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The extended Burr XII distribution: properties and applications

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Abstract

This paper introduces a new four-parameter lifetime model called the Marshall-Olkin generalized Burr XII (MOGBXII) distribution. We derive some of its mathematical properties, including quantile and generating functions, ordinary and incomplete moments, mean residual life, and mean waiting time and order statistics. The MOGBXII density can be expressed as a linear mixture of Burr XII densities. The maximum likelihood and least squares methods are used to estimate the MOGBXII parameters. Simulation results are obtained to compare the performances of the two estimation methods for both small and large samples. We empirically illustrate the flexibility and importance of the MOGBXII distribution in modeling various types of lifetime data.

Keywords: Burr XII, least squares, maximum likelihood, mean residual life, moments, order statistics.

2010 MSC: 62E10, 60E05.

1. Introduction

The quality of statistical analysis depends on the chosen statistical distribution. Modeling real life data is frequently used to describe various phenomena finance, economics, environmental, biomedical sciences and engineering, among others. Many extended distributions have been developed and applied in several fields to model survival data. However, there still remain many important problems involving real data, which are not contemplated by existing models.

The Burr XII (BXII) distribution is proposed by Burr [9] and it has many applications in many areas including failure time modeling, reliability and acceptance sampling plans. For example, Shao [23] used the three-parameter BXII distribution to model extreme events with applications to áood frequency. Tadikamalla [24] studied the BXII model and its related models, namely: loglogistic, compound Weibull gamma, Pareto II (Lomax) and Weibull exponential distributions.

Aiming at a more flexible BXII distribution, there are various generalizations modified forms of the BXII distribution with different numbers of parameters. For example, the beta BXII (ParanaÌba et al.,

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[21]), Kumaraswamy BXII (ParanaÌba et al., [20]), McDonald BXII (Gomes et al., [11]), Marshall-Olkin extended BXII (Al-Saiarie et al., [8]), beta exponentiated BXII (Mead, [18]), exponentiated Weibull BXII (Abouelmagd et al., [2]), Kumaraswamy exponentiated BXII (Mead and Afify, [19]), Weibull BXII (Afify et al., [4]), odd exponentiated half-logistic BXII (Aldahlan and Afify, [7]) and odd Lindley BXII (Abouelmagd et al., [1]) distributions.

The survival function (SF) and probability density function (PDF) of the BXII distribution are given by

$$\overline{G}(x;\alpha,\beta) = (1+x^{\alpha})^{-\beta} \quad \text{and} \quad g(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-\beta-1}, \; x>0, \tag{1.1}$$

where $\overline{G}(x;\alpha,\beta) = 1 - G(x;\alpha,\beta)$ and $\alpha > 0$ and $\beta > 0$ are shape parameters.

The aim of this paper is to propose and study a new lifetime model called the Marshall-Olkin generalized Burr XII (MOGBXII) distribution. Its main feature is that two additional positive shape parameters are inserted in equation (1.1) to provide more áexibility for the generated distribution. We construct the MOGBXII distribution using the Marshall-Olkin generalized-G (MOG-G) family of distributions due to Yousof et al. [25], and give some of its mathematical properties. We prove that the MOGBXII distribution is capable of modeling various shapes of data using two data sets. It can provide better fits to both data sets.

Let $\overline{G}(x;¸) = 1 - G(x;¸)$ denotes the SF of a baseline model with parameter vector ¸, Yousof et al. [25] defined the cumulative distribution function (CDF) of their MOG-G family by

$$F(x;\delta,a,¸) = \frac{1-\overline{G}(x,¸)^{a}}{1-(1-\delta)\,\overline{G}(x,¸)^{a}}, \quad x \in \mathbb{R}, \tag{1.2}$$

where are two positive shape parameters representing the different patterns of the MOG-G family. The corresponding PDF of (1.2) is

$$f(x;\delta,a,¸) = \frac{\delta a\, g(x;¸)\; \overline{G}(x,¸)^{a-1}}{\left[1-(1-\delta)\,\overline{G}(x,¸)^{a}\right]^{2}}, \quad x \in \mathbb{R}, \tag{1.3}$$

where $g(x;¸)$ is the baseline PDF, $\delta > 0$ and $a > 0$ are two shape parameters. Generally a random variable X with PDF (1.3) is denoted by X ~MOG-G$(\delta, a, ¸)$. If $\delta = 1$, then the MOG-G class reduces to the generalized-G (G-G) family (Gupta et al., [13]). We have the Marshall-Olkin-G (MO-G) class (Marshall and Olkin, [17]) for $a = 1$. Finally, for $\delta = a = 1$, we obtain the baseline distribution.

The hazard rate function (HRF) of the MOG-G family is

$$\tau(x;\delta,a,¸) = \frac{a\,\pi(x;¸)}{1-(1-\delta)\,\overline{G}(x,¸)^{a}}, \quad x \in \mathbb{R},$$

where $\pi(x;¸) = g(x;¸)/\overline{G}(x,¸)$ is the baseline HRF.

Using Equations (1.1) and (1.2), we obtain the four-parameter MOGBXII CDF

$$F(x;\alpha,\beta,\delta,a) = \frac{1-(1+x^{\alpha})^{-a\beta}}{1-(1-\delta)\,(1+x^{\alpha})^{-a\beta}}, \; x > 0. \tag{1.4}$$

The PDF amd HRF of the MOGBXII model are given by

$$f(x;\alpha,\beta,\delta,a) = \alpha\beta\delta a x^{\alpha-1}(1+x^{\alpha})^{-a\beta-1}\left[1-(1-\delta)(1+x^{\alpha})^{-a\beta}\right]^{-2}, \; x>0 \tag{1.5}$$

and

$$\tau(x;\alpha,\beta,\delta,a) = \frac{\alpha\beta a x^{\alpha-1}(1+x^{\alpha})^{-1}}{1-(1-\delta)(1+x^{\alpha})^{-a\beta}}, \; x>0,$$

where α, β, δ and a are positive shape parameters. The random variable X having PDF (1.5) is denoted by X ~MOGBXII$(\alpha, \beta, \delta, a)$.

The MOGBXII distribution contains some special cases.

1. For $\delta = 1$, the MOGBXII model reduces to the generalized BXII distribution.
2. For $a = 1$, we have the Marshall-Olkin BXII distribution.
3. The case $\alpha = 1$, refers to the MOG-Lomax (MOGLx) distribution.
4. The case $\beta = 1$, refers to the MOG-log-logistic (MOGLL) distribution.
5. The MOGBXII model reduces to the generalized Lx distribution for $\delta = \alpha = 1$.
6. The MOGBXII model reduces to the generalized LL distribution for $\delta = \beta = 1$.
7. The case $a = \alpha = 1$, refers to the MOLx distribution.
8. The case $a = \beta = 1$, refers to the MOLL distribution.
9. For $\delta = a = 1$, we obtain the BXII distribution.
10. For $\delta = a = \alpha = 1$, we obtain the standard Lx distribution.
11. For $\delta = a = \beta = 1$, we obtain the standard LL distribution.

Some plots of the PDF of the MOGBXII distribution for some selected parameter values are displayed in Figure 1. Figure 2 shows some plots of the MOGBXII HRF for some parameter values. One can see that, the MOGBXII HRF can be increasing, decreasing and upside down bathtub. Its PDF can be reversed-J shaped, symmetric, left skewed and right skewed.

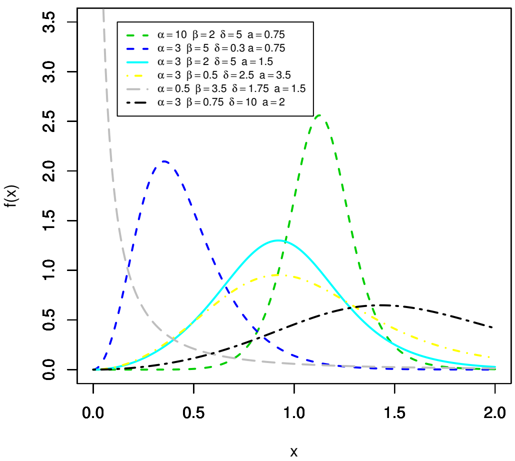

Figure 1: Plots of the PDF of the MOGBXII distribution for some parameter values.

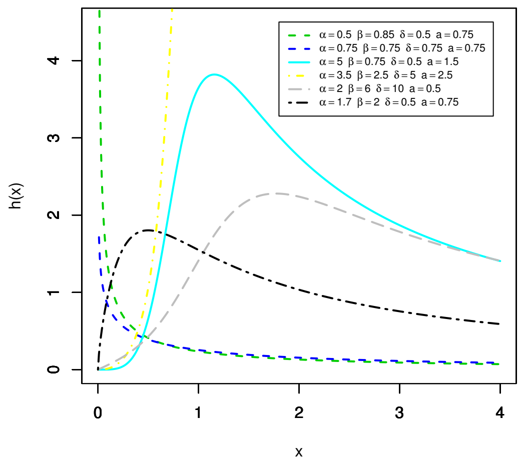

Figure 2: Plots of the HRF of the MOGBXII distribution for some parameter values.

The remainder of the paper is organized as follows. In Section 2, we investigate some mathematical properties of the MOGBXII distribution including linear representation for its PDF, quantile and generating functions, ordinary and incomplete moments, mean residual life, mean waiting time and order statistics. The maximum likelihood and least squares estimation methods are discussed in Section 3. A simulation study is carried out to compare the performance of the two methods of estimation in Section 4. In Section 5, the MOGBXII distribution is applied to two real data sets to illustrate its importance. Finally, in Section 6, we give some concluding remarks.

2. Properties of the MOGBXII distribution

We provide some mathematical properties of the MOGBXII distribution including linear representation, quantile function (QF), moment generating function (MGF), ordinary and incomplete moments, mean residual life, mean waiting time and order statistics.

2.1. Linear representation

According to Yousof et al. [25], the CDF of the MOG-G family can be rewritten as

$$F(x) = \sum_{k=0}^{\infty} \upsilon_k \; G(x)^k,$$

where $\upsilon_0 = (2/\delta)$ and for $k \geqslant 1$, we have

$$\upsilon_k = \frac{1}{\delta}\left(\varphi_k - \frac{1}{\delta}\sum_{r=1}^{k} c_r \, \upsilon_{k-r}\right),$$

where $\varphi_k = (-1)^{k+1}\binom{\eta}{k}$ and $c_r = (1-\delta)(-1)^{r+1}\binom{\eta}{r}$. Then, the PDF of the MOG-G family can be expressed as

$$f(x) = \sum_{k=0}^{\infty} \upsilon_{k+1} \; h_{k+1}(x),$$

where $h_{k+1}(x) = (k+1) g(x) G(x)^k$ is the exp-G density with power parameter $k > 0$.

Then, the PDF of the MOGBXII can be expressed as

$$f(x) = \sum_{k=0}^{\infty} \upsilon_{k+1} \; (k+1)\alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-\beta-1}\left[1-(1+x^{\alpha})^{-\beta}\right]^{k}.$$

Consider the power series $(1-z)^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} z^m$, $|z| < 1$ and n positive integer, for the term $\left[1-(1+x^{\alpha})^{-\beta}\right]^{k}$, then we have

$$f(x) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \upsilon_{k+1} \; (-1)^m \binom{k}{m} (k+1)\alpha\beta x^{\alpha-1}(1+x^{\alpha})^{-\beta(m+1)-1}.$$

Then, the MOGBXII density (1.5) can be expressed as a linear mixture of BXII densities as follows

$$f(x) = \sum_{m=0}^{\infty} \zeta_m \; g(x;\alpha,(m+1)\beta), \tag{2.1}$$

where $g(x;\alpha,(m+1)\beta) = \alpha(m+1)\beta x^{\alpha-1}(1+x^{\alpha})^{-\beta(m+1)-1}$ is the BXII PDF with parameters $\alpha > 0$ and $(m+1)\beta > 0$, and

$$\zeta_m = \sum_{k=m}^{\infty} \frac{\upsilon_{k+1} \; (-1)^m (k+1)}{m+1}\binom{k}{m}.$$

Based on equation (2.1), several properties of the MOGBXII distribution can be obtained from those properties of the BXII distribution.

Let Y be a random variable with BXII distribution (1.1) with positive parameters α and β. For $s < \alpha\beta$, the s^{th} ordinary and incomplete moments of Y are, respectively, given by

$$\mu'_{s,Y} = \beta\, B\left(\beta - \frac{s}{\alpha}, 1+\frac{s}{\alpha}\right) \qquad \text{and} \qquad \varphi_{s,Y}(t) = \beta\, B\left(t^{\alpha}; \beta - \frac{s}{\alpha}, 1+\frac{s}{\alpha}\right),$$

where $B(b,c) = \int_0^{\infty} t^{b-1}(1+t)^{-(b+c)}dt$ is the beta function and $B(z;b,c) = \int_0^{z} t^{b-1}(1+t)^{-(b+c)}dt$ is the incomplete beta function of the second type.

2.2. Quantile and generating functions

The QF of X is obtained by inverting (1.4) as

$$x_u = \left\{ \left[\frac{1-u}{1-(1-\delta)\,u} \right]^{\frac{-1}{\alpha\beta}} - 1 \right\}^{\frac{1}{\alpha}}, \quad 0 < u < 1. \tag{2.2}$$

By setting $u = 0.5$ in (2.2), we obtain the median of X. Simulating the MOGBXII random variable is straightforward. If U is a uniform variate on the unit interval $(0, 1)$, then the random variable $X = x_u$ at $u = U$ follows equation (1.5).

The MGF of X, $M_X(t) = E[\exp(tX)]$, can be obtained from (2.1) as

$$M_X(t) = \sum_{m=0}^{\infty} \zeta_m \, M_{m+1}(t),$$

where $M_{m+1}(t)$ is the MGF of the BXII distribution with parameters α and $(m+1)\beta$. We provide a simple representation for the MGF, $M(t)$, of the BXII(α, β) model similarly to Paranaíba et al. [21] who derived a simple representation for the MGF of the three-parameter BXII distribution.

For $t < 0$, one can write

$$M(t) = \alpha\beta \int_0^{\infty} \exp(wt)\, w^{\alpha-1} (1 + w^{\alpha})^{-\beta-1}\, dw.$$

The Meijer G-function is defined by

$$G^{r,n}_{p,q}\left(x \middle| \begin{array}{c} c_1, \ldots, c_p \\ d_1, \ldots, d_q \end{array}\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(d_j + t) \prod_{j=1}^{n} \Gamma(1 - c_j - t)}{\prod_{j=n+1}^{p} \Gamma(c_j + t) \prod_{j=r+1}^{p} \Gamma(1 - d_j - t)} x^{-t} dt,$$

where $i = \sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik, [12, Section 9.3]). The Meijer G-function contains as particular cases many integrals with elementary and special functions (Prudnikov et al., [22]). We now assume that $\alpha = r/\beta$, where r and β are positive integers.

Hence, we have the following result, which holds for r and k positive integers, $\mu > -1$ and $p > 0$ (Prudnikov et al., [22, p. 21]),

$$I\left(p, \mu, \frac{r}{\beta}, \nu\right) = \int_0^{\infty} \exp(-px)\, x^{\mu} \left(1 + x^{\frac{r}{\beta}}\right)^{\nu} dx = D G^{\beta, \beta+r}_{\beta+r, \beta}\left(\frac{r^r}{p^r} \middle| \begin{array}{c} \triangle(r, -\mu), \triangle(\beta, \nu+1) \\ \triangle(\beta, 0) \end{array} \right),$$

where $D = \frac{\beta^{-\nu} r^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{r-1}{2}} \Gamma(-\nu)\, p^{\mu+1}}$ and $\triangle(\beta, b) = \frac{b}{\beta}, \frac{b+1}{\beta}, \ldots, \frac{b+\beta}{\beta}$. Hence, we can write (for $t < 0$)

$$M(t) = r\, I\left(-t, \frac{r}{\beta} - 1, \frac{r}{\beta}, -\beta - 1\right).$$

Then, the MGF of X reduces to

$$M_X(t) = r \sum_{m=0}^{\infty} \zeta_m\, I\left(-t, \frac{r}{\beta(m+1)} - 1, \frac{r}{\beta(m+1)}, -\beta(m+1) - 1\right).$$

2.3. Ordinary and incomplete moments

The r^{th} ordinary moment of X is given by

$$\mu'_r = E(X^r) = \sum_{m=0}^{\infty} \zeta_m \int_{-\infty}^{\infty} x^r\, g(x; \alpha, (m+1)\beta)\, dx.$$

Then, we obtain

$$\mu_r' = \sum_{m=0}^{\infty} \zeta_m \ (m+1)\,\beta\, B\left((m+1)\,\beta - \frac{r}{\alpha}, 1+\frac{r}{\alpha}\right), \ r < \alpha\beta\,(m+1)\,. \tag{2.3}$$

We have the mean of X, by setting $r = 1$ in (2.3).

The r^{th} incomplete moment, $\varphi_r(t)$, of the MOGBXII distribution is defined using equation (2.1) as

$$\varphi_r(t) = \int_0^t x^r f(x)\,dx = \sum_{m=0}^{\infty} \zeta_m \int_0^t x^r\, g\,(x;\alpha,(m+1)\,\beta)\,dx.$$

Then, we can write

$$\varphi_r(t) = \sum_{m=0}^{\infty} \zeta_m \ (m+1)\,\beta\, B\left(z^\alpha;(m+1)\,\beta - \frac{r}{\alpha}, 1+\frac{r}{\alpha}\right), \ r < \alpha\beta\,(m+1)\,. \tag{2.4}$$

The first incomplete moment of X, is determined from equation (2.4) by setting $r = 1$ as

$$\varphi_1(t) = \sum_{m=0}^{\infty} \zeta_m \ (m+1)\,\beta\, B\left(z^\alpha;(m+1)\,\beta - \frac{1}{\alpha}, 1+\frac{1}{\alpha}\right), \tag{2.5}$$

which has important applications related to the mean residual life, mean waiting time, Bonferroni and Lorenz curves.

2.4. *Mean residual life and mean waiting time*

The mean residual life (MRL) (or life expectancy at age t) represents the expected additional life length for a unit, which is alive at age t and it is defined by $m_X(t) = E\,(X - t \mid X > t)\,,\ t > 0$.

The MRL has some applications in life insurance, biomedical sciences, economics, demography, product technology, maintenance and product quality control (Lai and Xie, [14]).

The MRL of X, can be defined as

$$m_X(t) = [1 - \varphi_1(t)]\,/S\,(t) - t, \tag{2.6}$$

where $S\,(t) = 1 - F(x)$ is the SF of the MOGBXII distribution and $\varphi_1(t)$ is given in (2.5).

By substituting (2.5) in equation (2.6), we have MRL of the MOGBXII distribution as

$$m_X(t) = \frac{1}{S\,(t)} \sum_{m=0}^{\infty} \zeta_m \ (m+1)\,\beta\, B\left(z^\alpha;(m+1)\,\beta - \frac{1}{\alpha}, 1+\frac{1}{\alpha}\right) - t.$$

The mean waiting time (MWT) (or mean inactivity time) is defined by $M_X(t) = E\,[t - X \mid X \leqslant t]$, $t > 0$, represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0, t)$.

The MWT of X can be defined as

$$M_X(t) = t - [\varphi_1(t)/F\,(t)]\,. \tag{2.7}$$

By inserting (2.5) in equation (2.7), the MWT of the MOGBXII distribution reduces to

$$M_X(t) = t - \frac{1}{F\,(t)} \sum_{m=0}^{\infty} \zeta_m \ (m+1)\,\beta\, B\left(z^\alpha;(m+1)\,\beta - \frac{1}{\alpha}, 1+\frac{1}{\alpha}\right).$$

2.5. Order statistics

Let $X_1, \ldots, X_n$ be a random sample of size n from the MOGBXII distribution and let $X_{1:n}, \ldots, X_{n:n}$ be the corresponding order statistics. Then, the PDF of the of i^{th} order statistic, $X_{i:n}$, is defined by

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!} f(x)\; F(x)^{i-1} [1-F(x)]^{n-i}.$$

Then, the PDF of the i^{th} order statistic of the MOBXII distribution reduces to

$$\begin{aligned} f_{i:n}(x) = &\frac{n!}{(i-1)!\,(n-i)!} \frac{\alpha\beta\delta a x^{\alpha-1}(1+x^\alpha)^{-a\beta-1}}{\left[1-(1-\delta)(1+x^\alpha)^{-a\beta}\right]^2} \\ &\times \left[\frac{1-(1+x^\alpha)^{-a\beta}}{1-(1-\delta)(1+x^\alpha)^{-a\beta}}\right]^{i-1} \left[1-\frac{1-(1+x^\alpha)^{-a\beta}}{1-(1-\delta)(1+x^\alpha)^{-a\beta}}\right]^{n-i}. \end{aligned} \tag{2.8}$$

Hence, the PDF of the first order statistic $X_{1:n}$ follows from (2.8) with $i = 1$, as

$$f_{1:n}(x) = \frac{n\delta^n \alpha\beta a x^{\alpha-1}(1+x^\alpha)^{-na\beta-1}}{\left[1-(1-\delta)(1+x^\alpha)^{-a\beta}\right]^{n+1}}.$$

The PDF of the largest order statistic $X_{n:n}$ is given by

$$f_{n:n}(x) = \frac{n\alpha\beta\delta a x^{\alpha-1}(1+x^\alpha)^{-a\beta-1}\left[1-(1+x^\alpha)^{-a\beta}\right]^{n-1}}{\left[1-(1-\delta)(1+x^\alpha)^{-a\beta}\right]^{n+1}}.$$

3. Estimation methods

3.1. Maximum likelihood estimation

The estimation of the MOGBXII parameters from complete samples only is considered by the maximum likelihood method. Let $x_1, \ldots, x_n$ be a random sample of the MOGBXII distribution with parameter vector $\theta = (\alpha, \beta, \delta, a)^T$. The log-likelihood function for θ is

$$\begin{aligned} \ell = &\, n\log\alpha + n\log\beta + n\log\delta + n\log a + (\alpha-1)\sum_{i=1}^n \log x_i \\ &- (a\beta+1)\sum_{i=1}^n \log(1+x_i^\alpha) - 2\sum_{i=1}^n \log\left[1-(1-\delta)(1+x_i^\alpha)^{-a\beta}\right]. \end{aligned} \tag{3.1}$$

The maximum likelihood estimators (MLEs) can be obtained by maximizing (3.1) either by using the different programs such as R, SAS or by solving the nonlinear likelihood equations obtained by differentiating (3.1).

The score vector elements, $\mathbf{U}(\Theta) = \frac{\partial\ell}{\partial\theta} = \left(\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial a}\right)^T$, can be obtained from the corresponding author upon request.

3.2. Least squares estimation

The estimation of the MOGBXII parameters is performed using least squares as an alternative method to maximum likelihood estimation. Let $x_{1:n} < x_{2:n} < \cdots < x_{n:n}$ be the sample order statistics of size n from the MOGBXII distribution, hence the least squares estimators (LSEs) of the MOGBXII parameters α, β, δ and a can be obtained by minimizing

$$V(\alpha, \beta, \delta, a) = \sum_{i=1}^n \left[\frac{1-\left(1+x_{i:n}^\alpha\right)^{-a\beta}}{1-(1-\delta)\left(1+x_{i:n}^\alpha\right)^{-a\beta}} - \frac{i}{n+1}\right]^2,$$

with respect to α, β, δ and a. Furthermore, the LSEs follow by solving the non-linear equations

$$\sum_{i=1}^{n}\left[\frac{1-\left(1+x_{i:n}^{\alpha}\right)^{-a\beta}}{1-(1-\delta)\left(1+x_{i:n}^{\alpha}\right)^{-a\beta}}-\frac{i}{n+1}\right]\Delta_s\left(x_{i:n}|\alpha,\beta,\delta,a\right)=0,\ s=1,2,3,4,$$

$$\Delta_1\left(x_{i:n}|\alpha,\beta,\delta,a\right)=\frac{\partial}{\partial\alpha}F\left(x_{i:n}|\alpha,\beta,\delta,a\right),$$

$$\Delta_2\left(x_{i:n}|\alpha,\beta,\delta,a\right)=\frac{\partial}{\partial\beta}F\left(x_{i:n}|\alpha,\beta,\lambda\right),$$

$$\Delta_3\left(x_{i:n}|\alpha,\beta,\delta,a\right)=\frac{\partial}{\partial\delta}F\left(x_{i:n}|\alpha,\beta,\delta,a\right),$$

$$\Delta_4\left(x_{i:n}|\alpha,\beta,\delta,a\right)=\frac{\partial}{\partial a}F\left(x_{i:n}|\alpha,\beta,\delta,a\right).$$

4. Simulation study

In this section, we conduct a simulation study for different sample sizes to assess the accuracy of the MLEs and LSEs for the parameters of the MOGBXII distribution. We consider eight different combinations (Comb) for the parameters α, β, δ and a. Comb 1: $\alpha = 3.5$, $\beta = 0.5$, $\delta = 1.5$, $a = 0.75$. Comb 2: $\alpha = 3.5$, $\beta = 0.5$, $\delta = 0.75$, $a = 0.75$. Comb 3: $\alpha = 3.5$, $\beta = 1.5$, $\delta = 0.75$, $a = 0.75$. Comb 4: $\alpha = 1.5$, $\beta = 1.5$, $\delta = 0.75$, $a = 0.75$. Comb 5: $\alpha = 1.5$, $\beta = 1.5$, $\delta = 5$, $a = 2.5$. Comb 6: $\alpha = 3.5$, $\beta = 1.5$, $\delta = 5$, $a = 2.5$. Comb 7: $\alpha = 3.5$, $\beta = 0.5$, $\delta = 5$, $a = 2.5$. Comb 8: $\alpha = 3.5$, $\beta = 0.5$, $\delta = 5$, $a = 0.75$.

The simulation results are obtained from 1000 Monte Carlo replications using the R software by taking sample sizes $n = 20$, 50, 100, and 200.

Tables 1-4 list the mean values of the estimates and mean square errors (MSEs) (in parentheses) of the MLEs and LSEs of the MOGBXII parameters. The values in these tables illustrate that the MSEs decrease when the sample size increases as expected under first-order asymptotic theory for the MLEs and LSEs.

Table 1: Average values of the estimates and their corresponding MSEs.

Parameters	$n = 20$		$n = 50$	
	MLEs	LSEs	MLEs	LSEs
$\alpha = 3.50$	3.7470(0.5967)	3.4506(0.6373)	3.6297(0.4264)	3.5040(0.4754)
$\beta = 0.50$	0.5196(0.0099)	0.5180(0.0097)	0.5230(0.0088)	0.5224(0.0082)
$\delta = 1.50$	1.5971(0.1554)	1.4914(0.1565)	1.5488(0.1366)	1.5048(0.1383)
$a = 0.75$	0.7356(0.0161)	0.7403(0.0155)	0.7377(0.0148)	0.7398(0.0142)
	$n = 100$		$n = 200$	
$\alpha = 3.50$	3.5890(0.2942)	3.5185(0.3594)	3.5743(0.1787)	3.5578(0.2375)
$\beta = 0.50$	0.5158(0.0079)	0.5161(0.0081)	0.5221(0.0067)	0.5193(0.0070)
$\delta = 1.50$	1.5408(0.1171)	1.5256(0.1240)	1.5321(0.0918)	1.5175(0.0979)
$a = 0.75$	0.7391(0.0135)	0.7416(0.0129)	0.7270(0.0135)	0.7284(0.0125)
	$n = 20$		$n = 50$	
$\alpha = 3.50$	3.7242(0.5096)	3.4965(0.5427)	3.6081(0.3311)	3.5001(0.3604)
$\beta = 0.50$	0.5134(0.0104)	0.4936(0.0106)	0.5131(0.0095)	0.5047(0.0089)
$\delta = 0.75$	0.7537(0.0375)	0.6871(0.0458)	0.7483(0.0336)	0.7165(0.0389)
$a = 0.75$	0.7340(0.0174)	0.7161(0.0186)	0.7302(0.0165)	0.7147(0.0167)
	$n = 100$		$n = 200$	
$\alpha = 3.50$	3.5849(0.1997)	3.5538(0.2381)	3.5474(0.1123)	3.5604(0.1404)
$\beta = 0.50$	0.5158(0.0080)	0.5026(0.0087)	0.5191(0.0071)	0.5083(0.0080)
$\delta = 0.75$	0.7531(0.0277)	0.7247(0.0328)	0.7482(0.0224)	0.7226(0.0283)
$a = 0.75$	0.7290(0.0131)	0.7219(0.0148)	0.7262(0.0129)	0.7159(0.0141)

Table 2: Average values of the estimates and their corresponding MSEs.

Parameters	$n = 20$		$n = 50$	
	MLEs	LSEs	MLEs	LSEs
$\alpha = 3.50$	3.7338(0.4084)	3.5265(0.4225)	3.6181(0.2155)	3.5294(0.2231)
$\beta = 1.50$	1.6188(0.0658)	1.5779(0.0567)	1.6095(0.0574)	1.5774(0.0477)
$\delta = 0.75$	0.7374(0.0416)	0.6975(0.0461)	0.7312(0.0398)	0.7103(0.0431)
$a = 0.75$	0.7037(0.0227)	0.6669(0.0270)	0.6976(0.0193)	0.6821(0.0218)
	$n = 100$		$n = 200$	
$\alpha = 3.50$	3.5831(0.1339)	3.5492(0.1496)	3.5571(0.0837)	3.5473(0.0908)
$\beta = 1.50$	1.5800(0.0473)	1.5679(0.0452)	1.5824(0.0455)	1.5728(0.0435)
$\delta = 0.75$	0.7436(0.0385)	0.7279(0.0423)	0.7370(0.0345)	0.7234(0.0390)
$a = 0.75$	0.7090(0.0183)	0.6953(0.0206)	0.7050(0.0163)	0.6952(0.0189)
	$n = 20$		$n = 50$	
$\alpha = 1.50$	1.5902(0.0603)	1.5049(0.0569)	1.5581(0.0396)	1.5220(0.0390)
$\beta = 1.50$	1.5779(0.0484)	1.5322(0.0449)	1.5583(0.0391)	1.5470(0.0372)
$\delta = 0.75$	0.7359(0.0477)	0.6932(0.0516)	0.7360(0.0453)	0.7115(0.0465)
$a = 0.75$	0.7193(0.0246)	0.6870(0.0273)	0.7123(0.0201)	0.6907(0.0227)
	$n = 100$		$n = 200$	
$\alpha = 1.50$	1.5382(0.0236)	1.5182(0.0255)	1.5174(0.0148)	1.5132(0.0168)
$\beta = 1.50$	1.5553(0.0366)	1.5507(0.0365)	1.5487(0.0359)	1.5458(0.0355)
$\delta = 0.75$	0.7406(0.0400)	0.7274(0.0433)	0.7582(0.0335)	0.7282(0.0373)
$a = 0.75$	0.7146(0.0177)	0.6998(0.0197)	0.7317(0.0145)	0.7071(0.0177)

Table 3: Average values of the estimates and their corresponding MSEs.

Parameters	$n = 20$		$n = 50$	
	MLEs	LSEs	MLEs	LSEs
$\alpha = 1.50$	1.5739(0.0216)	1.5371(0.0158)	1.5544(0.0155)	1.5324(0.0127)
$\beta = 1.50$	1.5549(0.0145)	1.5523(0.0143)	1.5358(0.0093)	1.5379(0.0098)
$\delta = 5.00$	5.1006(0.3603)	5.0812(0.3492)	5.0458(0.3306)	5.0653(0.3196)
$a = 2.50$	2.4944(0.0211)	2.4877(0.0205)	2.4682(0.0154)	2.4722(0.0158)
	$n = 100$		$n = 200$	
$\alpha = 1.50$	1.5367(0.0104)	1.5231(0.0089)	1.5204(0.0061)	1.5108(0.0057)
$\beta = 1.50$	1.5226(0.0058)	1.5270(0.0064)	1.5216(0.0047)	1.5220(0.0044)
$\delta = 5.00$	4.9702(0.3007)	4.9831(0.2780)	4.9564(0.2879)	4.9861(0.2635)
$a = 2.50$	2.4598(0.0121)	2.4603(0.0127)	2.4523(0.0109)	2.4595(0.0108)
	$n = 20$		$n = 50$	
$\alpha = 3.50$	3.6632(0.4254)	3.4344(0.3882)	3.5765(0.2043)	3.4491(0.2226)
$\beta = 1.50$	1.5543(0.0148)	1.5500(0.0135)	1.5434(0.0100)	1.5436(0.0098)
$\delta = 5.00$	5.1037(0.3538)	5.2474(0.3836)	5.1027(0.3624)	5.2676(0.3714)
$a = 2.50$	2.4927(0.0202)	2.4820(0.0189)	2.4761(0.0147)	2.4820(0.0148)
	$n = 100$		$n = 200$	
$\alpha = 3.50$	3.5207(0.1004)	3.4468(0.1168)	3.5300(0.0555)	3.4802(0.0638)
$\beta = 1.50$	1.5313(0.0068)	1.5367(0.0070)	1.5286 (0.0055)	1.5339(0.0057)
$\delta = 5.00$	5.1160(0.3671)	5.2257(0.3428)	5.0689(0.3510)	5.1916(0.3217)
$a = 2.50$	2.4707(0.0123)	2.4698(0.0116)	2.4626(0.0113)	2.4718(0.0103)

Table 4: Average values of the estimates and their corresponding MSEs.

Parameters	$n = 20$		$n = 50$	
	MLEs	LSEs	MLEs	LSEs
$\alpha = 3.50$	3.6384(0.4344)	3.3653(0.3974)	3.5883(0.3283)	3.4301(0.3217)
$\beta = 0.50$	0.5164(0.0061)	0.5249(0.0067)	0.5110(0.0044)	0.5195(0.0046)
$\delta = 5.00$	5.2787(0.4241)	5.1641(0.3808)	5.2082(0.3843)	5.1764(0.3577)
$a = 2.50$	2.4732(0.0162)	2.4872(0.0153)	2.4771(0.0150)	2.4890(0.0146)
	$n = 100$		$n = 200$	
$\alpha = 3.50$	3.5574(0.1993)	3.4742(0.2104)	3.5151(0.1204)	3.4712(0.1351)
$\beta = 0.50$	0.5087(0.0033)	0.5130(0.0034)	0.5055(0.0023)	0.5099(0.0024)
$\delta = 5.00$	5.1954(0.3654)	5.1834(0.3469)	5.1570(0.3455)	5.1824(0.3311)
$a = 2.50$	2.4877(0.0138)	2.4968(0.0139)	2.5055(0.0142)	2.5083(0.0131)
	$n = 20$		$n = 50$	
$\alpha = 3.50$	3.8010(0.6536)	3.3419(0.4971)	3.6868(0.5638)	3.3831(0.4880)
$\beta = 0.50$	0.5223(0.0080)	0.5577(0.0115)	0.5287(0.0077)	0.5553(0.0103)
$\delta = 5.00$	5.2373(0.4039)	5.0769(0.3543)	5.2130(0.3801)	5.0983(0.3481)
$a = 0.75$	0.7071(0.0060)	0.7312(0.0047)	0.7150(0.0051)	0.7293(0.0043)
	$n = 100$		$n = 200$	
$\alpha = 3.50$	3.6362(0.4524)	3.4557(0.4469)	3.6017(0.3412)	3.4615(0.3839)
$\beta = 0.50$	0.5289(0.0071)	0.5463(0.0089)	0.5265(0.0059)	0.5435(0.0078)
$\delta = 5.00$	5.1916(0.3544)	5.1315(0.3382)	5.1326(0.3124)	5.1133(0.3048)
$a = 0.75$	0.7131(0.0050)	0.7213(0.0042)	0.7160(0.0043)	0.7225(0.0041)

One can see, from Tables 2-4, that the maximum likelihood method performs better than least square method in most cases in terms of minimum MSEs, however, the two methods perform very well.

All in all, the results show that the MSEs tend to zero as the sample size increase in all cases which indicate that the MLEs and LSEs are consistent.

5. Two applications

In this section, we illustrate the flexibility and importance of the MOGBXII distribution empirically by two real data applications. The first data set consists of 128 observations of bladder cancer patients which represents the remission times (in months) (Lee and Wang, [15]). These data have been analyzed by Cordeiro et al. [10] and Afify et al. [6]. The second data set contains 346 observations and refers to nicotine measurements, made from several brands of cigarettes in 1998, collected by the Federal Trade. These data have been analyzed by Afify et al. [5].

Table 5 lists the competitive models of the MOGBXII distribution which will be compared with it.

Table 5: Fitted competitive distributions of the MOGBXII model.

Distribution	Abbreviation	Author(s)
Burr XII (special case of MOGBXII)	BXII	Burr [9]
Transmuted BXII	TBXII	Afify et al. [3]
Weibull BXII	WBXII	Afify et al. [4]
Odd exponentiated half-logistic BXII	OEHLBXII	Aldahlan and Afify [7]
Odd Lindley BXII	OLBXII	Abouelmagd et al. [1]
Transmuted transmuted BXII	TTBXII	Mansour et al. [16]
Generalized Lomax	GLx	Special case of MOGBXII
Marshall-Olkin log-logistic	MOLL	Special case of MOGBXII

We consider the Kolmogorov Smirnov (KS) statistic, its P-value (PV), Cramér-von Mises (W^*) and Anderson-Darling (A^*) statistics to compare the fitted distributions.

Tables 6 and 7 list the values of the MLEs and their corresponding standard errors (in parentheses) of the MOGBXII parameters and other fitted models parameters. These tables also show the values KS, PV, W^* and A^* statistics for both data sets.

In Tables 6 and 7, we compare the MOGBXII model with the TBXII, WBXII, OEHLBXII, OLBXII, TTBXII, GLx, MOLL and BXII distributions. We note that the MOGBXII model gives the lowest values for the KS, W^* and A^* statistics and the largest value of the PV among all fitted models. So, the MOGBXII model could be chosen as the best model to explain both data sets.

Table 6: The MLEs, their (SEs) and the KS, PV, W^* and A^* measures for cancer data.

Model	Estimates	KS	PV	W^*	A^*
MOGBXII	0.8125, 0.4586, 93.2647, 5.8515	0.0301	0.9998	0.0137	0.0871
$(\alpha, \beta, \delta, a)$	(0.6229)(7.4929)(195.018)(95.880)				
TBXII	6.5235, 6.3067, 3.5559, 0.211	0.0304	0.9997	0.0152	0.0987
$(\alpha, \beta, \theta, \lambda)$	(0.964)(6.009)(0.372)(1.077)				
WBXII	0.7186, 0.1317, 41.6406, 2.7679	0.0479	0.9300	0.0469	0.3105
(α, β, a, b)	(0.2890)(0.2015)(264.275)(1.0101)				
OEHLBXII	2.9623, 0.7077, 0.5081, 1.1006	0.0406	0.9839	0.0325	0.2136
(α, λ, a, b)	(4.8260)(1.7634)(1.1386)(3.1994)				
OLBXII	1.4463, 0.1016, 9.8827, 16.8443	0.0349	0.9976	0.0199	0.1353
$(\alpha, \beta, \sigma, \theta)$	(0.2040)(0.3702)(9.4003)(74.818)				
TTBXII	1.4215, 0.7035, -0.8918, -0.8521	0.1084	0.0983	0.3144	2.0176
$(\alpha, \beta, \lambda, a)$	(0.2759)(0.1490)(0.1369)(0.1029)				
GLx	0.6317, 0.7928	0.3157	0.0000	0.2738	1.7756
(a, β)	(19.446)(24.406)				
MOLL	0.5885, 2.8383	0.2460	0.0000	0.0969	0.6657
(α, δ)	(0.0446)(0.4342)				
BXII	2.3348, 0.2337	0.2506	0.0000	0.7497	4.5548
(α, β)	(0.3540)(0.0399)				

Table 7: The MLEs, their (SEs) and the KS, PV, W^* and A^* measures for nicotine data.

Model	Estimates	KS	PV	W^*	A^*
MOGBXII	1.8333, 0.4300, 19.1264, 12.6791	0.0969	0.0030	0.4072	2.2308
$(\alpha, \beta, \delta, a)$	(0.2994)(3.9518)(13.004)(116.51)				
TBXII	2.3141, 453.203, 11.7134, -0.5970	0.1094	0.0005	0.4481	2.5409
$(\alpha, \beta, \theta, \lambda)$	(0.1617)(3746.6)(42.499)(0.1504)				
WBXII	0.7219, 2.1970, 0.0339, 2.7274	0.1086	0.0005	0.4522	2.6372
(α, β, a, b)	(0.2657)(0.8762)(0.0526)(1.3047)				
OEHLBXII	1.6929, 0.0832, 0.7468, 4.7463	0.1000	0.0019	0.3813	2.1561
(α, λ, a, b)	(0.2605)(0.1383)(0.3093)(2.2878)				
OLBXII	1.6515, 1.5005, 0.4585, 0.2357	0.1014	0.0016	0.3967	2.2496
$(\alpha, \beta, \sigma, \theta)$	(0.4752)(0.4752)(0.1488)(0.1586)				
TTBXII	2.7767, 2.6801, -0.4546, -0.4733	0.1277	0.0000	0.6798	3.8391
$(\alpha, \beta, \lambda, a)$	(0.2093)(0.2063)(0.1576)(0.1518)				
GLx	1.6340, 1.0203	0.3902	0.0000	1.7429	10.384
(a, β)	(59.979)(37.454)				
MOLL	3.7511, 0.4695	0.1357	0.0000	1.3148	7.6356
(α, δ)	(0.1617)(0.0443)				
BXII	3.4864, 1.8087	0.1513	0.0000	0.8847	5.1203
(α, β)	(0.1538)(0.0973)				

The histogram of both data sets, and the estimated CDF, SF and PP plots for the MOGBXII distribution are shown in Figures 3 and 4. The plots in these figures show that the MOGBXII distribution has a close fits to both data sets.

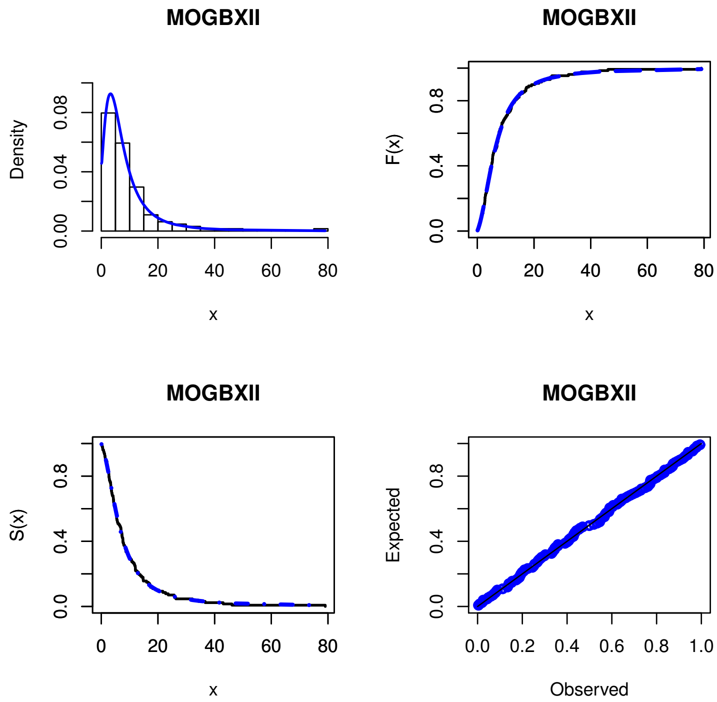

Figure 3: Fitted PDF, CDF, SF and PP plots of the MOGBXII distribution for cancer data.

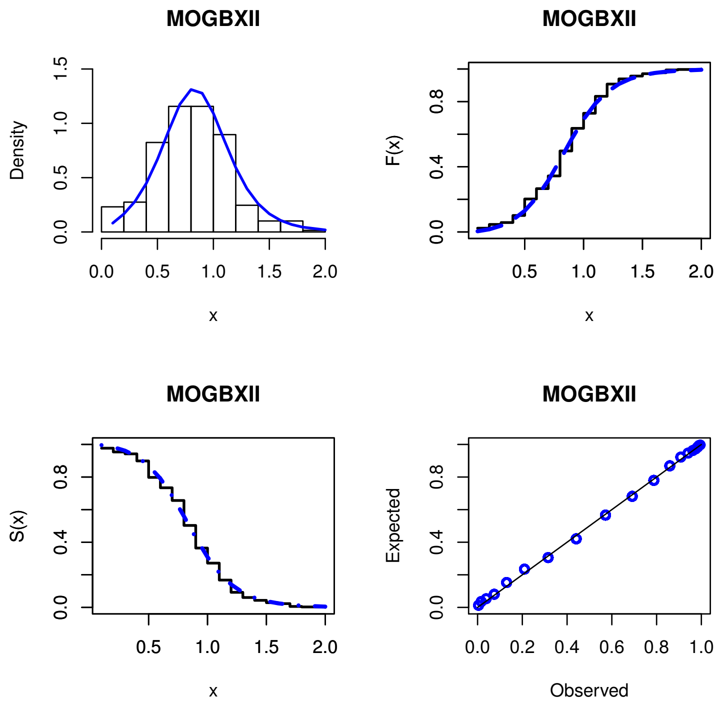

Figure 4: Fitted PDF, CDF, SF and PP plots of the MOGBXII distribution for nicotine data.

Figure 5 displays the HRF plots of the MOGBXII distribution for both data sets. It is seen that, the HRF is upside down bathtub for both data sets.

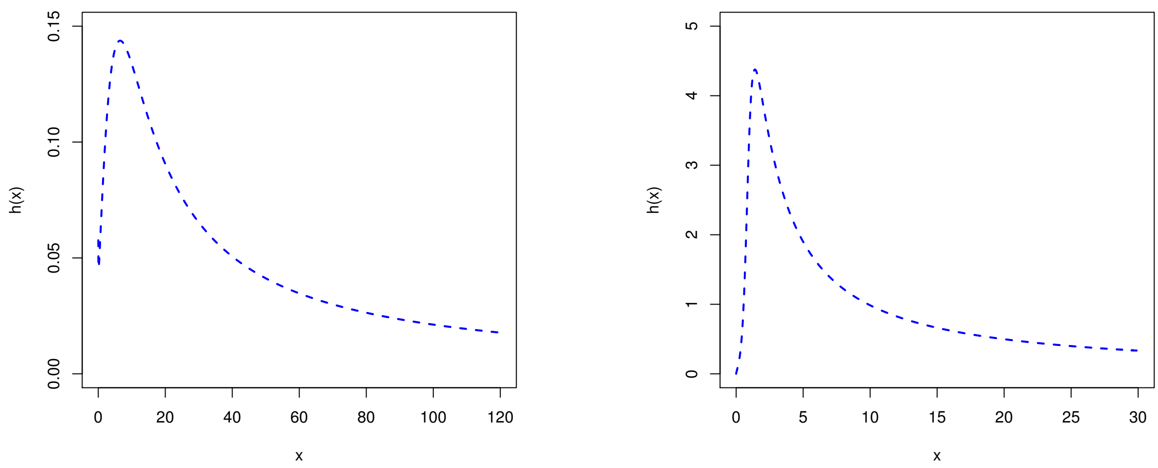

Figure 5: The HRF plots of the MOGBXII distribution for cancer data (left panel) and nicotine data (right panel).

6. Conclusions

In this paper, we propose a new four-parameter model called the MarshallOlkin generalized Burr XII (MOGBXII) distribution, which contains the Burr XII (BXII), Marshall-Olkin BXII, Marshall-Olkin log-logistic and generalized Lomax distributions, among others as special cases. The MOGBXII density function can be expressed as a linear mixture of BXII densities. Explicit expressions for some of its mathematical quantities including the quantile and generating functions ordinary and incomplete moments, mean residual life, mean waiting time and order statistics are derived. The MOGBXII parameters are estimated by the maximum likelihood and least squares methods. Monte Carlo simulation results are reported for both estimation methods. The proposed distribution provides better fits than some other nested and non-nested models by using two real data sets.

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