Fixed point theorems for rational type (α-Θ)-contractions in controlled metric spaces

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Abstract

This paper aims to define rational type (α-Θ)-contraction in controlled metric space and obtain some advanced fixed point theorems. The outcomes generalize and extend various famous results in the literature. An example and certain consequences are presented to illustrate the significance of established results.

Keywords: Fixed point, rational type (α-Θ)-contraction, controlled metric spaces.

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1. Introduction

In 1906 Frechet provided the conception of metric space which was an axiomatic development in functional analysis. Due to its simplicity, it has been generalized by several researchers\cite{1–19} in the recent past.

Czerwik \cite{9} defined the conception of b-metric space in this way.

\textbf{Definition 1.1} (\cite{9}). Let $W \neq \emptyset$ and $b \geq 1$. A mapping $\kappa_b : W \times W \rightarrow [0, \infty)$ is called a b-metric if these assertions hold:

\begin{enumerate}[(b1)]
\item $\kappa_b(t, s) = 0 \iff t = s$;
\item $\kappa_b(t, s) = \kappa_b(s, t)$ for all $t, s \in W$;
\item $\kappa_b(t, z) \leq b[\kappa_b(t, s) + \kappa_b(s, z)]$ for all $t, s, z \in W$.
\end{enumerate}

Then $(W, \kappa_b)$ is called a b-metric space (b-MS).

Kamran et al. \cite{14} gave the notion of extended b-metric spaces in 2017.

\textbf{Definition 1.2.} Let $W \neq \emptyset$, $p : W \times W \rightarrow [1, \infty)$ and $\kappa_e : W \times W \rightarrow [0, \infty)$. Then $\kappa_e$ is called an extended b-metric if

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for all \( \iota, s, z \in \mathcal{W} \). Then \( (\mathcal{W}, \kappa_\iota) \) is said to be an extended \( b \)-metric space (Eb-MS).

Currently, a new type of a space was given by Mlaiki et al. [16].

**Definition 1.3.** Let \( \mathcal{W} \neq \emptyset \), \( \rho : \mathcal{W} \times \mathcal{W} \to [1, \infty) \), and \( \kappa_p : \mathcal{W} \times \mathcal{W} \to [0, \infty) \). Then \( \kappa_p \) is said to be a controlled metric if

(i) \( \kappa_p(\iota, s) = 0 \) if and only if \( \iota = s \);
(ii) \( \kappa_p(\iota, s) = \kappa_p(s, \iota) \);
(iii) \( \kappa_p(\iota, s) \leq \rho(\iota, s)[\kappa_p(\iota, z) + \kappa_p(s, z)] \)

for all \( \iota, s, z \in \mathcal{W} \). The pair \( (\mathcal{W}, \kappa_p) \) is called a controlled metric space (CMS).

In 2015, Samet et al. [13] gave the notion of \( \Theta \)-contraction in this way.

**Definition 1.4.** Let \( \Theta : (0, \infty) \to (1, \infty) \) be a function satisfying:

- \( \Theta_1 \) \( \Theta \) is nondecreasing;
- \( \Theta_2 \) for each sequence \( \{\iota_n\} \subseteq \iota^+ \), \( \lim_{n \to \infty} \Theta(\iota_n) = 1 \iff \lim_{n \to \infty} (\iota_n) = 0 \);
- \( \Theta_3 \) \( \exists 0 < h < 1 \) and \( \theta \in (0, \infty) \) such that \( \lim_{n \to 0^+} \frac{\Theta(\iota)}{\iota} = \theta \).

A self mapping \( \mathcal{O} : \mathcal{W} \to \mathcal{W} \) is called \( \Theta \)-contraction if \( \exists \Theta \) satisfying \( \Theta_1 \)-\( \Theta_3 \) and \( k \in (0, 1) \) such that

\[ k(\mathcal{O} \iota, \mathcal{O}s) \neq 0 \Rightarrow \Theta(k(\mathcal{O} \iota, \mathcal{O}s)) \leq [\Theta(k(\iota, s))]^k \]

for all \( \iota, s \in \mathcal{W} \).

**Theorem 1.5** ([13]). If \( \mathcal{O} : \mathcal{W} \to \mathcal{W} \) be a \( \Theta \)-contraction on complete metric space \( (\mathcal{W}, \kappa) \), then \( \exists \iota^* \in \mathcal{W} \) such that \( \iota^* = \mathcal{O} \iota^* \).

We represent by the \( \Omega \), the family of all above mapping \( \Theta \) satisfying the above assertions \( \Theta_1 \)-\( \Theta_3 \) to be consistent with Samet et al. [13].

In this paper, we define rational type \( (\alpha, \Theta) \)-contraction in the setting of complete CMS to obtain some generalized results.

**2. Main results**

**Definition 2.1.** Let \( (\mathcal{W}, \kappa_p) \) be a CMS. A function \( \mathcal{O} : \mathcal{W} \to \mathcal{W} \) is said to be a rational type \( (\alpha, \Theta) \)-contraction if \( \exists \alpha : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+ \), \( k \in (0, 1) \) and \( \Theta \in \Omega \) such that

\[ \alpha(\iota, s)\Theta(\kappa_p(\mathcal{O} \iota, \mathcal{O}s)) \leq \Theta(\kappa_p(\iota, s))^k, \]

where

\[ M(\iota, s) = \max \left\{ \kappa_p(\iota, s), \kappa_p(\iota, \mathcal{O} \iota), \kappa_p(s, \mathcal{O}s), \frac{\kappa_p(\mathcal{O} \iota, \mathcal{O}s)\kappa_p(s, \mathcal{O}s)}{1 + \kappa_p(\iota, s)} \right\}, \]

\[ \forall \iota, s \in \mathcal{W} \] with \( \kappa_p(\mathcal{O} \iota, \mathcal{O}s) > 0 \).

From now onward, we consider \( (\mathcal{W}, \kappa_p) \) as complete controlled metric space.

**Theorem 2.2.** Let \( (\mathcal{W}, \kappa_p) \) is a complete CMS and \( \mathcal{O} : \mathcal{W} \to \mathcal{W} \) be a rational type \( (\alpha, \Theta) \)-contraction such that:

(i) \( \mathcal{O} \) is \( \alpha \)-admissible;
(ii) \( \exists t_0 \in \mathcal{W} \) such that \( \alpha(t_0, \mathcal{O} t_0) \geq 1 \);
(iii) \( \mathcal{O} \) is continuous;
(iv) \( \sup_{m \geq 1} \lim_{t \to \infty} \frac{p(t_{i+1}, t_{i+2}) p(t_{i+1}, t_n)}{p(t_i, t_{i+1})} < 1. \)

In addition, assume that, for every \( t \in \mathcal{W} \), we have \( \lim_{n \to \infty} p(t, t_n) \) and \( \lim_{n \to \infty} p(t_n, t) \) exist and are finite. Then, \( \exists t^* \in \mathcal{W} \) such that \( t^* = \alpha t^* \).

Proof. Suppose \( t_0 \in \mathcal{W} \) is such that \( \alpha(t_0, \partial t_0) \geq 1 \). We generate \( \{t_n\} \) in \( \mathcal{W} \) by \( t_{n+1} = \alpha t_n, \forall n \in \mathbb{N} \).

Obviously, if \( \exists n_0 \in \mathbb{N} \) for which \( t_{n_0+1} = t_{n_0} \), then \( \alpha t_{n_0} = t_{n_0} \) and the proof is finished. Thus, we assume that \( t_{n+1} \neq t_n, \forall n \in \mathbb{N} \). By using (i) and (ii), it is obvious that

\[ \alpha(t_n, t_{n+1}) \geq 1, \]

\( \forall n \in \mathbb{N} \). Thus by (2.1), we obtain

\[ 1 < \Theta(\kappa_p(t_n, t_{n+1})) = \Theta(\kappa_p(\partial t_{n-1}, \partial t_n)) \leq \alpha(t_n, t_{n+1}) \Theta(\kappa_p(\partial t_{n-1}, \partial t_n)). \]

Since \( \mathcal{O} \) is a rational type \((\alpha, \Theta)\)-contraction, so \( \forall n \in \mathbb{N} \), we can write

\[
1 < \Theta(\kappa_p(t_n, t_{n+1})) \leq \alpha(t_n, t_{n+1}) \Theta(\kappa_p(\partial t_{n-1}, \partial t_n)) \\
\leq \Theta(M(t_{n-1}, t_n))^k \\
= \Theta \left( \max \left\{ \kappa_p(t_{n-1}, t_n), \kappa_p(t_{n-1}, \partial t_{n-1}), \kappa_p(t_n, \partial t_n), \frac{\kappa_p(t_{n-1}, \partial t_{n-1}) \kappa_p(t_n, \partial t_n)}{1 + \kappa_p(t_{n-1}, t_n)} \right\} \right)^k \\
= \Theta \left( \max \left\{ \kappa_p(t_{n-1}, t_n), \kappa_p(t_{n-1}, t_n), \kappa_p(t_n, t_{n+1}), \frac{\kappa_p(t_{n-1}, t_n) \kappa_p(t_n, t_{n+1})}{1 + \kappa_p(t_{n-1}, t_n)} \right\} \right)^k \\
\leq \Theta(\max(\kappa_p(t_{n-1}, t_n), \kappa_p(t_{n-1}, t_n), \kappa_p(t_n, t_{n+1}), \kappa_p(t_n, t_{n+1})))^k \\
= \Theta(\max(\kappa_p(t_{n-1}, t_n), \kappa_p(t_n, t_{n+1})))^k. 
\]

Thus

\[ 1 < \Theta(\kappa_p(t_n, t_{n+1})) \leq \Theta(\max(\kappa_p(t_{n-1}, t_n), \kappa_p(t_n, t_{n+1})))^k. \tag{2.3} \]

If there exists \( n \in \mathbb{N} \) such that \( \Theta(\max(\kappa_p(t_{n-1}, t_n), \kappa_p(t_n, t_{n+1})))^k = \kappa_p(t_n, t_{n+1}) \), then (2.3) becomes

\[ 1 < \Theta(\kappa_p(t_n, t_{n+1})) \leq \Theta(\kappa_p(t_n, t_{n+1}))^k < \Theta(\kappa_p(t_n, t_{n+1})), \]

which is a contradiction. Therefore \( \Theta(\max(\kappa_p(t_{n-1}, t_n), \kappa_p(t_n, t_{n+1})))^k = \kappa_p(t_{n-1}, t_n), \forall n \in \mathbb{N} \).

Thus from (2.3), we get

\[ 1 < \Theta(\kappa_p(t_n, t_{n+1})) \leq \Theta(\kappa_p(t_{n-1}, t_n))^k \leq \Theta(\kappa_p(t_{n-2}, t_{n-1}))^k \leq \ldots \leq \Theta(\kappa_p(t_0, t_1))^k. \]

Thus by (2.3), we have

\[ 1 < \Theta(\kappa_p(t_n, t_{n+1})) \leq \Theta(\kappa_p(t_0, t_1))^k. \tag{2.4} \]

Taking \( n \to \infty \) in (2.4), we get

\[ \lim_{n \to \infty} \Theta(\kappa_p(t_n, t_{n+1})) = 1. \]
By (Θ₂), we get
\[
\lim_{n \to \infty} \kappa_p(t_n, t_{n+1}) = 0.
\]

By (Θ₃), \( \exists 0 < h < 1 \) and \( \vartheta \in (0, \infty) \) such that
\[
\lim_{n \to \infty} \frac{\Theta(\kappa_p(t_n, t_{n+1})) - 1}{\kappa_p(t_n, t_{n+1})^h} = \vartheta.
\]

Assume that \( \vartheta < \infty \). In this case, let \( \lambda = \frac{\vartheta}{2} > 0 \). By definition, \( \exists n_1 \in \mathbb{N} \) so that
\[
\left| \frac{\Theta(\kappa_p(t_n, t_{n+1})) - 1}{\kappa_p(t_n, t_{n+1})^h} - \vartheta \right| \leq \lambda, \quad \forall n > n_1.
\]

This implies that
\[
\frac{\Theta(\kappa_p(t_n, t_{n+1})) - 1}{\kappa_p(t_n, t_{n+1})^h} \geq \vartheta - \lambda = \frac{\vartheta}{2} = \lambda, \quad \forall n > n_1.
\]

Then
\[
n\kappa_p(t_n, t_{n+1})^h \leq \mu_n [\Theta(\kappa_p(t_n, t_{n+1})) - 1], \quad \forall n > n_1,
\]

with \( \mu = \frac{1}{\lambda} \). Now we assume that \( \vartheta = \infty \) and \( \lambda > 0 \). By definition, \( \exists n_1 \in \mathbb{N} \) so that
\[
\lambda \leq \frac{\Theta(\kappa_p(t_n, t_{n+1})) - 1}{\kappa_p(t_n, t_{n+1})^h}, \quad \forall n > n_1.
\]

This implies that
\[
n\kappa_p(t_n, t_{n+1})^h \leq \mu_n [\Theta(\kappa_p(t_n, t_{n+1})) - 1], \quad \forall n > n_1,
\]

where \( \mu = \frac{1}{\lambda} \). Hence, in all situation, \( \exists \mu > 0 \) and \( n_1 \in \mathbb{N} \) so that
\[
\lim_{n \to \infty} n\kappa_p(t_n, t_{n+1})^h = 0.
\]

for all \( n > n_1 \). Thus by (2.4) and (2.5), we get
\[
\frac{\Theta(\kappa_p(t_0, t_1)) - 1}{\kappa_p(t_0, t_1)^h} = 0.
\]

Taking \( n \to \infty \), we obtain
\[
\lim_{n \to \infty} n\kappa_p(t_n, t_{n+1})^h = 0.
\]

Hence, \( \exists n_2 \in \mathbb{N} \) so that
\[
\kappa_p(t_n, t_{n+1}) \leq \frac{1}{n^{1/h}}, \quad \forall n > n_2.
\]

Consider the triangle inequality for \( q \geq 1 \), we have
\[
k_p(t_n, t_{n+q}) \leq p(t_n, t_{n+1})k_p(t_n, t_{n+1}) + p(t_{n+1}, t_{n+q})k_p(t_{n+1}, t_{n+q})
\leq p(t_n, t_{n+1})k_p(t_n, t_{n+1}) + p(t_{n+1}, t_{n+q})p(t_{n+1}, t_{n+2})k_p(t_{n+1}, t_{n+2})
\quad + p(t_{n+2}, t_{n+q})p(t_{n+2}, t_{n+q})k_p(t_{n+2}, t_{n+q})
\leq p(t_n, t_{n+1})k_p(t_n, t_{n+1}) + p(t_{n+1}, t_{n+2})k_p(t_{n+1}, t_{n+2})
\quad + p(t_{n+2}, t_{n+3})k_p(t_{n+2}, t_{n+3})
\quad + p(t_{n+3}, t_{n+q})k_p(t_{n+3}, t_{n+q})
\quad : 
\leq p(t_n, t_{n+1})k_p(t_n, t_{n+1}) + \sum_{i=n+1}^{n+q-2} \left( \prod_{j=n+1}^{i} p(t_j, t_{j+1}) \right) p(t_i, t_{i+1})k_p(t_i, t_{i+1})
\]
Thus
\[
\sum_{i=n+1}^{n+q-1} p(t_i, t_{n+q}) + \prod_{i=n+1}^{n+q-1} p(t_{n+q-1}, t_{n+q}).
\]
which further implies that
\[
\kappa_p(t_n, t_{n+q}) \leq p(t_n, t_{n+1}) \kappa_p(t_n, t_{n+1}) + \sum_{i=n+1}^{n+q-2} \left( \prod_{j=n+1}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \kappa_p(t_i, t_{i+1})
\]
\[
+ \left( \prod_{i=n+1}^{n+q-1} p(t_i, t_{n+q}) \right) p(t_{n+q-1}, t_{n+q}) \kappa_p(t_{n+q-1}, t_{n+q})
\]
\[
= p(t_n, t_{n+1}) \kappa_p(t_n, t_{n+1}) + \sum_{i=n+1}^{n+q-1} \left( \prod_{j=n+1}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \kappa_p(t_i, t_{i+1})
\]
\[
\leq p(t_n, t_{n+1}) \kappa_p(t_n, t_{n+1}) + \sum_{i=n+1}^{n+q-1} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \kappa_p(t_i, t_{i+1})
\]
\[
\leq p(t_n, t_{n+1}) \kappa_p(t_n, t_{n+1}) + \sum_{i=n+1}^{\infty} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \frac{1}{i^{1-x}}.
\]
Thus
\[
\kappa_p(t_n, t_{n+q}) \leq p(t_n, t_{n+1}) \kappa_p(t_n, t_{n+1}) + \sum_{i=n+1}^{n+q-1} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \frac{1}{i^{1-x}}. \tag{2.6}
\]
Now, consider
\[
\sum_{i=n+1}^{n+q-1} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \frac{1}{i^{1-x}} = \sum_{i=n+1}^{n+q-1} \frac{1}{i^{1-x}} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1})
\]
\[
\leq \sum_{i=n+1}^{\infty} \frac{1}{i^{1-x}} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) = \sum_{i=n+1}^{\infty} U_i V_i,
\]
where
\[
U_i = \frac{1}{i^{1-x}}
\]
and
\[
V_i = \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}).
\]
Since \( \frac{1}{i^{1-x}} > 0 \) \( \sum_{i=n+1}^{\infty} \frac{1}{i^{1-x}} \) converges and also \( V_i = \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \) is increasing and bounded above. Thus \( \lim_{i \to \infty} V_i = \sup(V_i) \), exists and is non zero. Hence, the product \( \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \) converges. Thus \( \sum_{i=n+1}^{\infty} U_i V_i \) converges. Let us consider the partial sum
\[
S_q = \sum_{i=0}^{q} \left( \prod_{j=0}^{i} p(t_j, t_{n+q}) \right) p(t_i, t_{i+1}) \frac{1}{i^{1-x}}.
\]
Now from (2.6), we have

$$\kappa_p(t_n, t_{n+q}) \leq p(t_n, t_{n+1})\kappa_p(t_{n+1}, t_{n+1}) + (S_{n+q-1} - S_n).$$  \hspace{1cm} (2.7)

By ratio test and using the condition (2.2), we get guaranty of existence of \(\lim_{n \to \infty} S_n\). Hence \(\{S_n\}\) is Cauchy. Now taking \(n \to +\infty\) in (2.7), we have

$$\lim_{n \to \infty} \kappa_p(t_n, t_{n+q}) = 0,$$

that is, \(\{t_n\}\) is a Cauchy sequence in \(\mathcal{W}, \kappa_p\), so \(\{t_n\}\) converges to some \(u \in \mathcal{W}\). Now we prove that \(u = \Theta u\). Since \(t_n \to u\) as \(n \to \infty\) and the mapping \(\Theta\) is continuous, so we have \(\Theta t_n \to \Theta u\) as \(n \to \infty\). Thus we have

$$\kappa_p(u, \Theta u) = \lim_{n \to \infty} \kappa_p(t_{n+1}, \Theta u) \lim_{n \to \infty} \kappa_p(t_n, \Theta u) = 0,$$

and hence \(u = \Theta u\).

\[\square\]

**Theorem 2.3.** Let \((\mathcal{W}, \kappa_e)\) be a complete Eb-MS and \(\Theta : \mathcal{W} \to \mathcal{W}\). Assume that \(\exists k \in (0, 1)\) and \(\Theta \in \Omega\) such that

$$\Theta(\kappa_p(0, t)) \leq \Theta(M(t)) \kappa_e^k,$$

where

$$M(t) = \max \left\{ \kappa_p(t, t), \kappa_p(t, \Theta t), \kappa_p(\Theta t, \Theta t), \frac{\kappa_p(t, \Theta t) \kappa_p(t, \Theta t) \kappa_p(\Theta t, \Theta t)}{1 + \kappa_p(t, t)} \right\}, \forall t, s \in \mathcal{W}$$

with \(\kappa_p(0, t) > 0\). Suppose that these assertions also hold

(i) \(\Theta\) is continuous;

(ii) \(\sup_{m \geq 1} \lim_{m \to \infty} \frac{p(t_{i+1} + 1, t_{i+1, 2})}{p(t_{i+1}, t_{i+1})} < 1\).

In addition, assume that, for every \(t \in \mathcal{W}\), we have \(\lim_{n \to \infty} p(t_n, t)\) and \(\lim_{n \to \infty} p(t, t_n)\) exist and are finite. Then, \(\exists t^* \in \mathcal{W}\) such that \(t^* = \Theta t^*\).

**Corollary 2.4.** Let \((\mathcal{W}, \kappa_e)\) be a complete Eb-MS and \(\Theta : \mathcal{W} \to \mathcal{W}\) be a rational type \((\alpha, \Theta)\)-contraction such that:

(i) \(\Theta\) is \(\alpha\)-admissible;

(ii) \(\exists u_0 \in \mathcal{W}\) such that \(\alpha(u_0, \Theta u_0) \geq 1\);

(iii) \(\Theta\) is continuous;

(iv) \(\sup_{m \geq 1} \lim_{m \to \infty} \frac{p(t_{i+1} + 1, t_{i+1, 2})}{p(t_{i+1}, t_{i+1})} < 1\).

In addition, assume that, for every \(t \in \mathcal{W}\), we have \(\lim_{n \to \infty} p(t_n, t)\) and \(\lim_{n \to \infty} p(t, t_n)\) exist and are finite. Then, \(\exists t^* \in \mathcal{W}\) such that \(t^* = \Theta t^*\).

**Proof.** If we take \(p(t, z) = p(z, s)\) in above Theorem 2.2, we get the conclusion. \[\square\]

**Corollary 2.5.** Let \((\mathcal{W}, \kappa_p)\) be a complete b-MS and \(\Theta : \mathcal{W} \to \mathcal{W}\) be a rational type \((\alpha, \Theta)\)-contraction such that:

(i) \(\Theta\) is \(\alpha\)-admissible;

(ii) \(\exists u_0 \in \mathcal{W}\) such that \(\alpha(u_0, \Theta u_0) \geq 1\);

(iii) \(\Theta\) is continuous.

Then, \(\exists u \in \mathcal{W}\) such that \(\Theta u = u\).

**Proof.** If we take \(p(t, z) = p(z, s) = b \geq 1\) in above Theorem 2.2. \[\square\]

**Corollary 2.6.** Let \((\mathcal{W}, \kappa)\) be a complete metric space and \(\Theta : \mathcal{W} \to \mathcal{W}\) be a rational type \((\alpha, \Theta)\)-contraction such that:
Case 01: If \( O \) is \( \alpha \)-admissible;

(ii) \( \exists u_0 \in W \text{ such that } \alpha(u_0, 0, u_0) \geq 1; \)

(iii) \( O \) is continuous.

Then, \( \exists u \in W \text{ such that } Ou = u. \)

**Proof.** If we take \( p(t, z) = p(z, s) = 1 \) in above Theorem 2.2.

**Example 2.7.** Let \( W = \{0, 1, 2\} \). Define \( p : W \times W \to [1, \infty) \) and \( \kappa_p : W \times W \to [1, \infty) \) as \( p(t, s) = 1 + ts \) and

\[
\begin{align*}
\kappa_p(2, 2) &= \kappa_p(0, 0) = \kappa_p(1, 1) = 0, \\
\kappa_p(2, 0) &= \kappa_p(0, 2) = 5, \quad \kappa_p(1, 0) = \kappa_p(0, 1) = 10, \\
\kappa_p(1, 2) &= \kappa_p(2, 1) = 30.
\end{align*}
\]

Now, define

\[
O : W \to W
\]

by

\[
O_0 = \begin{cases} 0, & \text{if } t \in \{0, 2\}, \\ 2, & \text{if } t = 1, \end{cases}
\]

and choose \( k = \frac{3}{4} \). Define \( \Theta(\beta) = e^{\sqrt{\beta}} \). Now we discuss various cases to prove the assumptions of our main result.

**Case 01:** If \( t = 0, s = 1 \), we have

\[
\Theta(\kappa_p(0, 01)) = \Theta(\kappa_p(0, 2)) = e^{\sqrt{\frac{3}{4}}} < (e^{\sqrt{\frac{10}{3})}})^{\frac{3}{4}} = \left[ \Theta \left( \max \left\{ \kappa_p(0, 1), \kappa_p(0, 00), \kappa_p(1, 01), \frac{\kappa_p(0, 00) \kappa_p(1, 01)}{1 + \kappa_p(0, 1)} \right\} \right) \right]^{\frac{3}{4}}.
\]

**Case 02:** If \( t = 0, s = 2 \), we have

\[
\Theta(\kappa_p(00, 02)) = \Theta(\kappa_p(0, 0)) = e^0 < \left[ \Theta \left( \max \left\{ \kappa_p(0, 2), \kappa_p(0, 00), \kappa_p(2, 02), \frac{\kappa_p(0, 00) \kappa_p(2, 02)}{1 + \kappa_p(0, 2)} \right\} \right) \right]^{\frac{3}{4}}.
\]

**Case 03:** If \( t = 1, s = 2 \), we have

\[
\Theta(\kappa_p(01, 02)) = \Theta(\kappa_p(2, 0)) = e^{\sqrt{\frac{5}{4}}} < (e^{\sqrt{\frac{10}{3})}})^{\frac{3}{4}} < \left[ \Theta \left( \max \left\{ \kappa_p(1, 2), \kappa_p(1, 01), \kappa_p(2, 02), \frac{\kappa_p(1, 01) \kappa_p(2, 02)}{1 + \kappa_p(1, 2)} \right\} \right) \right]^{\frac{3}{4}}.
\]

**Case 04:** If \( t = s = 0, t = 1, t = s = 2 \), we have

\[
\Theta(\kappa_p(0t, 0s)) = \Theta(\kappa_p(0, 0)) = e^0 < \left[ \Theta \left( \max \left\{ \kappa_p(t, s), \kappa_p(t, 0t), \kappa_p(s, 0s), \frac{\kappa_p(t, 0t) \kappa_p(s, 0s)}{1 + \kappa_p(t, s)} \right\} \right) \right]^{\frac{3}{4}}, \forall s, t \in W.
\]

Hence, all the conditions of above theorem are satisfied and \( O \) has a unique fixed point, which is, \( t = 0 \).

We can establish variety of results as special cases of our main Theorem 2.2.
References


