, t_1, t_1).
\end{aligned}$$

Since $\delta\kappa^n < 1$, we have $\lim_{n \longrightarrow \infty} G_1(t_n, t_{n+p}, t_{n+p}) = 0$, this due to $\lim_{n \longrightarrow \infty} G_1(t_n, t_m, t_m) = 0$, follows that t_n and s_n are G_b-Cauchy sequences with limits ξ in X and q in Y. Using the inequality (2.1), we have

$$\begin{aligned}
&h(\beta(s_n, \varphi\psi s_n)G_2(\varphi t_n, \varphi\psi s_{n-1}, \varphi\psi s_{n-1}), G_1(t_n, \psi s_{n-1}, \psi s_{n-1}), G_2(s_{n-1}, s_{n-1}, \varphi t_n),\\
&\quad G_2(s_{n-1}, \varphi\psi s_{n-1}, \varphi\psi s_{n-1}), G_1(t_{n-1}, t_{n-1}, \psi\varphi t_{n-1})) \leqslant 0,\\
&h(\beta(s_n, s_{n+1})G_2(\varphi t_n, s_n, s_n), G_1(t_n, t_{n-1}, t_{n-1}), G_2(s_{n-1}, s_{n-1}, \varphi t_n), G_2(s_{n-1}, s_n, s_n), G_1(t_{n-1}, t_{n-1}, t_n)) \leqslant 0
\end{aligned}$$

Taking n tend to ∞, we have

$$h(\beta(q, q)G_2(\varphi\xi, q, q), 0, G_2(q, q, \varphi\xi), 0, 0) \leqslant 0,$$

by the condition(iii), we get $h(G_2(\varphi\xi, q, q), 0, G_2(q, q, \varphi\xi), 0, 0) \leqslant 0$, hence $q = \varphi\xi$. Using the inequality (2.2), we have

$$\begin{aligned}
&g(\alpha(t_n, \psi\varphi t_n)G_1(\psi s_n, \psi s_n, \psi\varphi t_{n-1}), G_2(s_n, s_n, \varphi t_{n-1}), G_1(t_{n-1}, \psi s_n, \psi s_n),\\
&\quad G_1(t_{n-1}, t_{n-1}, \psi\varphi t_{n-1}), G_2(s_{n-1}, \varphi\psi s_{n-1}, \varphi\psi s_{n-1})) \leqslant 0,\\
&g(\alpha(t_n, t_{n+1})G_1(\psi s_n, \psi s_n, t_n), G_2(s_n, s_n, \varphi t_{n-1}), G_1(t_{n-1}, \psi s_n, \psi s_n),\\
&\quad G_1(t_{n-1}, t_{n-1}, t_n), G_2(s_{n-1}, s_n, s_n)) \leqslant 0.
\end{aligned}$$

Taking n tend to ∞, we obtain

$$g(G_1(\psi q, \psi q, \xi), 0, G_1(\xi, \psi q, \psi q), 0, 0) \leqslant 0,$$

we get $\xi = \psi q$. Thus $\psi\varphi\xi = \psi q = \xi$, $\varphi\psi q = \varphi\xi = q$, and so $\psi\varphi$ has a fixed point ξ and $\varphi\psi$ has a fixed point q.

Suppose that $\psi\varphi$ has a another fixed point ξ_1 and $\varphi\psi$ has a another fixed point q_1. We apply (2.1), (iii) and using the property (π2), we get

$$\begin{aligned}
&h(\beta(q_1, \varphi\psi q_1)G_2(\varphi\xi, \varphi\psi q_1, \varphi\psi q_1), G_1(\xi, \psi q_1, \psi q_1), G_2(q_1, q_1, \varphi\xi),\\
&\quad G_2(q_1, \varphi\psi q_1, \varphi\psi q_1), G_1(\psi q, \psi q, \psi\varphi\psi q)) \leqslant 0,\\
&h(\beta(q_1, q_1)G_2(\varphi\psi q, \varphi\psi q_1, \varphi\psi q_1), G_1(\xi, \psi q_1, \psi q_1), G_2(q_1, q_1, \varphi\xi),\\
&\quad G_2(q_1, \varphi\psi q_1, \varphi\psi q_1), G_1(\psi q, \psi q, \psi\varphi\psi q)) \leqslant 0,\\
&h(G_2(q, q_1, q_1), G_1(\psi q, \psi q_1, \psi q_1), G_2(q_1, q_1, q), 0, 0) \leqslant 0,
\end{aligned}$$

from which it easy to see that

$$G_2(q, q_1, q_1) \leqslant \frac{1}{4\delta^3} G_1(\psi q, \psi q_1, \psi q_1). \tag{2.8}$$

We apply the inequality (2.2) and (iii), we obtain

$$\begin{aligned}
&g(\alpha(\psi q_1, \psi\varphi\psi q_1)G_1(\psi\varphi\psi q, \psi\varphi\psi q, \psi\varphi\psi q_1), G_2(\varphi\psi q, \varphi\psi q, \varphi\psi q_1),\\
&\quad G_1(\psi q_1, \psi\varphi\psi q, \psi\varphi\psi q), G_1(\psi q, \psi q, \psi\varphi\psi q), G_2(q_1, \varphi\psi q_1, \varphi\psi q_1)) \leqslant 0,\\
&g(\alpha(\psi q_1, \psi q_1)G_1(\psi q, \psi q, \psi q_1), G_2(q, q, q_1), G_1(\psi q_1, \psi q, \psi q), 0, 0) \leqslant 0,\\
&g(G_1(\psi q, \psi q, \psi q_1), G_2(q, q, q_1), G_1(\psi q_1, \psi q, \psi q), 0, 0) \leqslant 0,
\end{aligned}$$

which implies that

$$G_1(\psi q, \psi q, \psi q_1) \leqslant \frac{1}{4\delta^3} G_2(q, q, q_1).$$

Again by using the proposition1.4, we get,

$$\begin{aligned}
&\frac{1}{2\delta} G_1(\psi q, \psi q_1, \psi q_1) \leqslant G_1(\psi q, \psi q, \psi q_1) \leqslant \frac{1}{4\delta^3} G_2(q, q, q_1) \leqslant \frac{1}{2\delta^3} G_2(q, q_1, q_1),\\
&G_1(\psi q, \psi q_1, \psi q_1) \leqslant \frac{1}{\delta^2} G_2(q, q_1, q_1).
\end{aligned} \tag{2.9}$$

Now it follows from the inequalities (2.8) and (2.9) that

$$G_2(q, q_1, q_1) \leqslant \frac{1}{4\delta^3} G_1(\psi q, \psi q_1, \psi q_1) < \frac{1}{4\delta^5} G_2(q, q_1, q_1) < G_2(q, q_1, q_1)$$

and so $q = q_1$ since $\frac{1}{4\delta^5} < 1$. Now $\varphi\psi\xi_1 = \xi_1$ implies $\varphi\psi\varphi\xi_1 = \varphi\xi_1$ and so $\varphi\xi_1 = q$. Thus $\xi = \psi\varphi\xi = \psi q = \psi\varphi\xi_1 = \xi_1$. □

Corollary 2.8. *Let* (X, G_1) *and* (Y, G_2) *be complete* G_b*-metric spaces, and* φ*:* $X \longrightarrow Y$, ψ *:Y* $\longrightarrow$*X be a mappings. Suppose that there exists a functions* α*:* $X \times X \to [0, \infty)$, $\beta : Y \times Y \to [0, \infty)$, *such that*

$$\begin{aligned}\beta(s_1, \varphi\psi s_1)G_2(\varphi t, \varphi\psi s_1, \varphi\psi s_2) &\leqslant \frac{1}{4\delta^3}\max(G_1(t, \psi s_1, \psi s_2), G_2(s_1, s_2, \varphi t\ ),\\ &\qquad G_2(s_1, \varphi\psi s_1, \varphi\psi s_2), \frac{1}{2\delta}G_1(t, t, \psi\varphi t))\\ \alpha(t, \psi\varphi t)G_1(\psi s_1, \psi s_2, \psi\varphi t\ ) &\leqslant \frac{1}{4\delta^3}\max G_2(s_1, s_2, \varphi t),\\ &\qquad G_1(t, \psi s_1, \psi s_2), G_1(t, t, \psi\varphi t), \frac{1}{2\delta}G_2(s_1, \varphi\psi s_1, \varphi\psi s_2))\end{aligned}$$

for all $t \in X$, $s_1, s_2 \in Y$. *Suppose also that:*

(i) $\varphi\psi$ *is an* β*-admissible and* $\psi\varphi$ *is an* α*-admissible;*
(ii) *there exists* $x_0 \in X$ *and* $y_0 \in Y$ *such that* $\alpha(x_0, \psi\varphi(x_0)) \geqslant 1, \beta(y_0, \varphi\psi(y_0)) \geqslant 1$*;*
(iii) $\alpha(x, x)) \geqslant 1$, $\beta(y, y) \geqslant 1, x \in X, y \in Y$.

Then $\psi\varphi$ *and* $\varphi\psi$ *have a unique fixed points* ξ *in X and* q *in* Y*, respectively. Further* $\varphi\xi = q$ *and* $\psi q = \xi$*.*

We prove an analogous results for compact G_b-metric spaces.

Definition 2.9. Let π^* be the set of all upper semi-continuous function in each variable $h(r_1, r_2, r_3, r_4, r_5) : \mathbb{R}^5_+ \to \mathbb{R}$ satisfying:

(π^*1) h is non-decreasing in variable r_1 and non-increasing in variables r_3,r_4, r_5;
(π^*2) if either $h(\mu, \nu, 0, \mu, 2\delta\nu) < 0$ or $h(\mu, \nu, \mu, 0, 0) < 0$ for all $\mu, \nu \geqslant 0$, then $\mu < \frac{1}{2\delta}\nu$.

Definition 2.10. Let (X, G_1) and (Y, G_2) be compact G_b-metric spaces, φ: $X \longrightarrow Y$, ψ :Y $\longrightarrow$X be two given mappings and α: $X \times X \to [0, \infty)$, $\beta : Y \times Y \to [0, \infty)$. The pair (φ, ψ) is said to be an (β, α)-implicit

contractive pair of mappings whenever there exists $h, g \in \pi^*$ such that

$$h(\beta(s_1,\varphi\psi s_1)G_2(\ \varphi t,\varphi\psi s_1,\varphi\psi s_2),G_1(t,\ \psi s_1,\ \psi s_2),G_2(s_1,\ s_2,\varphi t),G_2(s_1,\varphi\psi s_1,\varphi\psi s_2),G_1(t,t,\ \psi\varphi t)) < 0\ , \tag{2.10}$$

$$g(\alpha(t,\ \psi\varphi t)G_1(\ \psi s_1,\psi s_2,\ \psi\varphi t),G_2(s_1,\ s_2,\varphi t),G_1(t,\ \psi s_1,\ \psi\ s_2),G_1(t,t,\ \psi\varphi t),G_2(s_1,\varphi\psi s_1,\varphi\psi s_2)) < 0 \tag{2.11}$$

for all $t \in X$, $s_1, s_2 \in Y$.

Theorem 2.11. *Let* (X, G_1) *and* (Y, G_2) *be compact* G_b*-metric spaces, and* φ*:* $X \longrightarrow Y$, ψ *:Y* $\longrightarrow$*X be an* (β, α)*-implicit contractive continuous mappings satisfying the conditions:*

(i) $\varphi\psi$ *is an* β*-admissible and* $\psi\varphi$*is an* α*-admissible;*
(ii) *there exists* $x_0 \in X$ *and* $y_0 \in Y$ *such that* $\alpha(x_0, \psi\varphi(x_0)) \geqslant 1, \beta(y_0, \varphi\psi(y_0)) \geqslant 1$*;*
(iii) $\alpha(x, x) \geqslant 1$, $\beta(y, y) \geqslant 1, x \in X, y \in Y$.

Then $\psi\varphi$ *has a unique fixed point* ξ_1 *in* X *and* $\varphi\psi$ *has a unique fixed point q in* Y*. Further,* $\varphi\xi_1 = q$ *and* $\psi q = \xi_1$*.*

Proof. Let $\omega : X \to \mathbb{R}^+$ defined by $\omega(t) = G_1(t, \psi\varphi t, \psi\varphi t)$ is G_b-continuous on X. Since X is compact, there exists a point ξ in X such that

$$\omega(\xi) = G_1(\xi, \psi\varphi\xi, \psi\varphi\xi) = \min\{G_1(t, \psi\varphi t, \psi\varphi t); t \in X\}.$$

Suppose that $\varphi\xi \neq\ \varphi\psi\varphi\xi$. Then $\xi \neq \psi\varphi\xi$. Put $s_1 = s_2 = \varphi\xi, t = \psi s = \psi\varphi\xi$ in (2.11), we get

$$\begin{aligned}&g(\alpha(\psi\varphi\xi, \psi\varphi\psi\varphi\xi)G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\ \varphi\xi, \varphi\xi, \varphi\psi\varphi\xi), G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\xi),\\ &\quad G_1(\psi\varphi\xi, \psi\varphi\xi, \psi\varphi\psi\varphi\xi), G_2(\varphi\xi, \varphi\psi\varphi\xi, \varphi\psi\varphi\xi)) < 0,\end{aligned}$$

$$g(\alpha(\psi\varphi\xi,\psi\varphi\psi\varphi\xi)G_1(\psi\varphi\xi,\psi\varphi\xi,\psi\varphi\psi\varphi\xi),G_2(\varphi\xi,\varphi\xi,\varphi\psi\varphi\xi),0,$$
$$G_1(\psi\varphi\xi,\psi\varphi\xi,\psi\varphi\psi\varphi\xi),2\delta G_2(\varphi\xi,\varphi\xi,\varphi\psi\varphi\xi)<0.$$

By using the condition (i) and (ii), we obtain $\alpha(\xi,\ \psi\varphi\xi)\geqslant 1 \Longrightarrow \alpha(\psi\varphi\xi,\psi\varphi\psi\varphi\xi)\geqslant 1$, and using (π^*1),(π^*2), we get

$$G_1(\psi\varphi\xi,\psi\varphi\xi,\psi\varphi\psi\varphi\xi\ )<\frac{1}{2\delta}G_2(\varphi\xi,\varphi\xi,\varphi\psi\varphi\xi). \tag{2.12}$$

Using Proposition 1.4 and (2.12), we get

$$G_2(\varphi\psi\varphi\xi,\varphi\xi,\varphi\xi)\leqslant 2\delta G_2(\varphi\xi,\varphi\psi\varphi\xi,\varphi\psi\varphi\xi),\quad G_1(\psi\varphi\xi,\psi\varphi\xi,\psi\varphi\psi\varphi\xi)<G_2(\varphi\xi,\varphi\psi\varphi\xi,\varphi\psi\varphi\xi). \tag{2.13}$$

Putting $s_1=s_2=\varphi\xi$, $t=\xi$ in (2.10), we have

$$h(\beta(\varphi\xi,\varphi\psi\varphi\xi)G_2(\varphi\xi,\varphi\psi\varphi\xi,\varphi\psi\varphi\xi),G_1(\xi,\psi\varphi\xi,\psi\varphi\xi),G_2(\varphi\xi,\varphi\xi,\varphi\xi),$$
$$G_2(\varphi\xi,\varphi\psi\varphi\xi,\varphi\psi\varphi\xi),G_1(\xi,\xi,\psi\varphi\xi))<0,$$
$$h(\beta(\varphi\xi,\varphi\psi\varphi\xi)G_2(\varphi\xi,\varphi\psi\varphi\xi,\varphi\psi\varphi\xi),G_1(\xi,\psi\varphi\xi,\psi\varphi\xi),0,G_2(\varphi\xi,\varphi\psi\varphi\xi,\varphi\psi\varphi\xi),$$
$$2\delta G_1(\xi,\psi\varphi\xi,\psi\varphi\xi))<0.$$

Applying conditions (i) and (ii), put $x_0=\varphi\xi$, $\beta(\ \varphi\xi,\varphi\psi\varphi\xi\ )\geqslant 1$, and using (π^*1),(π^*2), we obtain

$$G_2(\varphi\xi,\ \varphi\psi\varphi\xi,\varphi\psi\varphi\xi)\ <\frac{1}{2\delta}G_1(\xi,\ \psi\varphi\xi,\psi\varphi\xi). \tag{2.14}$$

From (2.13) and (2.14), we obtain

$$G_1(\psi\varphi\xi,\psi\varphi\xi,\psi\varphi\psi\varphi\xi\ )\ <\frac{1}{2\delta}G_1(\xi,\psi\varphi\xi,\psi\varphi\xi),$$
$$\frac{1}{2\delta}G_1(\psi\varphi\xi,\psi\varphi\psi\varphi\xi,\psi\varphi\psi\varphi\xi)\leqslant G_1(\psi\varphi\psi\varphi\xi,\psi\varphi\xi,\psi\varphi\xi),$$
$$\frac{1}{2\delta}G_1(\psi\varphi\xi,\psi\varphi\psi\varphi\xi,\psi\varphi\psi\varphi\xi)<\frac{1}{2\delta}G_1(\xi,\ \psi\varphi\xi,\psi\varphi\xi).$$

Hence $\varpi(\psi\varphi\xi)<\omega(\xi)$, and we have a contradiction. So $\varphi\psi\varphi\xi=\varphi\xi$. If $\varphi\xi=q$ and $\psi q=\xi_1$, then we have $\psi\varphi(\psi\varphi\xi)=\psi(\varphi\psi\varphi\xi)=\psi\varphi\xi=\psi q=\xi_1$, and $q=\varphi\xi=\varphi\psi(\varphi\xi)=\varphi(\psi\varphi\xi)=\varphi\xi_1$. Then $\psi q=\xi_1$ is a fixed point of $\psi\varphi$ and $\varphi\xi_1=q$ is a fixed point of $\varphi\psi$.

Suppose that $\psi\varphi$ has a another fixed point ξ_2. Then applying (2.11), we have,

$$g(\alpha(\xi_2,\psi\varphi\xi_2)G_1(\psi\varphi\xi_1,\psi\varphi\xi_1,\psi\varphi\xi_2),G_2(\varphi\xi_1,\varphi\xi_1,\varphi\xi_2),G_1(\xi_2,\xi_1,\xi_1),$$
$$G_1(\xi_2,\xi_2,\psi\varphi\xi_2),G_2(\varphi\xi_2,\varphi\psi\varphi\xi_2,\varphi\psi\varphi\xi_2))<0,$$
$$g(\alpha(\xi_2,\psi\varphi\xi_2)G_1(\xi_1,\xi_1,\xi_2),G_2(\varphi\xi_1,\varphi\xi_1,\varphi\xi_2)\ ,G_1(\xi_2,\xi_1,\xi_1),$$
$$G_1(\xi_2,\xi_2,\psi\varphi\xi_2),G_2(\varphi\xi_2,\ \varphi\xi_1,\varphi\xi_1))<0,$$
$$g(\alpha(\xi_2,\xi_2)G_1(\xi_1,\xi_1,\xi_2)\ ,G_2(\varphi\xi_1,\varphi\xi_1,\varphi\xi_2),G_1(\xi_2,\xi_1,\xi_1),0,2\delta G_2(\varphi\xi_1,\varphi\xi_2,\varphi\xi_2))<0.$$

Applying condition (iii) and using (π^*1),(π^*2), we obtain

$$G_1(\xi_1,\xi_1,\xi_2)<\frac{1}{2\delta}G_2(\varphi\xi_1,\varphi\xi_1,\varphi\xi_2). \tag{2.15}$$

Using (2.10) we have,

$$h(\beta(\varphi\xi_2,\varphi\psi\varphi\xi_2)G_2(\ \varphi\xi_1,\varphi\psi\varphi\xi_2,\varphi\psi\varphi\xi_2),G_1(\xi_1,\psi\varphi\xi_2,\psi\varphi\xi_2),$$
$$G_2(\varphi\xi_2,\varphi\xi_2,\varphi\xi_1),G_2(\varphi\xi_2,\varphi\psi\varphi\xi_2,\varphi\psi\varphi\xi_2),G_1(\xi_2,\xi_2,\psi\varphi\xi_2))<0,$$
$$h(\beta(\varphi\xi_2,\varphi\psi\varphi\xi_2)G_2(\varphi\xi_1,\varphi\xi_2,\varphi\xi_2),G_1(\xi_1,\xi_2,\xi_2),G_2(\varphi\xi_2,\varphi\xi_2,\varphi\xi_1),0,0)<0,$$

and it follows that

$$\begin{aligned} G_2(\varphi\xi_1, \varphi\xi_2, \varphi\xi_2) &< \frac{1}{2\delta} G_1(\xi_1, \xi_2, \xi_2), \\ \frac{1}{2\delta} G_2(\varphi\xi_2, \varphi\xi_1, \varphi\xi_1) \leqslant G_2(\varphi\xi_1 \varphi\xi_2, \varphi\xi_2) &< \frac{1}{2\delta} G_1(\xi_1, \xi_2, \xi_2). \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), it follows that

$$G_1(\xi_1, \xi_1, \xi_2) < \frac{1}{2\delta} G_1(\xi_1, \xi_2, \xi_2) \leqslant \frac{1}{2\delta} 2\delta G_1(\xi_1, \xi_1, \xi_2),$$

which gives a contradiction and so the fixed point ξ_1 must be unique. Similarly, q is the unique fixed point of $\varphi\psi$. □

Corollary 2.12. *Let* (X, G_1) *and* (Y, G_2) *be compact* G_b*-metric spaces, and* φ: $X \longrightarrow Y$, ψ :*Y* $\longrightarrow$*X be a continuous mappings. Suppose that there exists a functions* $\alpha : X \times X \to [0, \infty)$, $\beta : Y \times Y \to [0, \infty)$, *such that*

$$\begin{aligned} \beta(s_1, \varphi\psi s_1) G_2(\varphi t, \varphi\psi s_1, \varphi\psi s_2) &< \frac{1}{2\delta} \max(G_1(t, \psi s_1, \psi s_2), G_2(s_1, s_2, \varphi t), \\ &\qquad G_2(s_1, \varphi\psi s_1, \varphi\psi s_2), \frac{1}{2\delta} G_1(t, t, \psi\varphi t)), \\ \alpha(t, \psi\varphi t) G_1(\psi s_1, \psi s_2, \psi\varphi t) &< \frac{1}{2\delta} \max G_2(s_1, s_2, \varphi t), G_1(t, \psi s_1, \psi s_2), \\ &\qquad G_1(t, t, \psi\varphi t), \frac{1}{2\delta} G_2(s_1, \varphi\psi s_1, \varphi\psi s_2)) \end{aligned}$$

for all $t \in X$, $s_1, s_2 \in Y$. *Suppose also that:*

(i) $\varphi\psi$ *is an* β*-admissible and* $\psi\varphi$ *is an* α*-admissible;*
(ii) *there exists* $x_0 \in X$ *and* $y_0 \in Y$ *such that* $\alpha(x_0, \psi\varphi(x_0)) \geqslant 1, \beta(y_0, \varphi\psi(y_0)) \geqslant 1$;
(iii) $\alpha(x, x)) \geqslant 1$, $\beta(y, y) \geqslant 1, x \in X, y \in Y$.

Then $\psi\varphi$ *has a unique fixed point* ξ *in X and* $\varphi\psi$ *has a unique fixed point* q *in Y, furthermore* $\varphi\xi = q$ *and* $\psi q = \xi$.

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References

[1] M. Abbas, B. E. Rhoades, *Common fixed point results for non-commuting mappings without continuity in generalized metric spaces*, Appl. Math. Comput., **215** (2009), 262–269. 1
[2] G. M. Abd-Elhamed, *Fixed point theorems for contractions and generalized contractions in G-metric spaces*, J. Interpolat. Approx. Sci. Comput., **2015** (2015), 20–27. 1
[3] G. M. Abd-Elhamed, *Related fixed point theorems on two complete and compact G-metric spaces*, Int. J. Eng. Res. Tech., **12** (2019), 446–456.
[4] R. P. Agarwal, Z. Kadelburg, S. Radenović, *On coupled fixed point results in asymmetric G-metric spaces*, J. Inequal. Appl., **2013** (2013), 12 pages. 1
[5] A. Aghajani, M. Abbas, J. R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered* G_b*-metric spaces*, Filomat, **28** (2014), 1087–1101. 1
[6] A. Amini-Harandi, *Fixed point theory for quasi-contraction maps in b-metric spaces*, Fixed Point Theory, **15** (2014), 351–358. 1
[7] H. Aydi, W. Shatanawi, C. Vetro, *On generalized weakly g-contraction mapping in g-metric spaces*, Comput. Math. Appl., **62** (2011), 4222–4229. 1

[8] S. Czerwik, *Nonlinear set-valued contraction mappings in* b*-metric spaces*, Atti Sem. Mat. Fis. Univ. Modena, **46** (1998), 263–276. 1

[9] N. Hussain, C. Vetro, F. Vetro, *Fixed point results for* α*-implicit contractions with application to integral equations*, Nonlinear Anal. Model. Control, **21** (2016), 362–378. 1

[10] Z. Mustafa, V. Parvaneh, M. Abbas, J. R. Roshan, *Some coincidence point results for generalized* (ψ, ϕ)*-weakly contractive mappings in ordered* G*-metric spaces*, Fixed Point Theory Appl., **2013** (2013), 23 pages. 1

[11] Z. Mustafa, B. Sims, *A new approach to generalized metric spaces*, J. Nonlinear Convex Anal., **7** (2006), 289–297. 1

[12] H. K. Nashine, Z. Kadelburg, *Cyclic generalized* ϕ*-contractions in* b*-metric spaces and an application to integral equations*, Filomat, **28** (2014), 2047–2057. 1

[13] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet. Stiint. Ser. Mat. Univ. Bacau, **7** (1997), 129–133. 1

[14] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math., **32** (1999), 157–163.

[15] V. Popa, A.-M. Patriciu, *A general fixed point theorem for mappings satisfying an implicit relation in complete* G*-metric spaces*, Gazi Univ. J. Sci., **25** (2012), 403–408.

[16] V. Popa, A.-M. Patriciu, *A general fixed point theorem for pairs of weakly compatible mappings in* G*-metric spaces*, J. Nonlinear Sci. Appl., **5** (2012), 151–160.

[17] V. Popa, A.-M. Patriciu, *Two general fixed point theorems for pairs of weakly compatible mappings in* G*-metric spaces*, Novi Sad J. Math., **42** (2012), 49–60.

[18] V. Popa, A.-M. Patriciu, *Fixed point theorems for mappings satisfying an implicit relation in complete* G*-metric spaces*, Bul. Inst. Politeh. Iasi, Sect. I, Mat. Mec. Teor. Fiz., **59** (2013), 97–123. 1

[19] S. Radenović, T. Došnović, T. A. Lampert, Z. Golubović, *A note on some recent fixed point results for cyclic contractions in* b*-metric spaces and an application to integral equations*, Appl. Math. Comput., **273** (2016), 155–164. 1