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Fixed point approximations of noncommutative generic 2-generalized Bregman nonspreading mappings with equilibriums

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Abstract

In this paper, a Halpern type iterative scheme for finding a common element in the set of fixed points of generic 2-generalized Bregman nonspreading mappings and the solution set of equilibrium problem have been proposed. We also prove that the sequence generated by the scheme converges strongly to the element in a real reflexive Banach space. Our results improve and generalize some announced results in the literature.

Keywords: 2-generalized hybrid mapping, normally 2-generalized hybrid mapping, fixed point, generic 2-generalized Bregman nonspreading mapping, equilibrium problem.

2010 MSC: 47H09, 47H10, 47J25.

1. Introduction

Let C be a nonempty subset of a real Hilbert space H and $T : C \to H$ be a nonlinear map. A point $x \in H$ is called a fixed point of T if $Tx = x$. Let the set of fixed points of T be denoted by $F(T)$, i.e., $F(T) = \{x \in C : Tx = x\}$. A mapping $T : C \to H$ is called 2-generalized hybrid [22] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \leqslant \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2, \ \forall\, x, y \in C.\end{aligned}$$

Let $g : C \times C \to \mathbb{R}$ be a bifunction, the equilibrium problem with respect to g is to find a point $x \in C$ such that $g(x, y) \geqslant 0$ for all $y \in C$. Let the set of solutions of the equilibrium problem be denoted by $EP(g)$. Numerous problems can be reduced to finding solution of the equilibrium problem among which can be found in physics, optimization and economics. To solve equilibrium problems, some of the methods have been proposed; see, for example, Blum and Oettli [9] and Combettes and Hirstoaga [14].

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In 2016, Alizardeh and Moradlou [5] obtained weak convergence theorems for finding common element of the set of solutions of an equilibrium problem and the set of fixed points of 2-generalized hybrid mappings in Hilbert spaces.They proved that the sequence generated by

$$\begin{cases} x_1 = x \in E, \\ u_n \in E \ \text{ such that } \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geqslant 0, \quad \forall y \in C, \\ y_n = (1-\beta_n)x_n + \beta_n S x_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n \quad \forall n \in \mathbb{N}, \end{cases}$$

converges weakly to $v = \lim_{n\to\infty} P_{F(S)\cap EP(f)}x_1$, where $P_{F(S)\cap EP(f)}x_1$ is the metric projection of C on $F(S) \cap EP(f)$, E is a nonempty closed convex subset of a real Hilbert space H, S is a 2-generalized hybrid mapping and f is a bifunction from $E \times E$ to $\mathbb{R}$. Takahashi [28] in 2018 proved weak and strong convergence theorems for noncommutative 2-generalized hybrid mappings in Hilbert spaces.

Kondo and Takahahasi [19] introduced a mapping which contain 2-generalized hybrid mapping in Hilbert spaces. A mapping $T : C \to C$ is called normally 2-generalized hybrid [19] if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$ such that

$$\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + \alpha_3\|x - Ty\|^2 \\ + \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + \beta_3\|x - y\|^2 \leqslant 0, \ \forall\, x, y \in C,$$

where (a) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geqslant 0$ and (b) $\sum_{i=1}^{3}\alpha_i > 0$.

In 2018, Hojo, Kondo and Takahashi [17] proved weak and strong convergence theorems for commutative normally 2-generalized hybrid mappings in Hilbert spaces. Recently, Takahashi et al. [29] proved strong convergence theorem by hybrid method for two noncommutative normally 2-generalized hybrid mappings in Hilbert spaces. They established that the sequence $\{x_n\} \subset C$ defined by

$$\begin{cases} x_1 = x \in C, \\ y_n = a_n x_n + b_n(\gamma_n S + (1-\gamma_n)T)x_n + c_n(\delta_n S^2 + (1-\delta_n)T^2)x_n, \\ C_n = \{z \in C : \|y_n - z\| \leqslant \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geqslant 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_1, \ \ \forall n \in \mathbb{N}, \end{cases}$$

converges strongly to $z_0 = P_{F(S)\cap F(T)}$, where $P_{F(S)\cap F(T)}$ is the metric projection of C on $F(S) \cap F(T)$, C is a nonempty closed convex subset of a real Hilbert space H, S and T are normally 2-generalized hybrid mappings.

Let E be a real Banach space and $f : E \to (-\infty, +\infty]$ be a convex function. We denote by domf the domain of f; that is domf $= \{x \in E : f(x) < \infty\}$. For any $x \in$ int(dom(f)) and $y \in E$, the derivative of f at x in the direction y is defined by

$$f^{\circ}(x, y) := \lim_{t\to 0} \frac{f(x+ty) - f(x)}{t}. \tag{1.1}$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, the gradient of f at x is the linear functional $\nabla f(x) : E \to (-\infty, +\infty]$ defined by $\langle \nabla f(x), y\rangle = f^{\circ}(x, y)$, for any $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at every $x \in$ int(dom(f)). The function f is said to be Fréchet differentiable at x if the limit in (1.1) is attained uniformly in y, $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset $C \subset$ int(dom(f)) if the limit (1.1) is attained uniformly for $x \in E$ and $\|y\| = 1$. It is well known that if a continuous convex function f is Gâteaux differentiable (resp. Fréchet differentiable) in int(dom(f)), then ∇f is norm-to-weak* continuous (resp. continuous) in int(dom(f)) (see also [6]).

Let E be a real Banach space and $f : E \to (-\infty, +\infty]$ a strictly convex and Gâteaux differentiable function. The function $D_f :$ domf$\times$ int(dom(f)) $\to [0, +\infty)$, defined by

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \tag{1.2}$$

is called the Bregman distance with respect to f (see [13]).

Remark 1.1. If E is a smooth Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have $\nabla f(x) = 2Jx$ for all $x \in E$ where $J : E \to E^*$ is the normalized duality mapping. Hence $D_f(x, y) = \phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, for all $x, y \in E$. Also if E is a Hilbert space, then $D_f(x, y) = \|x - y\|^2$, $\forall x, y \in E$.

Observe that from (1.2), we have for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}(\operatorname{dom}(f))$.

$$D_f(x, z) = D_f(x, y) + D_f(y, z) + \langle x - y, \nabla f(y) - \nabla f(z)\rangle,$$

which is called the three point identity. As an extension and generalization of the normally 2-generalized hybrid mapping, Ali and Haruna [3] introduced a generic 2-generalized Bregman nonspreading mapping in a real reflexive Banach space. A mapping $T : C \to C$ is called a generic 2-generalized Bregman nonspreading mapping if there exist $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

(i) $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geqslant 0$;
(ii) $\sum_{i=1}^{3} \alpha_i > 0$;
(iii) for all $x, y \in C$,

$$\begin{aligned}\alpha_1 D_f(T^2x, Ty) + \alpha_2 D_f(Tx, Ty) + \alpha_3 D_f(x, Ty) + \beta_1 D_f(T^2x, y) + \beta_2 D_f(Tx, y) + \beta_3 D_f(x, y)\\ \leqslant \gamma_1\big(D_f(Ty, T^2x) - D_f(Ty, x)\big) + \gamma_2\big(D_f(Ty, Tx) - D_f(Ty, x)\big)\\ + \delta_1\big(D_f(y, T^2x) - D_f(y, x)\big) + \delta_2\big(D_f(y, Tx) - D_f(y, x)\big).\end{aligned}$$

Such mapping is called $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2)$-generic 2-generalized Bregman nonspreading mapping. For some mappings in which the generic 2-generalized Bregman nonspreading mapping contained as special cases in the space, see, for example, [2, 4, 15, 23].

Remark 1.2. If $E = H$ is a real Hibert space, then $D_f(x, y) = \|x - y\|^2$ and consequently the generic 2-generalized Bregman nonspreading mapping reduces to $(\alpha_1', \alpha_2', \alpha_3', \beta_1', \beta_2', \beta_3')$ normally 2-generalized hybrid in the sense of [19] where $\alpha_1' = \alpha_1 - \gamma_1, \alpha_2' = \alpha_2 - \gamma_2, \alpha_3' = \alpha_3 + \gamma_1 + \gamma_2$ and $\beta_1' = \beta_1 - \delta_1, \beta_2' = \beta_2 - \delta_2, \beta_3' = \beta_3 + \delta_1 + \delta_2$.

Motivated and inspired by the above results, it is our purpose in this paper to prove that the sequence generated by the proposed iterative scheme converges strongly to the common element of the set of fixed point of noncommutative generic 2-generalized Bregman nonspreading mappings and the set of solutions of the equilibrium problem in Banach spaces. Our result improves and generalizes the results of Alizadeh and Moradlou [5] and Takahashi et al. [29].

2. Preliminaries

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the dual space of E. Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. The Fenchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

Observe that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leqslant f(x) + f^*(x^*), \ \forall x \in E, \ x^* \in E^*.$$

It is well known that if $f : E \to (-\infty, +\infty]$ is a proper, convex and lower semi-continuous, then $f^* : E^* \to (-\infty, +\infty]$ is proper, convex and weak* lower semi-continuous function; see for example [27].

A sublevel of f is the set of the form $\operatorname{lev}^f_{\leqslant} r := \{x \in E : f(x) \leqslant r\}$ for $r \in \mathbb{R}$.

A function f on E is coercive [16] if every sublevel of f is bounded, equivalently

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

Let $B_r := \{x \in E : \|x\| \leqslant r\}$ for all $r > 0$ and $S_E := \{x \in E : \|x\| = 1\}$. A function f on E is said to be

(i) strongly coercive [31] if
$$\lim_{\|x\|\to+\infty}\frac{f(x)}{\|x\|}=+\infty;$$
(ii) locally bounded if $f(B_r)$ is bounded for all $r>0$;
(iii) locally uniformly smooth ([31]) if for all $r>0$, the $\lim_{t\to 0}\frac{\sigma_r(t)}{t}=0$, where $\sigma_r:[0,+\infty)\to[0,+\infty]$ is the function defined by
$$\sigma_r(t)=\sup_{x\in B_r, y\in S_E, \alpha\in(0,1)}(\alpha f(x+(1-\alpha)ty)+(1-\alpha)f(x-\alpha ty)-f(x))\times(\alpha(1-\alpha))^{-1}$$
for all $t\geqslant 0$.
(iv) locally uniformly convex (or uniformly convex on bounded subsets of E ([31])) if for all $r,t>0$, $\rho_r(t)>0$, where $\rho_r:[0,+\infty)\to[0,+\infty]$ is the gauge of uniform convexity of f, defined by
$$\rho_r(t)=\inf_{x,y\in B_r,\|x-y\|=t,\alpha\in(0,1)}(\alpha f(x)+(1-\alpha)f(y)-f(\alpha x+(1-\alpha)y))\times(\alpha(1-\alpha))^{-1}$$
for all $t\geqslant 0$.

Let $x\in \text{int}(\text{dom}(f))$, the subdifferential of f at x is the convex set defined by
$$\partial f(x)=\{x^*\in E^*: f(x)+\langle x^*, y-x\rangle\leqslant f(y),\ \forall y\in E\}.$$

Definition 2.1 ([8])**.** The function f is said to be:

(i) essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $\text{dom} f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2.2. Let E be a reflexive Banach space. Then we have:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [8] Theorem 5.4);
(ii) $(\partial f)^{-1}=\partial f^*$;
(iii) f is Legendre if and only if f^* is Legendre (see [8, Corrolary 5.5];
(iv) if f is Legendre, then ∇f is a bijection satisfying $\nabla f=(\nabla f^*)^{-1}$, $\text{ran}\nabla f=\text{dom}\nabla f^*=\text{int}(\text{dom}(f^*))$ and $\text{ran}\,\nabla f^*=\text{dom}\nabla f=\text{int}(\text{dom}(f))$, (see [8, Theorem 5.10].

Various examples of Legendre functions were given in [7, 8] . One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1<p<\infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f coincides with the generalized duality mapping of E, i.e, $\nabla f=J_p$ $(1<p<\infty)$. In particular, $\nabla f=I$ the identity mapping in Hilbert spaces.

Definition 2.3 ([11, 18])**.** Let E be a Banach space. The function $f:E\to\mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(i) f is continuous, strictly convex and Gâteaux differentiable;
(ii) the set $\{y\in E: D_f(x,y)<r\}$ is bounded for all $x\in E$ and $r>0$.

The following result can be found in [1] (see also [12, 18]).

Lemma 2.4. *Let E be a reflexive Banach space and $f:E\to\mathbb{R}$ be a strongly coercive Bregman function. Let $V_f: E\times E^*\to[0,+\infty)$ be a function associated with f defined by*
$$V_f(x,x^*)=f(x)-\langle x,x^*\rangle+f^*(x^*),\quad \forall x\in E,\ x^*\in E^*. \tag{2.1}$$
Then the following assertions hold:

(i) $V_f(x,x^*) = D_f(x,\nabla f^*(x^*))$, $\forall x \in E, x^* \in E^*$;
(ii) $V_f(x,x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \leqslant V_f(x,x^*+y^*)$, $\forall x \in E$, $x^* \in E^*$.

Also from equation (2.1), it is obvious that $D_f(x,y) = V_f(x,\nabla f(y))$ and V_f is convex in the second variable. Therefore for $t \in (0,1)$ and $x,y \in E$, we have

$$D_f(z,\nabla f^*(t\nabla f(x) + (1-t)\nabla f(y))) \leqslant tD_f(z,x) + (1-t)D_f(z,y).$$

A Bregman projection [10] of $x \in \text{int}(\text{dom}(f))$ onto the nonempty, closed and convex set $C \subset \text{dom}f$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x),x) = \inf\{D_f(y,x) : y \in C\}.$$

The following is well-known concerning Bregman projections.

Lemma 2.5 ([12]). *Let C be nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(a) $z = P_C^f x$ *if and only if* $\langle \nabla f(x) - \nabla f(z), y - z\rangle \leqslant 0, \forall y \in C$;
(b) $D_f(y,P_C^f x) + D_f(P_C^f x,x) \leqslant D_f(y,x)$, $\forall x \in E, y \in C$.

The following result is proved in [31].

Lemma 2.6 ([31]). *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent*

(1) *f is bounded on bounded sets and uniformly smooth on bounded sets;*
(2) *f^* is Fréchet differentiable and f^* is uniformly norm-to-norm continuous on bounded sets;*
(3) $\text{dom}f^* = E^*$, *f^* is strongly coercive and uniformly convex on bounded sets.*

Lemma 2.7 ([24]). *Let E be a Banach space and let $g : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then the following are equivalent.*

(1) $\lim_{n\to\infty} D_f(x_n,y_n) = 0$;
(2) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let $f : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in \text{int}(\text{dom}(f))$ is the function $\nu_f(x,.) : \text{int}(\text{dom}(f)) \times [0,+\infty] \to [0,+\infty]$ defined by

$$\nu_f(x,t) = \inf\{D_f(y,x) : y \in \text{dom}f, \|y-x\| = t\}.$$

The function f is totally convex at x if $\nu_f(x,t) > 0$ whenever $t > 0$. The function f is called totally convex if it is totally convex at every point $x \in \text{int}(\text{dom}(f))$ and is said to be totally convex on bounded sets if $\nu_f(B,t) > 0$, for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $V_f : \text{int}(\text{dom}(f)) \times [0,+\infty] \to [0,+\infty]$ defined by

$$V_f(B,t) = \inf\{\nu_f(x,t) : x \in B \cap \text{dom}f\}.$$

Lemma 2.8 ([26]). *If $x \in \text{int}(\text{dom}(f))$, then the following statements are equivalent:*

(i) *the function f is totally convex at x;*
(ii) *for any sequence $\{y_n\} \subset \text{dom}f$,*

$$\lim_{n\to+\infty} D_f(y_n,x) = 0 \Rightarrow \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Lemma 2.9 ([21]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function such that ∇f^* is bounded on bounded subsets of $\text{int}(\text{dom} f^*)$. Let $x \in \text{int}(\text{dom}(f))$. If $\{D_f(x, x_n)\}_{n\in\mathbb{N}}$ is bounded, then so is the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Lemma 2.10 ([25]). *Let $f : E \to (-\infty, +\infty]$ be a Legendre function. Let C be a nonempty closed convex subset of $\text{int}(\text{dom} f)$ and $T : C \to C$ be a quasi -Bregman nonexpansive mapping. Then $F(T)$ is closed and convex.*

The following results will play vital roles in establishing our main results

Lemma 2.11 ([30]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leqslant (1 - \alpha_n)a_n + \alpha_n\delta_n,\ n \geqslant n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a real sequence satisfying the following conditions $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leqslant 0$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12 ([20]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \leqslant a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leqslant a_{m_k+1}, \quad a_k \leqslant a_{m_k+1}.$$

Infact, $m_k = \max\{j \leqslant k : a_j < a_{j+1}\}$.

To solve equilibrium problem, the bifunction $g : C \times C \to \mathbb{R}$ is assumed to satisfy the following conditions as can be seen in [9]:

(A1) $g(x, x) = 0, \quad \forall x \in C$;
(A2) g is monotone that is, $g(x, y) + g(y, x) \leqslant 0, \quad \forall x, y \in C$;
(A3) $\limsup_{t\to\infty} g(x + t(z - x), y) \leqslant g(x, y), \quad \forall x, y, z \in C$;
(A4) the function $y \to g(x, y)$ is convex and lower semi continuous.

The resolvent of the bifunction g [14] is the operator $T_r : E \to 2^C$ defined by

$$T_r x = \{x \in C : g(x, y) + \frac{1}{r}\langle \nabla f x - \nabla f z, y - x\rangle \geqslant 0,\ \forall y \in C\}.$$

Lemma 2.13 ([25]). *Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies conditions (A1)-(A4), then the following hold:*

(i) *T_r is single-valued;*
(ii) *T_r is a Bregman firmly nonexpansive operator;*
(iii) *$F(T_r) = EP(g)$;*
(iv) *$EP(g)$ is closed and convex;*
(v) *for all $x \in E$ and $p \in F(T_r)$ we have $D_f(p, T_r x) + D_f(T_r x, x) \leqslant D_f(p, x)$.*

3. Main results

In this section, E is consider to be a real reflexive Banach space and by $x_n \to x$ and $x_n \rightharpoonup x$ in E we mean that the sequence $\{x_n\}$ converges strongly and weakly to x respectively. We propose a Halpern type iterative scheme for noncommutative generic 2-generalized Bregman nonspreading mappings with equilibrium in Banach spaces. We then prove that the sequence generated by such algorithm converges strongly to the common element of the set of fixed point of noncommutative generic 2-generalized Bregman nonspreading mappings and the set of solutions of the equilibrium problem in the space. We begin with the following Lemma.

Lemma 3.1. *Let $f : E \to \mathbb{R}$ be a strictly convex function which is uniformly Fréchet differentiable on bounded subsets of E. Let C be a nonempty subset of $\text{int}(\text{dom}(f))$ and $T : C \to C$ be a generic 2-generalized Bregman nonspreading mapping. If $x_n \rightharpoonup p$, $(x_n - Tx_n) \to 0$ and $(x_n - T^2x_n) \to 0$ as $n \to \infty$, then $p \in F(T)$.*

Proof. Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightharpoonup p$, $(x_n - Tx_n) \to 0$ and $(x_n - T^2x_n) \to 0$ as $n \to \infty$. Since $T : C \to C$ is a generic 2-generalized Bregman nonspreading mapping then following similar techniques as in [3, Lemma 3.6] , we see that $D_f(p, Ty) \leqslant D_f(p, y)$ for all $y \in C$. Thus, setting $y = p$ together with strict convexity of f we get $p \in F(T)$. This completes the proof. □

Proposition 3.2. *Let C be a nonempty subset of $\text{int}(\text{dom}(f))$ and $T : C \to C$ be a generic 2-generalized Bregman nonspreading mapping. If $F(T) \neq \emptyset$, then T is quasi Bregman nonexpansive.*

Proof. Since $T : C \to C$ is a generic 2-generalized Bregman nonspreading mapping with $F(T) \neq \emptyset$ then let $x \in F(T)$ so that $x = Tx = T^2x$. Thus, from the definition of T we get

$$\begin{aligned}\alpha_1 D_f(x, Ty) + \alpha_2 D_f(x, Ty) + \alpha_3 D_f(x, Ty) + \beta_1 D_f(x, y) + \beta_2 D_f(x, y) + \beta_3 D_f(x, y)\\ \leqslant \gamma_1\big(D_f(Ty, x) - D_f(Ty, x)\big) + \gamma_2\big(D_f(Ty, x) - D_f(Ty, x)\big)\\ + \delta_1\big(D_f(y, x) - D_f(y, x)\big) + \delta_2\big(D_f(y, x) - D_f(y, x)\big)\end{aligned}$$

for all $y \in C$. This implies

$$(\alpha_1 + \alpha_2 + \alpha_3)D_f(x, Ty) + (\beta_1 + \beta_2 + \beta_3)D_f(x, y) \leqslant 0, \ \forall y \in C.$$

Since $\sum_{i=1}^{3} \alpha_i > 0$, then it holds that

$$D_f(x, Ty) \leqslant \frac{-(\beta_1 + \beta_2 + \beta_3)}{(\alpha_1 + \alpha_2 + \alpha_3)} D_f(x, y), \ \forall y \in C.$$

Also, using the fact that $\sum_{i=1}^{3}(\alpha_i + \beta_i) \geqslant 0$, we get

$$D_f(x, Ty) \leqslant D_f(x, y), \ \forall y \in C.$$

Hence T is quasi Bregman nonexpansive. This completes the proof □

Theorem 3.3. *Let $f : E \to \mathbb{R}$ be strongly coercive, Legendre, uniformly Fréchet differentiable and totally convex function which is bounded on bounded subsets of E. Let C be a nonempty, closed and convex subset of $\text{int}(\text{dom}f)$ and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Let $S, T : C \to C$ generic 2-generalized Bregman nonspreading mappings such that $\mathcal{F} = F(S) \cap F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$*

$$\begin{cases} y_n = \nabla f^*(\alpha_n \nabla f x_n + \beta_n \nabla f T_{r_n} u_n + \gamma_n \nabla f v_n), \\ x_{n+1} = P_C^f \nabla f^*(\delta_n \nabla f u + (1 - \delta_n)\nabla f y_n), \ \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $u_n = \nabla f^*(\epsilon_n \nabla f S x_n + (1 - \epsilon_n)\nabla f T x_n)$, $v_n = \nabla f^*(\lambda_n \nabla f S^2 x_n + (1 - \lambda_n)\nabla f T^2 x_n)$ with the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\epsilon_n\}, \{\lambda_n\} \subset [a, b] \subset (0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $(C_1) : \lim_{n\to\infty} \delta_n = 0$, $(C_2) : \sum_{n=1}^{\infty} \delta_n = +\infty$. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}^f(u)$.

Proof. Since S and T are generic 2-generalized Bregman nonspreading mappings with nonempty fixed points then by Proposition 3.2, they are quasi Bregman nonexpansive mappings. Thus, it follows from Lemmas 2.10 and 2.13 that $\mathcal{F}$ is closed and convex. Hence $P_{\mathcal{F}}^f(u)$ is well defined. Now, let $z = P_{\mathcal{F}}^f(u) \subset \mathcal{F}$ so that

$$\begin{aligned} D_f(z, u_n) &= D_f(z, \nabla f^*(\epsilon_n \nabla f S x_n + (1 - \epsilon_n)\nabla f T x_n)) \\ &\leqslant \epsilon_n D_f(z, S x_n) + (1 - \epsilon_n) D_f(z, T x_n) \\ &\leqslant \epsilon_n D_f(z, x_n) + (1 - \epsilon_n) D_f(z, x_n) = D_f(z, x_n). \end{aligned}$$

Put $z_n = T_{r_n} u_n$ so that

$$D_f(z, z_n) = D_f(z, T_{r_n} u_n) \leqslant D_f(z, u_n) - D_f(T_{r_n} u_n, u_n) \leqslant D_f(z, x_n).$$

Similarly,

$$\begin{aligned} D_f(z, v_n) &= D_f(z, \nabla f^*(\lambda_n \nabla f S^2 x_n + (1-\lambda_n)\nabla f T^2 x_n)) \\ &\leqslant \lambda_n D_f(z, Sx_n) + (1-\lambda_n) D_f(z, Tx_n) \\ &\leqslant \lambda_n D_f(z, x_n) + (1-\lambda_n) D_f(z, x_n) = D_f(z, x_n). \end{aligned}$$

Also,

$$\begin{aligned} D_f(z, y_n) &= D_f(z, \alpha_n \nabla f x_n + \beta_n \nabla f z_n + \gamma_n \nabla f v_n)) \\ &\leqslant \alpha_n D_f(z, x_n) + \beta_n D_f(z, z_n) + \gamma D_f(z, v_n) \\ &\leqslant \alpha_n D_f(z, x_n) + \beta D_f(z, x_n) + \delta D_f(z, x_n) = D_f(z, x_n). \end{aligned}$$

And

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, \nabla P_C^f f^*(\delta_n \nabla f u + (1-\delta_n)\nabla f y_n)) \\ &\leqslant \delta_n D_f(z, u) + (1-\delta_n) D_f(z, y_n) \\ &\leqslant \delta_n D_f(z, u) + (1-\delta_n) D_f(z, x_n). \end{aligned} \tag{3.2}$$

Thus, by induction

$$D_f(z, x_{n+1}) \leqslant \max\{D_f(z, u), D_f(z, x_n)\}, \quad \forall n \geqslant 1.$$

This implies that the sequence $\{D_f(z, x_n)\}$ is bounded. Therefore, by Lemma 2.9 the sequence $\{x_n\}$ is bounded. Hence, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{z_n\}$ are all bounded. Since f is bounded on a bounded subsets of E then by proposition 1.1.11 of [11], ∇f is also bounded on bounded subsets of E^*. Hence the sequences $\{\nabla f(x_n)\}$, $\{\nabla f(u_n)\}$, $\{\nabla f(v_n)\}$ and $\{\nabla f(z_n)\}$ are bounded in E^*. We know from [31, Proposition 3.6.3] that $\operatorname{dom} f^* = E^*$ and f^* is strongly coercive and uniformly convex on bounded subsets of E^*.

Let $s = \sup\{\|\nabla f(x_n)\|, \|\nabla f(u_n)\|, \|\nabla f(v_n)\|, \|\nabla f(z_n)\|\}$ and $p_s^* : E^* \to \mathbb{R}$ be the gauge function of uniform convexity of the conjugate function f^*. Thus,

$$\begin{aligned} D_f(z, y_n) &= D_f(z, \nabla f^*(\alpha_n \nabla f x_n + \beta_n \nabla f z_n + \gamma_n \nabla f v_n)) \\ &= V_f(z, \alpha_n \nabla f x_n + \beta_n \nabla f z_n + \gamma_n \nabla f v_n)) \\ &= f(z) - \langle z, \alpha_n \nabla f x_n + \beta_n \nabla f z_n + \gamma_n \nabla f v_n \rangle + f^*(\alpha_n \nabla f x_n + \beta_n \nabla f z_n + \gamma_n \nabla f v_n) \\ &\leqslant f(z) - \alpha_n \langle z, \nabla f x_n \rangle - \beta_n \langle z, \nabla f(z_n) \rangle + \gamma_n \langle z, \nabla f(v_n) \rangle \\ &\quad + \alpha_n f^*(\nabla f(x_n)) + \beta_n f^*(\nabla f(z_n)) + \gamma_n f^*(\nabla f(v_n)) - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|) \\ &= \alpha_n (f(z) - \langle z, \nabla f(x_n) \rangle + f^*(\nabla f(x_n))) + \beta_n (f(z) - \langle z, \nabla f(z_n) \rangle + f^*(\nabla f(z_n)) \\ &\quad + \gamma_n (f(z) - \langle z, \nabla f v_n \rangle + f^*(\nabla f(z_n))] - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|) \\ &= \alpha_n V_f(z, \nabla f(x_n)) + \beta_n V_f(z, \nabla f(z_n)) + \gamma_n V_f(z, \nabla f(v_n)) \\ &\quad - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|) \\ &= \alpha_n D_f(z, x_n) + \beta_n D_f(z, z_n) + \gamma_n D_f(z, v_n) - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|) \\ &\leqslant \alpha_n D_f(z, x_n) + \beta_n D_f(z, x_n) + \gamma_n D_f(z, x_n) - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|) \\ &= D_f(z, x_n) - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|). \end{aligned}$$

Thus,

$$D_f(z, y_n) \leqslant D_f(z, x_n) - \alpha_n \beta_n p_s^*(\|\nabla f(x_n) - \nabla f(z_n)\|). \tag{3.3}$$

Similarly,

$$D_f(z, y_n) \leqslant D_f(z, x_n) - \alpha_n \gamma_n p_s^*(\|\nabla f(x_n) - \nabla f(v_n)\|). \tag{3.4}$$

It follows from (3.2), (3.3), and (3.4) that

$$\begin{aligned}D_f(z,x_{n+1}) &\leqslant \delta_n D_f(z,u)+(1-\delta_n)D_f(z,y_n)\\ &\leqslant \delta_n D_f(z,u)+(1-\delta_n)D_f(z,x_n)-(1-\delta_n)\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|)\\ &= \delta_n[D_f(z,u)-D_f(z,x_n)+\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|)]\\ &\quad + D_f(z,x_n)-\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|).\end{aligned}$$

Put $k_1=\sup\{|D_f(z,u)-D_f(z,x_n)|+\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|)\}$ and $k_2=\sup\{|D_f(z,u)-D_f(z,x_n)|+\alpha_n\gamma_n p_s^*(\|\nabla f(x_n)-\nabla f(v_n)\|)\}$, then we get

$$D_f(z,x_{n+1})\leqslant D_f(z,x_n)-\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|)+\delta_n k_1$$

and

$$D_f(z,x_{n+1})\leqslant D_f(z,x_n)-\alpha_n\gamma_n p_s^*(\|\nabla f(x_n)-\nabla f(v_n)\|)+\delta_n k_2.$$

These imply

$$\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|)\leqslant D_f(z,x_n)-D_f(z,x_{n+1})+\delta_n k_1 \tag{3.5}$$

and

$$\alpha_n\gamma_n p_s^*(\|\nabla f(x_n)-\nabla f(v_n)\|)\leqslant D_f(z,x_n)-D_f(z,x_{n+1})+\delta_n k_2. \tag{3.6}$$

We now consider the following cases.

Case 1. Assume the sequence $\{D_f(z,x_n)\}$ is non-increasing. Since it is bounded, then it is convergent. Thus, we have that

$$D_f(z,x_n)-D_f(z,x_{n+1})\to 0 \text{ as } n\to\infty. \tag{3.7}$$

Now, with the use of (C_1), equations (3.5), (3.6), and (3.7) we have

$$\alpha_n\beta_n p_s^*(\|\nabla f(x_n)-\nabla f(z_n)\|)\to 0 \text{ as } n\to\infty$$

and

$$\alpha_n\gamma_n p_s^*(\|\nabla f(x_n)-\nabla f(v_n)\|)\to 0 \text{ as } n\to\infty.$$

By using the property of p_s^* and the fact that $\alpha_n,\beta_n,\gamma_n\in[a,b]\subset(0,1)$, we obtain

$$\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(z_n)\|=0,\quad \lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(v_n)\|=0. \tag{3.8}$$

Also, from (3.1) and (3.8) we get

$$\begin{aligned}\|\nabla f(x_n)-\nabla f(y_n)\| &\leqslant \alpha_n\|\nabla f(x_n)-\nabla f(x_n)\|+\beta_n\|\nabla f(x_n)-\nabla f(z_n)\|\\ &\quad+\gamma_n\|\nabla f(x_n)-\nabla f(v_n)\|\to 0 \text{ as } n\to\infty.\end{aligned} \tag{3.9}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, then by Lemma 2.6 (2), f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E. Thus, we obtain from (3.8) and (3.9) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0,\quad \lim_{n\to\infty}\|x_n-v_n\|=0,\quad \lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.10}$$

Thus, by Lemma 2.7, we get that $\lim_{n\to\infty}D_f(v_n,x_n)=0$. Since $D_f(z,z_n)+D_f(z_n,u_n)\leqslant D_f(z,u_n)\leqslant D_f(z,x_n)$, then

$$\begin{aligned}D_f(z_n,u_n) &\leqslant D_f(z,x_n)-D_f(z,z_n)\\ &\leqslant |D_f(z_n,x_n)+\langle\nabla fz_n-\nabla fx_n, z-x_n\rangle|\\ &\leqslant D_f(z_n,x_n)+\|\nabla fz_n-\nabla fx_n\|\|z-x_n\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus, by Lemma 2.8 we get

$$\lim_{n\to\infty} \|z_n - u_n\| = 0.$$

This together with (3.10) implies

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.11}$$

Thus, by Lemma 2.7, we get that $\lim_{n\to\infty} D_f(u_n, x_n) = 0$. On the other hand,

$$\begin{aligned} D_f(z, u_n) &\leqslant \epsilon_n D_f(z, Sx_n) + (1-\epsilon_n)D_f(z, Tx_n) - \epsilon_n(1-\epsilon_n)\rho_s^*(\|\nabla fSx_n - \nabla fTx_n\|) \\ &= D_f(z, x_n) - \epsilon_n(1-\epsilon_n)\rho_s^*(\|\nabla fSx_n - \nabla fTx_n\|). \end{aligned}$$

This implies,

$$\begin{aligned} \epsilon_n(1-\epsilon_n)\rho_s^*(\|\nabla fSx_n - \nabla fTx_n\|) &\leqslant D_f(z, x_n) - D_f(z, u_n) \\ &\leqslant D_f(u_n, x_n) + \|\nabla fu_n - \nabla fx_n\|\|z - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \lambda_n(1-\lambda_n)\rho_s^*(\|\nabla fS^2x_n - \nabla fT^2x_n\|) &\leqslant D_f(z, x_n) - D_f(z, v_n) \\ &\leqslant D_f(v_n, x_n) + \|\nabla fv_n - \nabla fx_n\|\|z - x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus,

$$\|\nabla fSx_n - \nabla fTx_n\| \to 0 \quad \text{and} \quad \|\nabla fS^2x_n - \nabla fT^2x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.12}$$

From (3.12), we get

$$\|Sx_n - Tx_n\| \to 0 \quad \text{and} \quad \|S^2x_n - T^2x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.13}$$

Using (3.11) and (3.12), we see that

$$\begin{aligned} \|\nabla fx_n - \nabla fTx_n\| &\leqslant \|\nabla fx_n - \nabla fu_n\| + \|fu_n - \nabla fTx_n\| \\ &= \|fx_n - \nabla fu_n\| + \epsilon_n\|\nabla fSx_n - \nabla fTx_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.14}$$

Similarly, using (3.8) and (3.12) we have

$$\begin{aligned} \|\nabla fx_n - \nabla fT^2x_n\| &\leqslant \|\nabla fx_n - \nabla fv_n\| + \|fv_n - \nabla fT^2x_n\| \\ &= \|fx_n - \nabla fv_n\| + \lambda_n\|\nabla fS^2x_n - \nabla fT^2x_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we get

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - T^2x_n\| = 0. \tag{3.16}$$

Using (3.13) and (3.16), we get

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - S^2x_n\| = 0. \tag{3.17}$$

Since E is reflexive and the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. This together equations (3.16), (3.17), and Lemma 3.1 implies that $x \in F(S) \cap F(T)$.

Also, using (3.10), we obtain $z_{n_k} \rightharpoonup x$. Since $z_n = T_{r_n} u_n$ from the definition of T_r, we get

$$g(z_n, y) + \frac{1}{r_n}\langle \nabla f z_n - \nabla f u_n, y - z_n \rangle \geqslant 0, \quad \forall y \in C.$$

Hence

$$g(z_{n_k}, y) + \frac{1}{r_{n_k}}\langle \nabla f z_{n_k} - \nabla f u_{n_k}, y - z_{n_k} \rangle \geqslant 0, \quad \forall y \in C.$$

Using (A2), we get

$$\begin{aligned} g(y, z_{n_k}) \leqslant -g(z_{n_k}, y) &\leqslant \frac{1}{r_{n_k}}\langle \nabla f z_{n_k} - \nabla f u_{n_k}, y - z_{n_k} \rangle \\ &\leqslant \frac{1}{r_{n_k}}\|\nabla f z_{n_k} - \nabla f u_{n_k}\|\|y - z_{n_k}\|, \quad \forall y \in C. \end{aligned}$$

Taking limit as $k \to \infty$ of the above inequality and with use of (A4) and the fact that $z_{n_k} \rightharpoonup x$, we get $g(y, x) \leqslant 0$. Define $y_t = ty + (1-t)x$ for $0 < t < 1$ and $y \in C$. Since $x, y \in C$ then $y_t \in C$ which yields that $g(y_t, x) \leqslant 0$.

Using (A1) we see that

$$0 = g(y_t, y_t) \leqslant t g(y_t, y) + (1-t) g(y_t, x) \leqslant t g(y_t, y).$$

Thus, $g(y_t, y) \geqslant 0$. Now letting $t \to 0$ and using (A3) we see that $g(x, y) \geqslant 0$ for any $y \in C$. This implies $x \in EP(g)$. Hence $x \in \mathcal{F}$.

Now, let

$$w_n = \nabla f^*[\delta_n \nabla f(u) + (1-\delta_n)\nabla f(y_n)],$$

so that

$$\begin{aligned} D_f(y_n, w_n) &= D_f(y_n, \nabla f^*[\delta_n \nabla f(u) + (1-\delta_n)\nabla f y_n]) \\ &\leqslant \delta_n D_f(y_n, u) + (1-\delta_n) D_f(y_n, y_n) \\ &= \delta_n D_f(y_n, u) \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows from Lemma 2.8 that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.18}$$

Also, from (3.10) and (3.18) we see that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{3.19}$$

Using (a) of Lemma 2.5 and equation (3.19), we can conclude that

$$\begin{aligned} \limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), w_n - z \rangle &= \limsup_{n\to\infty}[\langle \nabla f(u) - \nabla f(z), w_n - x_n \rangle + \langle \nabla f(u) - \nabla f(z), x_n - z \rangle] \\ &= \limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_n - z \rangle \\ &= \lim_{k\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n_k} - z \rangle \\ &= \langle \nabla f(u) - \nabla f(z), x - z \rangle \leqslant 0,, \ \forall\, x \in \mathcal{F}. \end{aligned} \tag{3.20}$$

Using Lemmas 2.4 and 2.5, we see that

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f(z, P_C^f(w_n)) \\ &\leqslant D_f(z, w_n) \end{aligned}$$

$$
\begin{aligned}
&= V_f(z, \nabla f(w_n)) \\
&\leqslant V_f(z, \nabla f(w_n) - \alpha_n(\nabla f(u) - \nabla f(z)) + \delta_n\langle \nabla f(u) - \nabla f(z), w_n - z\rangle \\
&= V_f(z, \delta_n \nabla f(u) + (1-\delta_n)\nabla f(y_n) - \alpha_n(\nabla f(u) - \nabla f(z)) + \delta_n\langle \nabla f(u) - \nabla f(z), w_n - z\rangle \\
&= V_f(z, (1-\delta_n)\nabla f(y_n) + \delta_n \nabla f(z)) + \delta_n\langle \nabla f(u) - \nabla f(z), w_n - z\rangle \\
&\leqslant (1-\delta_n)V_f(z, \nabla f(y_n)) + \delta_n V_f(z, \nabla f(z)) + \delta_n\langle \nabla f(u) - \nabla f(z), w_n - z\rangle \\
&= (1-\delta_n)D_f(z, y_n) + \delta_n D_f(z,z) + \delta_n\langle \nabla f(u) - \nabla f(z), w_n - z\rangle \\
&\leqslant (1-\alpha_n)D_f(z, x_n) + \delta_n\langle \nabla f(u) - \nabla f(z), w_n - z\rangle.
\end{aligned} \tag{3.21}
$$

Therefore by Lemma 2.11, inequalities (3.20) and (3.21), we obtain that $D_f(z, x_n) \to 0$ as $n \to \infty$. Hence by Lemma 2.8, $x_n \to z = P^f_{\mathcal{F}}(u)$ as $n \to \infty$.

Case 2. Let assume that the sequence $\{D_f(z, x_n)\}$ is not non-increasing and take $\{\Phi_n\} = \{D_f(z, x_n)\}$. Let there exists a subsequence $\{n_i\}$ of $\{n\}$ such that for all $i \in \mathbb{N}$, $\Phi_{n_i} \leqslant \Phi_{n_i+1}$. For some sufficiently large N and for all $n \geqslant N$, let the map $\tau : \mathbb{N} \to \mathbb{N}$ be defined by

$$\tau(n) = \max\{k \leqslant n : \Phi_k \leqslant \Phi_{k+1}\},$$

so that from Lemma 2.12 we see that the sequence $\tau(n)$ is non-decreasing with $\tau(n) \to \infty$ as $n \to \infty$ and $\Phi_{\tau(n)} \leqslant \Phi_{\tau(n)+1}$, $\Phi_n \leqslant \Phi_{\tau(n)+1}$. Using the fact that $\delta_{\tau(n)} \to 0$ as $\tau(n) \to \infty$ and by equation (3.5) and (3.6) we obtain

$$p^*_s(\|\nabla f(x_{\tau(n)}) - \nabla f(z_{\tau(n)})\|) \to 0,\ p^*_s(\|\nabla f(x_{\tau(n)}) - \nabla f(v_{\tau(n)})\|) \to 0.$$

Following similar argument as in Case 1, we see that

$$\lim_{n\to\infty} \|x_{\tau(n)} - Tx_{\tau(n)}\| = 0,\ \lim_{n\to\infty} \|x_{\tau(n)} - T^2 x_{\tau(n)}\| = 0$$

and

$$\lim_{n\to\infty} \|x_{\tau(n)} - Sx_{\tau(n)}\| = 0,\ \lim_{n\to\infty} \|x_{\tau(n)} - S^2 x_{\tau(n)}\| = 0.$$

Also,

$$\limsup_{\tau(n)\to\infty} \langle \nabla f(u) - \nabla f(z), w_{\tau(n)} - z\rangle \leqslant 0.$$

It follows from equation (3.21) that

$$\Phi_{\tau(n)+1} \leqslant \Phi_{\tau(n)} + \delta_{\tau(n)}[\langle \nabla f(u) - \nabla f(z), x_{\tau(n)} - z\rangle - \Phi_{\tau(n)}].$$

From the fact that $\Phi_{\tau(n)} \leqslant \Phi_{\tau(n)+1}$ and $\Phi_{\tau(n)} > 0$, the previous inequality yields

$$\Phi_{\tau(n)} \leqslant \langle \nabla f(u) - \nabla f(z), w_{\tau(n)} - z\rangle \to 0 \text{ as } \tau(n) \to \infty.$$

Thus, $\lim_{\tau(n)\to\infty} \Phi_{\tau(n)} = \lim_{\tau(n)\to\infty} \Phi_{\tau(n)+1} = 0$. Since $0 \leqslant \Phi_n \leqslant \Phi_{\tau(n)+1}$, it implies that $\lim_{n\to\infty} \Phi_n = \lim_{n\to\infty} D_f(z, x_n) = 0$. Therefore, by Lemma 2.8, we arrived at $x_n \to z$ as $n \to \infty$. Hence in view of the above two cases, we see that the sequence $\{x_n\}$ converges strongly to $z = P^f_{\mathcal{F}}(u)$. This completes the proof. □

As a consequence, in view of Remark 1.2, the following result is obtained by applying Theorem 3.3.

Corollary 3.4. *Let C be a subset of a real Hilbert space and $g : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $S, T : C \to C$ be normally 2-generalized hybrid mappings with $f(x) = \|x\|^2$ such that $\mathcal{F} = F(S) \cap F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $u, x_1 \in C$,*

$$\begin{cases} y_n = \alpha_n x_n + \beta_n T_{r_n} u_n + \gamma_n v_n, \\ x_{n+1} = \delta_n u + (1-\delta_n) y_n, \end{cases}$$

where $u_n = \epsilon_n S x_n + (1-\epsilon_n) T x_n$, $v_n = \lambda_n S^2 x_n + (1-\lambda_n) T^2 x_n$ with the real sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\epsilon_n\}$, $\{\lambda_n\} \subset [a, b] \subset (0, 1)$ satisfying $\alpha_n + \beta_n + \gamma_n = 1$ and $(C_1) : \lim_{n\to\infty} \delta_n = 0$, $(C_2) : \sum_{n=1}^{\infty} \delta_n = +\infty$. Then $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(u)$, where $P_{\mathcal{F}}(u)$ is the metric projection of C onto $\mathcal{F}$.

Proof. By remark 1.2, the generic 2-generalized Bregman nonspreading mapping reduces to normally 2-generalized hybrid mapping in Hilbert space, i.e., there exists $\alpha_1', \alpha_2', \alpha_3', \beta_1', \beta_2', \beta_3' \in \mathbb{R}$ such that

$$\alpha_1'\|T^2x - Ty\|^2 + \alpha_2'\|Tx - Ty\|^2 + \alpha_3'\|x - Ty\|^2 + \beta_1'\|T^2x - y\|^2 + \beta_2'\|Tx - y\|^2 + \beta_3'\|x - y\|^2 \leqslant 0, \ \forall\, x, y \in C,$$

where $\alpha_1' = \alpha_1 - \gamma_1, \alpha_2' = \alpha_2 - \gamma_2, \alpha_3' = \alpha_3 + \gamma_1 + \gamma_2$ and $\beta_1' = \beta_1 - \delta_1, \beta_2' = \beta_2 - \delta_2, \beta_3' = \beta_3 + \delta_1 + \delta_2$ satisfying $\sum_{i=1}^{3}(\alpha_i' + \beta_i') = \sum_{i=1}^{3}(\alpha_i + \beta_i) \geqslant 0$ and $\sum_{i=1}^{3}\alpha_i' = \sum_{i=1}^{3}\alpha_i > 0$. Thus, by Theorem 3.3 we see that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}(u)$. This completes the proof. □

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