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The Marshall-Olkin exponentiated generalized G family of distributions: properties, applications, and characterizations

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Abstract

In this paper, we propose and study a new class of continuous distributions called the Marshall-Olkin exponentiated generalized G (MOEG-G) family which extends the Marshall-Olkin-G family introduced by Marshall and Olkin [A. W. Marshall, I. Olkin, Biometrika, **84** (1997), 641–652]. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, order statistics and probability weighted moments are derived. Some characterizations for the new family are presented. Maximum likelihood estimation for the model parameters under uncensored and censored data is addressed in Section 5 as well as a simulation study to assess the performance of the estimators. The importance and flexibility of the new family are illustrated by means of two applications to real data sets.

Keywords: Marshall-Olkin family, characterizations, censored Data, generating function, order statistics, maximum likelihood estimation.

2010 MSC: 60E05, 62E10.

1. Introduction

There has been a great deal of interest in extending and developing more flexible models through expanding the classical distributions by introducing additional shape parameters to the baseline distribution. Many generalized families of distributions have been studied and proposed over the last two decades for modeling data in many applied areas such as engineering, economics, biological studies, medical sciences, environmental sciences and finance. So, several classes of continuous distributions have been constructed by extending common G families. These generalized models give more flexibility by adding one (or more) shape parameters to the baseline model. For instance, Gupta et al. [17] introduced the exponentiated-G class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes can be cited like the beta generalized-G by Eugene et al. (2002),

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the gamma-G by Zografos and Balakrishnan [42], the exponentiated generalized-G by Cordeiro et al. [12], T-X by Alzaatreh et al. [8], exponentiated T-X by Alzaghal et al. [9], Kumaraswamy Marshall-Olkin-G by Alizadeh et al. [6], beta Marshall-Olkin-G by Alizadeh et al. [5], transmuted exponentiated generalized-G by Yousof et al. [39], transmuted geometric-G by Afify et al. [2], Kumaraswamy transmuted-G by Afify et al. [3], Burr X-G by Yousof et al. [40], exponentiated transmuted-G family by Merovci et al. [26], the complementary generalized transmuted Poisson family by Alizadeh et al. [7], beta transmuted-H by Afify et al. [4], generalized transmuted-G by Nofal et al. [31], the beta Weibull-G family by Yousof et al. [41], the Topp-Leone odd log-logistic family by Brito et al. [11], the Burr XII system of densities by Cordeiro et al. [14], new extended G family by Hamedani et al. [19], and odd power Lindley generator of probability distributions by Korkmaz et al. [20], among others.

Cordeiro et al. [12] proposed and studied the exponentiated generalized (EG) family with cdf and probability density function (pdf) given by

$$H_{a,b,\boldsymbol{\varphi}}(x) = \left[1-\overline{G}\left(x;\boldsymbol{\varphi}\right)^{a}\right]^{b}, \tag{1.1}$$

and

$$h_{a,b,\boldsymbol{\varphi}}(x) = abg\left(x;\boldsymbol{\varphi}\right)\overline{G}\left(x;\boldsymbol{\varphi}\right)^{a-1}\left[1-\overline{G}\left(x;\boldsymbol{\varphi}\right)^{a}\right]^{b-1},$$

respectively, where $\boldsymbol{\varphi}$ is the parameter vector, $\overline{G}\left(x;\boldsymbol{\varphi}\right) = 1-G\left(x;\boldsymbol{\varphi}\right)$ is the reliability function (rf) of a parent distribution. Marshall and Olkin [25] introduced a new method of adding a parameter into a family of distributions called the Marshall-Olkin-G (MO-G) family. They indicated that if $\overline{H}\left(x\right)$,$h\left(x\right)$ and $r\left(x\right)$ denote the rf, pdf and hazard rate function (hrf) of a continuous random variable X, then the MO-G family has rf defined by

$$\overline{F}_{\theta,\boldsymbol{\varphi}}(x) = \frac{\theta\overline{H}(x;\boldsymbol{\varphi})}{\theta-(1-\theta)\,\overline{H}(x;\boldsymbol{\varphi})},\ x\in\mathbb{R},\ \theta>0. \tag{1.2}$$

Clearly, when $\theta=1$, $\overline{F}(x)=\overline{H}(x)$. The corresponding pdf of (1.2) is given by

$$f_{\theta,\boldsymbol{\varphi}}(x) = \frac{\theta h(x;\boldsymbol{\varphi})}{\left[\theta-(1-\theta)\,\overline{H}(x;\boldsymbol{\varphi})\right]^{2}},\ x\in\mathbb{R},\ \theta>0.$$

By inserting (1.1) in (1.2), we obtain the cdf of the MOEG-G class

$$F\left(x\right) = \frac{\left[1-\overline{G}\left(x\right)^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}\left(x\right)^{a}\right]^{b}}, \tag{1.3}$$

with corresponding pdf given by

$$f\left(x\right) = \frac{ab\theta g\left(x\right)\overline{G}\left(x\right)^{a-1}\left[1-\overline{G}\left(x\right)^{a}\right]^{b-1}}{\left\{\theta+(1-\theta)\left[1-\overline{G}\left(x\right)^{a}\right]^{b}\right\}^{2}}, \tag{1.4}$$

where $\overline{G}\left(x\right)=\overline{G}\left(x;\boldsymbol{\varphi}\right)$ and $F\left(x\right)=F\left(x;\theta,a,b\right)$ etc. Henceforth, $X\sim$ MOEG-G($\theta,a,b,\boldsymbol{\varphi}$) denotes a random variable having density function (1.4).

Some special cases of the new family were studied recently. For instance, generalized exponential geometric extreme distribution by Ristic and Kundu [32] and Marshall-Olkin generalized Weibull distribution, Marshall-Olkin generalized Lindley distribution, Marshall-Olkin generalized Lomax distribution and Marshall-Olkin generalized Lomax distribution by Yousof et al. [40]. Furthermore, the basic motivations for using the MOEG-G family in practice are the following: to produce a skewness for symmetrical models; to define special models with all types of hrf; to construct heavy-tailed distributions for modeling various real data sets; to make the kurtosis more flexible compared to that of the baseline distribution; to

Table 1: Sub-families of the MOEG-G family

Sub-model	θ	a	b	$G(x)$	Author
EG-G family	1	a	b	$G(x)$	Cordeiro et al. [12]
G-G family	1	1	b	$G(x)$	Gupta et al. [17]
E-G family	1	a	1	$G(x)$	Cordeiro et al.[12]
MOG-G family	θ	1	b	$G(x)$	New
MOE-G family	θ	a	1	$G(x)$	New
$G(x)$	1	1	1	$G(x)$	–

generate distributions with left-skewed, right-skewed, symmetric, or reversed-J shape; to provide consistently better fits than other generated distributions under the same underlying model.

This paper is organized as follows. In Section 2 we formulate and plot two special MOEG models. In Section 3, we derive some of mathematical properties of the new family. Some useful characterizations are presented in Section 4. Maximum likelihood estimation for the model parameters under uncensored and censored data is addressed in Section 5 as well as a simulation study to assess the performance of the estimators. In Section 6, potentiality of the proposed class is illustrated by means of two real data sets. Finally, Section 7 provides some conclusions.

2. Special MOEG-G models

In this section, we provide two special models of the MOEG-G family of distributions. The pdf (1.4) will be most tractable when $G_{\Phi}(x)$ and $g_{\Phi}(x)$ have simple analytic expressions. These sub-models generalize some well-known distributions.

2.1. *The MOEG-Weibull (MOEG-W) distribution*

Consider the cdf and pdf (for $x > 0$) $G(x) = 1-\exp[-(\alpha x)^{\beta}]$ and $g(x) = \beta\alpha^{\beta}x^{\beta-1}\exp[-(\alpha x)^{\beta}]$, respectively, of the Weibull distribution with positive parameters α and β. Then, the pdf of the MOEG-W model is given by

$$f(x) = \frac{ab\theta\beta\alpha^{\beta}x^{\beta-1}\exp[-a(\alpha x)^{\beta}]\left[1-\exp[-a(\alpha x)^{\beta}]\right]^{b-1}}{\left\{\theta+(1-\theta)\left[1-\exp[-a(\alpha x)^{\beta}]\right]^{b}\right\}^{2}}.$$

The MOEG-W distribution includes EG-W if $\theta = 1$, G-W if $a = \theta = 1$, E-W if $b = \theta = 1$, MOG-W if $a = 1$, and MOE-W if $b = 1$. Plots of the density and hrf of the MOEG-W distribution for some parameter values are displayed in Figure 1.

2.2. *The MOEG-Lomax (MOEG-L) distribution*

The cdf and pdf (for $x > 0$) of the Lomax distribution with positive parameter α are $G(x) = 1-[1+x]^{-\alpha}$ and $g(x) = \alpha[1+x]^{-(\alpha+1)}$, respectively. Then, the pdf of the MOG-L distribution becomes

$$f(x) = \frac{ab\theta\alpha[1+x]^{-\alpha a-1}\{1-[1+x]^{-\alpha a}\}^{b-1}}{\left\{\theta+(1-\theta)\{1-[1+x]^{-\alpha a}\}^{b}\right\}^{2}}.$$

The MOEG-L distribution includes EG-L if $\theta = 1$, G-Lo if $a = \theta = 1$, E-L if $b = \theta = 1$, MOG-L if $a = 1$, and MOE-L if $b = 1$. Plots of the density and hrf of the MOEG-L distribution for some parameter values are displayed in Figure 2.

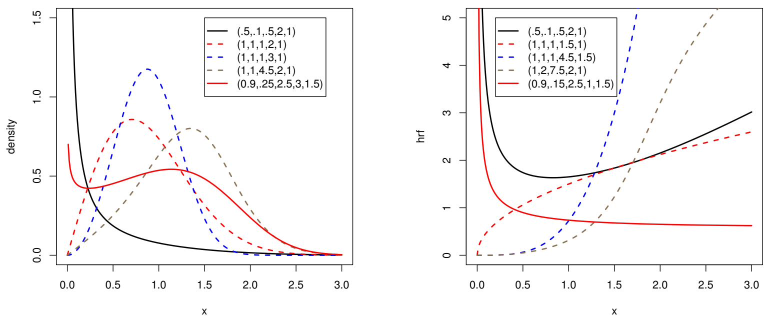

Figure 1: MOEG-W distribution: pdf (left figure), hrf (right figure).

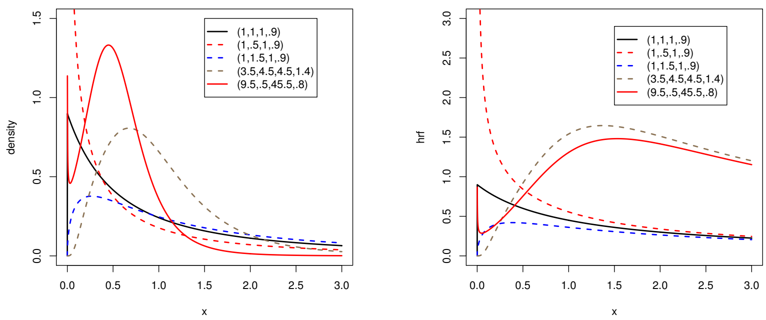

Figure 2: MOEG-L distribution: pdf (left figure), hrf (right figure).

3. Some mathematical properties

3.1. Extreme values

If $\overline{X} = (X_1 + \cdots + X_n)/n$ denotes the mean of a random sample from (1.4), then by the usual central limit theorem $\sqrt{n}(\overline{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$ under suitable conditions. Sometimes one would be interested in the asymptotic of the extreme values $M_n = \max(X_1, \ldots, X_n)$ and $m_n = \min(X_1, \ldots, X_n)$. First, suppose that G belongs to the max domain of attraction of Gumbel extreme value distribution. Then by Leadbetter et al. [21, chapter 1], there must exist a strictly positive function, say $h(t)$, such that

$$\lim_{t\to\infty} \frac{1 - G(t + xh(t))}{1 - G(t)} = e^{-x}, \forall x \in (-\infty, \infty).$$

But

$$\lim_{t\to\infty} \frac{1 - F(t + xh(t))}{1 - F(t)} = \lim_{x\to\infty} \frac{xf(tx)}{f(t)} = e^{-ax}, \forall\, x \in (-\infty, \infty),$$

so, it follows from Leadbetter et al. [21, chapter 1] that F belongs to the max domain of attraction of the Gumbel extreme value distribution with

$$\lim_{n\to\infty} P[a_n(M_n - b_n \leqslant x)] = \exp\left[-\exp(-ax)\right]$$

for some suitable norming constants $a_n > 0$ and b_n. Secondly, suppose that G belongs to the max domain of attraction of the Fréchet extreme value distribution. Then by Leadbetter et al. [21, chapter 1], there must exist $\beta > 0$ such that

$$\lim_{t\to\infty} \frac{1 - G(t + xh(t))}{1 - G(t)} = x^{c}, \forall\, x \in (-\infty, \infty).$$

But

$$\lim_{t\to\infty}\frac{1-F(t+xh(t))}{1-F(t)}=\lim_{t\to\infty}\frac{xf(tx)}{f(t)}=x^{a\,c},\forall\, x>0,$$

so, it follows from Leadbetter et al. [21, chapter 1] that F belongs to the max domain of attraction of the Fréchet extreme value distribution with

$$\lim_{n\to\infty}P[a_n(M_n-b_n\leqslant x)]=\exp(-x^{a\,c})$$

for some suitable norming constants $a_n>0$ and b_n. Thirdly, suppose that G belongs to the max domain of attraction of the Weibull extreme value distribution. Then, by Leadbetter et al. [21], there must exist a $\beta>0$ such that

$$\lim_{t\to 0}\frac{G(tx)}{G(t)}=x^{\beta},\forall\, x<0.$$

But

$$\lim_{t\to 0}\frac{F(tx)}{F(t)}=\lim_{t\to 0}\frac{xf(tx)}{f(t)}=x^{b\,\beta},\forall\, x<0,$$

so, it follows from Leadbetter et al. [21, chapter 1] that F belongs to the max domain of attraction of the Weibull extreme value distribution with

$$\lim_{n\to\infty}P[a_n(M_n-b_n\leqslant x)]=\exp\left[-(-x)^{b\,\beta}\right]$$

for some suitable norming constants $a_n>0$ and b_n. We conclude that F belongs to the same min domain of attraction as that of G. The same argument applies to max domain of attraction. That is, F belongs to the same min domain of attraction as that of G.

3.2. *Linear representation*

The MOEG-G cdf in (1.3) can be written as

$$F(x)=\frac{\sum_{k=0}^{\infty}a_k\,G(x)^k}{\sum_{k=0}^{\infty}b_k\,G(x)^k}=\sum_{k=0}^{\infty}c_k\,G(x)^k=\sum_{k=0}^{\infty}c_k\mathbf{M}_k(x),$$

where

$$a_k=\sum_{i=0}^{\infty}(-1)^{i+k}\binom{b}{i}\binom{ai}{k},\quad b_0=\theta+(1-\theta)\,a_0,c_0=\frac{a_0}{b_0},$$

for $k\geqslant 1$

$$c_k=\frac{1}{b_0}\left(a_k-\frac{1}{b_0}\sum_{r=1}^{k}b_k\,c_{k-r}\right)$$

and $\mathbf{M}_k(x)$ is the cdf of the Exp-G family with power parameter k. The pdf in (1.4) can also be expressed as a mixture of Exp-G densities

$$f(x)=\sum_{k=0}^{\infty}c_{k+1}\pi_{k+1}(x),\tag{3.1}$$

where $\pi_{k+1}(x)=(k+1)\,g(x)\,G(x)^k$ the Exp-G pdf with power parameter $k+1$. The statistical properties of the Exp-G models have been studied by many authors in recent years, for instance Gupta and Kundu [18] for exponentiated exponential, Nadarajah [28] for exponentiated Gumbel, Shirke and Kakade [36] for exponentiated log-normal and Nadarajah and Gupta [30] for exponentiated gamma distributions. Thus, some structural properties of the new family such as the ordinary and incomplete moments and generating function can be determined from well-established properties of the Exp-G distributions.

3.3. Probability weighted moments

The probability weighted moments (PWMs) are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. The PWMs method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly. The (s,z)th PWMs of X following the MOEG-G family, say $\rho_{s,z}$, is formally defined by

$$\rho_{s,z} = E\{X^s F(X)^z\} = \int_{-\infty}^{\infty} x^s F(x)^z f(x)\, dx.$$

Using equations (1.3) and (1.4), we can write

$$f(x)\, F(x)^z = ab\theta g(x) \frac{\sum_{k=0}^{\infty} \Phi_k G(x)^k}{\sum_{k=0}^{\infty} \Omega_k G(x)^k} = ab\theta g(x) \sum_{k=0}^{\infty} m_{k+1} G(x)^k,$$

where

$$\Phi_k = \sum_{h=0}^{\infty} (-1)^{h+k} \binom{a(h+1)-1}{k} \binom{b(z+1)-1}{h}$$

and

$$\Omega_k = \sum_{w=0}^{\infty} \sum_{h=0}^{\infty} (\theta-1)^h \theta^{z+h+2} (-1)^{w+k} \binom{aw}{k} \binom{bh}{w} \binom{z+2}{h}.$$

For $k \geqslant 1$ we have

$$d_k = \frac{1}{\Omega_0} \left[\Phi_k - \frac{1}{\Omega_0} \sum_{r=1}^{k} \Omega_k d_{k-r} \right]$$

and $m_0 = \frac{\Phi_0}{\Omega_0}$. Then, the (s,z)th PWMs of X can be expressed as

$$\rho_{s,r} = ab\theta \sum_{k=0}^{\infty} m_{k+1} \int_0^{\infty} x^s g(x) G(x)^k dx = ab\theta \sum_{k=0}^{\infty} m_{k+1} E\left(Y_{k+1}^s\right),$$

where Y_{k+1} is has Exp-G distribution with power parameter $k+1$.

3.4. Order statistics

Let $X_1, \dots, X_n$ be a random sample from the MOEG-G family of distributions and let $X_{1:n}, \dots, X_{n:n}$ be the corresponding order statistics. The pdf of i^{th} order statistic, $X_{i:n}$, can be written as

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x),$$

where $B(\cdot,\cdot)$ is the beta function. Using (1.3), (1.4), and the above equation, we have

$$f_{i:n}(x) = \frac{\sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j}}{B(i, n-i+1)} \left[\sum_{r=0}^{\infty} c_r r g(x) G(x)^{r-1} \right] \left[\sum_{k=0}^{\infty} c_k G(x)^k \right]^{j+i-1}.$$

Further,

$$\left[\sum_{k=0}^{\infty} c_k G(x)^k \right]^{j+i-1} = \sum_{k=0}^{\infty} \varphi_{j+i-1,k} G(x)^{j+i+k-1},$$

where $\varphi_{j+i-1,0} = c_0^{j+i-1}$ and (for $k \geqslant 1$)

$$\varphi_{j+i-1,k} = (kc_0)^{-1} \sum_{m=1}^{k} [m(j+i)-k]\, c_m \varphi_{j+i-1,k-m}.$$

Hence, the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i} \sum_{k,r=0}^{\infty} d_{i,j,k,r}\, \pi_{i+j+k+r}(x),$$

where

$$d_{i,j,k,r} = \frac{(-1)^j}{B(i, n-i+1)} \binom{n-i}{j} \frac{(1+r)\,\varphi_{j+i-1,k}}{i+j+k+r}.$$

Then, the density function of the MOEG-G order statistics is a mixture of Exp-G densities. Based on $f_{i:n}(x)$, we can easily obtain ordinary and incomplete moments and generating function of $X_{i:n}$ for any parent G distribution.

3.5. *Moments of residual and reversed residual life*

The n^{th} moment of the residual life, say

$$m_n(t) = E[(X-t)^n \,|_{(X>t)}],\ n = 1, 2, \ldots,$$

uniquely determined $F(x)$. The n^{th} moment of the residual life of X is given by

$$m_n(t) = \frac{1}{1-F(t)} \int_t^{\infty} (x-t)^n dF(x).$$

Therefore

$$m_n(t) = \frac{1}{1-F(t)} \sum_{k=0}^{\infty} c_{k+1}^{\star} \int_t^{\infty} x^r \pi_{k+1}(x),$$

where

$$c_{k+1}^{\star} = c_{k+1} \sum_{r=0}^{n} \binom{n}{r} (-t)^{n-r}.$$

Another interesting function is the mean residual life (MRL) function or the life expectation at age t defined by $m_1(t) = E[(X-t)\,|_{X>t}]$, which represents the expected additional life length for a unit which is alive at age t.

The MRL of X can be obtained by setting $n = 1$ in the last equation. The n^{th} moment of the reversed residual life, say

$$M_n(t) = E\left[(t-X)^n \,|_{(X\leqslant t)}\right] \text{ for } t > 0 \text{ and } n = 1, 2, \ldots$$

uniquely determines $F(x)$. We obtain

$$M_n(t) = \frac{1}{F(t)} \int_0^{t} (t-x)^n dF(x).$$

Therefore, the n^{th} moment of the reversed residual life of X becomes

$$M_n(t) = \frac{1}{F(t)} \sum_{k=0}^{\infty} c_{k+1}^{\star\star} \int_0^{t} x^r \pi_{k+1}(x),$$

where

$$c^{\star\star}_{k+1} = c_{k+1}\sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}.$$

The mean waiting time (MWT) or mean inactivity time (MIT) also called the mean reversed residual life function is given by $M_1(t) = E[(t-X) \mid X \leqslant t]$ and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in $(0,t)$. The MIT of the MOEG-G family of distributions can be obtained easily by setting $n = 1$ in the above equation.

3.6. General statistical results

The r^{th} ordinary moment of X is given by

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx.$$

Then, we obtain

$$\mu'_r = \sum_{k=0}^{\infty} c_{k+1} E(Y^r_{k+1}). \tag{3.2}$$

Setting $r = 1$ in (3.2), we get the mean of X. The last integration can be computed numerically for most parent models. The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. Here, we provide two formulae for the mgf $M_X(t) = E\left(e^{tX}\right)$ of X. Clearly, the first one can be derived from equation (3.1) as $M_X(t) = \sum\limits_{k=0}^{\infty} c_{k+1} M_{k+1}(t)$, where $M_{k+1}(t)$ is the mgf of Y_{k+1}. Hence, $M_X(t)$ can be determined from the Exp-G generating function. A second formula for $M_X(t)$ follows from (3.1) as $M_X(t) = \sum\limits_{k=0}^{\infty} c_{k+1}\, \tau(t,k)$, where $\tau(t,k) = \int_0^1 \exp\left[t\, Q_G(u)\right] u^{k-1} du$ and $Q_G(u)$ is the qf corresponding to $G(x; \boldsymbol{\phi})$, i.e., $Q_G(u) = G^{-1}(u; \boldsymbol{\phi})$. The main applications of the first incomplete moment refer to the mean deviations and the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. The s^{th} incomplete moment, say $I_s(t)$, of X can be expressed from (3.1) as

$$I_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx = \sum_{k=0}^{\infty} c_{k+1} \int_{-\infty}^{t} x^s\, \pi_{k+1}(x)\, dx. \tag{3.3}$$

The mean deviations about the mean $\delta_1 = E(|X - \mu'_1|)$ and about the median $\delta_2 = E(|X - M|)$ of X are given by $\delta_1 = 2\mu'_1 F(\mu'_1) - 2I_1(\mu'_1)$ and $\delta_2 = \mu'_1 - 2I_1(M)$, respectively, where $\mu'_1 = E(X)$, $M = \text{Median}(X) = Q(0.5)$ is the median, $F(\mu'_1)$ is easily calculated from (1.3) and $I_1(t)$ is the first incomplete moment given by (3.3) with $s = 1$. Now, we provide two ways to determine δ_1 and δ_2. First, a general equation for $I_1(t)$ can be derived from (3.3) as $I_1(t) = \sum\limits_{k=0}^{\infty} c_{k+1} J_{k+1}(x)$, where $J_{k+1}(x) = \int_{-\infty}^{t} x\, \pi_{k+1}(x)\, dx$ is the first incomplete moment of the Exp-G distribution. A second general formula for $I_1(t)$ is given by $I_1(t) = \sum\limits_{k=0}^{\infty} c_{k+1}\, \zeta_{k+1}(t)$, where $\zeta_k(t) = (k) \int_0^{G(t)} Q_G(u)\, u^{k-1} du$ can be computed numerically. These equations for $I_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = I_1(q)/(\pi\mu'_1)$ and $L(\pi) = I_1(q)/\mu'_1$, respectively, where $\mu'_1 = E(X)$ and $q = Q(\pi)$ is the qf of X at π.

4. Characterizations

This section deals with various characterizations of MOEG-G distribution. These characterizations are based on: (i) a simple relationship between two truncated moments and (ii) the reverse (or reversed) hazard function. It should be mentioned that for characterization (i) the cdf may not have a closed form.

We present our characterizations (i) and (ii) in two subsections.

4.1. Characterizations based on two truncated moments

In this subsection we present characterizations of MOEG-G distribution in terms of the ratio of two truncated moments. This characterization result employs a theorem due to Glänzel [15], see Theorem Appendix .1. Note that the result holds also when the interval H is not closed. As shown in Glänzel [16], this characterization is stable in the sense of weak convergence.

Proposition 4.1. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let*

$$q_1(x) = \left\{\theta + (1-\theta)\left[1-\overline{G}(x)^a\right]^b\right\}^2$$

and

$$q_2(x) = q_1(x)\left[1-\overline{G}(x)^a\right]^b \quad \textit{for } x \in \mathbb{R}.$$

The random variable X has pdf (1.4) if and only if the function ξ defined in Theorem Appendix .1 has the form

$$\xi(x) = \frac{1}{2}\left\{1+\left[1-\overline{G}(x)^a\right]^b\right\}, \quad x \in \mathbb{R}.$$

Proof. If X has pdf (5), then

$$(1-F(x))\,E\left[q_1(x) \mid X \geqslant x\right] = \theta\left\{1-\left[1-\overline{G}(x)^a\right]^b\right\}, \quad x \in \mathbb{R},$$

and

$$(1-F(x))\,E\left[q_2(x) \mid X \geqslant x\right] = \frac{\theta}{2}\left\{1-\left[1-\overline{G}(x)^a\right]^{2b}\right\}, \quad x \in \mathbb{R},$$

and finally

$$\xi(x)\,q_1(x) - q_2(x) = \frac{1}{2}q_1(x)\left\{1-\left[1-\overline{G}(x)^a\right]^b\right\} > 0 \quad \text{for } x \in \mathbb{R}.$$

Conversely, if ξ is given as above, then

$$s'(x) = \frac{\xi'(x)\,h(x)}{\xi(x)\,h(x)-g(x)} = \frac{abg(x)\,\overline{G}(x)^{a-1}\left[1-\overline{G}(x)^a\right]^{b-1}}{1-\left[1-\overline{G}(x)^a\right]^b}, \quad x \in \mathbb{R},$$

and hence

$$s(x) = -\log\left\{\left\{1-\left[1-\overline{G}(x)^a\right]^b\right\}\right\}, \quad x \in \mathbb{R}.$$

Now, in view of Theorem Appendix .1, X has density (1.4). □

Corollary 4.2. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 4.1. Then, X has pdf (5) if and only if there exist functions q_2 and ξ defined in Theorem Appendix .1 satisfying the differential equation*

$$\frac{\xi'(x)\,h(x)}{\xi(x)\,h(x)-g(x)} = \frac{abg(x)\,\overline{G}(x)^{a-1}\left[1-\overline{G}(x)^a\right]^{b-1}}{1-\left[1-\overline{G}(x)^a\right]^b} \quad x \in \mathbb{R}.$$

The general solution of the differential equation in Corollary 4.2 is

$$\xi(x) = \left\{1-\left[1-\overline{G}(x)^a\right]^b\right\}^{-1}\left[-\int abg(x)\,\overline{G}(x)^{a-1}\left[1-\overline{G}(x)^a\right]^{b-1}(q_1(x))^{-1}\,q_2(x) + D\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 4.1 with $D = \frac{1}{2}$. However, it should be also noted that there are other triplets (q_1, q_2, ξ) satisfying the conditions of Theorem Appendix .1.

4.2. Characterization in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F, is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in \text{support of } F.$$

Proposition 4.3. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable. The pdf of X is (1.4) if and only if its reverse hazard function $r_F(x)$ satisfies the differential equation*

$$r_F'(x) - g'(x)(g(x))^{-1} r_F(x) = ab\theta g(x) \frac{d}{dx} \left\{ \frac{\overline{G}(x)^{a-1}}{\left[1-\overline{G}(x)^a\right]\left\{\theta + (1-\theta)\left[1-\overline{G}(x)^a\right]^b\right\}} \right\},$$

with boundary condition $\lim_{x\to\infty} r_F(x) = 0$.

Proof. If X has pdf (5), then clearly the above differential equation holds. Now, if this differential equation holds, then

$$\frac{d}{dx}\left\{(g(x))^{-1} r_F(x)\right\} = ab\theta \frac{d}{dx} \left\{ \frac{\overline{G}(x)^{a-1}}{\left[1-\overline{G}(x)^a\right]\left\{\theta + (1-\theta)\left[1-\overline{G}(x)^a\right]^b\right\}} \right\},$$

or

$$r_F(x) = \frac{ab\theta g(x)\overline{G}(x)^{a-1}}{\left[1-\overline{G}(x)^a\right]\left\{\theta + (1-\theta)\left[1-\overline{G}(x)^a\right]^b\right\}},$$

which is the reverse hazard function of the MOEG-G distribution. □

Remark 4.4. For the special cases of $a = 1$ or $b = 1$, the formulas in the above subsections will be quite simplified.

5. Estimation

In this section, we apply the maximum likelihood method to estimate the parameters of the MOEG-G family for uncensored and censored data. We also assess the performance of the maximum likelihood estimators (MLEs) in terms of biases and mean squared errors by means of a simulation study.

5.1. Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. So, we consider the estimation of the unknown parameters of this family from complete samples only by maximum likelihood. Let $x_1, \ldots, x_n$ be a random sample from the MOEG-G distribution with parameters θ, a, b and $\boldsymbol{\phi}$. Let $\mathbf{P}=(\theta, a, b, \boldsymbol{\phi}^\intercal)^\intercal$ be the $p \times 1$ parameter vector. For determining the MLE of $\mathbf{P}$, we have the log-likelihood function

$$\begin{aligned}\ell = \ell(\mathbf{P}) &= n\log a + n\log b + n\log\theta + \sum_{i=1}^n \log g(x_i;\boldsymbol{\phi}) \\ &\quad + (a-1)\sum_{i=1}^n \log\overline{G}(x_i;\boldsymbol{\phi}) + (b-1)\sum_{i=1}^n \log(p_i) - 2\sum_{i=1}^n \log(z_i),\end{aligned}$$

where $p_i = 1 - \overline{G}(x_i; \boldsymbol{\phi})^a$ and $z_i = \theta + (1-\theta)\left[1 - \overline{G}(x_i; \boldsymbol{\phi})^a\right]^b$. The components of the score vector, $U(\mathbf{P}) = \frac{\partial \ell}{\partial \mathbf{P}} = \left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \boldsymbol{\phi}}\right)^T$, are given in Appendix B. Setting the nonlinear system of equations $U_\theta = U_a = U_b = 0$ and $U_{\boldsymbol{\phi}} = 0$ and solving them simultaneously yields the MLE $\widehat{\mathbf{P}} = (\widehat{\theta}, \widehat{a}, \widehat{b}, \widehat{\boldsymbol{\phi}}^T)^T$. To solve these equations, it is usually more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize ℓ. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\mathbf{P}) = \{\frac{\partial^2 \ell}{\partial r \, \partial s}\}$ (for $r, s = \theta, a, b, \boldsymbol{\phi}$), whose elements can be computed numerically. Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\mathbf{P}}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\mathbf{P}})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\mathbf{P}})$ is the total observed information matrix evaluated at $\widehat{\mathbf{P}}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

5.2. *Censored maximum likelihood estimation*

Often with lifetime data, we encounter censored observations. There are different forms of censoring: type I censoring, type II censoring, etc.. Here, we consider the general case of multi-censored data: there are n subjects of which n_0 are known to have failed at the times $x_1, \ldots, x_{n_0}$, n_1 are known to have failed in the interval $\left[s_{j-1}, s_j\right]$, $j = 1, \ldots, n_1$, n_2 survived to a time r_j, $j = 1, \ldots, n_2$, but not observed any longer. Note that $n = n_0 + n_1 + n_2$ and that type I censoring and type II censoring are included as particular cases of multi-censoring. The log-likelihood function for $\mathbf{P}$ is

$$\begin{aligned}
\ell_n(\mathbf{P}) &= n_0 \log a + n_0 \log b + n_0 \log \theta + \sum_{i=1}^{n_0} \log g(x_i; \boldsymbol{\phi}) + (a-1) \sum_{i=1}^{n_0} \log \overline{G}(x_i; \boldsymbol{\phi}) \\
&\quad + (b-1) \sum_{i=1}^{n_0} \log\left[1 - \overline{G}(x_i; \boldsymbol{\phi})^a\right] - 2 \sum_{i=1}^{n_0} \log\left\{\theta + (1-\theta)\left[1 - \overline{G}(x_i; \boldsymbol{\phi})^a\right]^b\right\} \\
&\quad + \sum_{i=1}^{n_1} \log\left\{\frac{\left[1 - \overline{G}(s_i; \boldsymbol{\phi})^a\right]^b}{\theta + (1-\theta)\left[1 - \overline{G}(s_i; \boldsymbol{\phi})^a\right]^b} - \frac{\left[1 - \overline{G}(s_{i-1}; \boldsymbol{\phi})^a\right]^b}{\theta + (1-\theta)\left[1 - \overline{G}(s_{i-1}; \boldsymbol{\phi})^a\right]^b}\right\} \\
&\quad + \sum_{i=1}^{n_2} \log\left\{1 - \frac{\left[1 - \overline{G}(r_i; \boldsymbol{\phi})^a\right]^b}{\theta + (1-\theta)\left[1 - \overline{G}(r_i; \boldsymbol{\phi})^a\right]^b}\right\}.
\end{aligned}$$

The normal equations are given in Appendix C.

5.3. *Simulation study*

In this section, we study the performance and accuracy of maximum likelihood estimates of the MOEG-L distribution parameters by conducting various simulations for different sample sizes and different parameter values. The method for generating sample from the MOEG-L model is performed by inverse cdf of MOEG-L and uniform random variable as follows:
If

$$X = Q\left(1 - \left\{1 - \left[\frac{\theta U}{1 - (1-\theta) U}\right]^{\frac{1}{b}}\right\}^{\frac{1}{a}}\right),$$

where $U \sim U(0,1)$ and Q is quantile function of Lomax distribution, then $X \sim$MOEG-L(a, b, θ, η). The simulation study is repeated for $N = 5000$ times each with sample size $n = 100, 300, 500$ and parameter values $(a, b, \theta, \eta) = (1,1,1,1), (2,1,3,2), (4,2,1,1)$, and $(1,1,2,3)$.

Two quantities are computed in this simulation study.

1) Average bias of the MLE $\hat{\varepsilon}$ of the parameter $\varepsilon = a, b, \theta, \eta$:

$$\mathrm{Bias}(\varepsilon) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\varepsilon}_i - \varepsilon).$$

2) Mean squared error (MSE) of the MLE $\hat{\varepsilon}$ of the parameter $\varepsilon = a, b, \theta, \eta$:

$$\mathrm{MSE}(\varepsilon) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{\varepsilon}_i - \varepsilon)^2.$$

Table 2 presents the Bias and MSE values of the parameters for different sample sizes. From the results, we can verify that as the sample size n increases, the MSEs decay toward zero.

Table 2: Monte Carlo simulation results: Average bias and MSE in parenthesis.

Sample Size	(a, b, θ, η)	$\widehat{a}$	$\widehat{b}$	$\widehat{\theta}$	$\widehat{\eta}$
$n = 100$	$(1, 1, 1, 1)$	0.026(0.015)	0.001(0.038)	0.281(0.862)	0.025(0.016)
	$(2, 1, 3, 2)$	−0.005(0.028)	0.073(0.116)	0.459(6.333)	−0.005(0.028)
	$(4, 2, 1, 1)$	−0.060(0.033)	0.105(0.222)	0.220(1.069)	0.041(0.056)
	$(1, 1, 2, 3)$	0.040(0.038)	0.035(0.073)	0.185(1.341)	−0.114(0.088)
$n = 300$	$(1, 1, 1, 1)$	−0.005(0.003)	0.006(0.017)	0.035(0.104)	−0.005(0.003)
	$(2, 1, 3, 2)$	0.007(0.007)	0.020(0.036)	0.292(1.615)	0.007(0.007)
	$(4, 2, 1, 1)$	−0.023(0.006)	0.034(0.065)	0.053(0.248)	0.011(0.016)
	$(1, 1, 2, 3)$	0.008(0.007)	0.024(0.031)	0.026(0.480)	−0.028(0.011)
$n = 500$	$(1, 1, 1, 1)$	−0.001(0.002)	0.017(0.012)	0.013(0.071)	−0.001(0.002)
	$(2, 1, 3, 2)$	−0.003(0.005)	0.023(0.020)	0.058(0.785)	−0.003(0.005)
	$(4, 2, 1, 1)$	−0.013(0.002)	−0.003(0.031)	0.053(0.101)	0.008(0.007)
	$(1, 1, 2, 3)$	0.012(0.005)	0.025(0.018)	0.004(0.292)	−0.036(0.004)

6. Applications

In this section, we illustrate the applicability of the MOEG-G family of distribution via two real data sets. We use on the MOEG-L distributions presented in Section 2. The method of maximum likelihood is used to estimate the model parameters.

Example 6.1. The first data set, is dealing with the remission times (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [23]. Lemonte and Cordeiro [24] used this data for illustrating flexibility of a proposed distribution.

Example 6.2. The second data set is concerning time between failures of secondary reactor pumps (thousand of hours) (Salman et al., [34]).

In the both examples, we shall compare the MOEG-L model with other comparative models: the Gamma Lomax (G-L) (Cordeiro et al., [13]), Kumaraswamy Lomax (Kw-L) (Lemonte and Cordeiro, [24]), and Beta Lomax (B-L) (Lemonte and Cordeiro, [24]) distributions. In order to compare the fits of the distributions, we consider various measures of goodness-of-fit including the maximized log-likelihood under the model ($-\widehat{\ell}$), Anderson-Darling (A^*) and Cramér-Von Mises (W^*) statistics. The smaller these statistics show better model for fitting. The estimated parameters based on MLE procedure are ginen in Tables 3 and 4, whereas the values of goodness-of-fit statistics are given in Tables 3 and 4. In the applications, the information about the hazard shape can help in selecting a particular model. To do so, a device called the total time on test (TTT) plot (Aarset, [1]) is useful. The TTT plot is obtained by plotting

$$G(r/n) = \left[\left(\sum_{i=1}^{r} y_{i,n}\right) + (n-r)y_{(r)}\right] / \sum_{i=1}^{n} y_{i,n},$$

where $r = 1, \ldots, n$ and $y_{i,n}$ $(i = 1, \ldots, n)$ are the order statistics of the sample, against r/n. If the shape is a straight diagonal the hazard is constant. If the shape is a straight diagonal the hazard is constant. It is convex shape for decreasing hazards and concave shape for increasing hazards. The bathtub-shaped hazard is obtained when the first convex and then concave. Both of used data sets have bathtub-shaped hazard. In both real data sets, the results show that the MOEG-L distribution yields a better fit than other distributions.

Table 3: The MLEs of the parameters and SEs in parentheses and the goodness-of-fit statistics for first data set.

Model	Estimates				$-\widehat{\ell}$	W^*	A^*
MOEG-L(a, b, θ, η)	4.02 (0.13)	0.55 (0.08)	111.70 (17.23)	0.51 (0.51)	409.76	0.014	0.093
Kw-L(a, b, β, α)	1.52 (0.05)	11.00 (0.97)	11.99 (1.04)	0.41 (0.02)	409.94	0.022	0.158
B-L(a, b, β, α)	1.58 (0.13)	1.14 (0.09)	23.89 (2.09)	3.95 (0.28)	410.07	0.026	0.179
G-L(a, β, α)	1.58 (0.09)	20.58 (1.80)	4.75 (0.33)		410.08	0.026	0.181

Table 4: The MLEs of the parameters and SEs in parentheses and the goodness-of-fit statistics for second data set.

Model	Estimates				$-\widehat{\ell}$	W^*	A^*
MOEG-L(a, b, θ, η)	0.79 (0.15)	1.34 (0.26)	0.83 (0.30)	1.90 (0.36)	32.409	0.029	0.253
Kw-L(a, b, β, α)	0.89 (0.05)	34.12 (7.11)	6.15 (1.70)	0.10 (0.02)	32.535	0.052	0.357
B-L(a, b, β, α)	0.93 (0.15)	34.00 (7.33)	4.15 (1.14)	0.10 (0.02)	32.568	0.057	0.390
G-L(a, β, α)	0.97 (0.15)	0.70 (0.18)	1.35 (0.28)		33.940	0.231	1.393

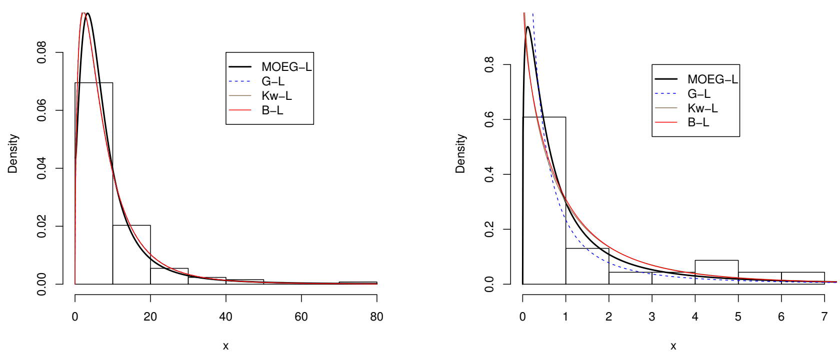

Figure 3: (Left panel): fitted models on histogram of first data set, (Right panel): fitted models on histogram of second data set.

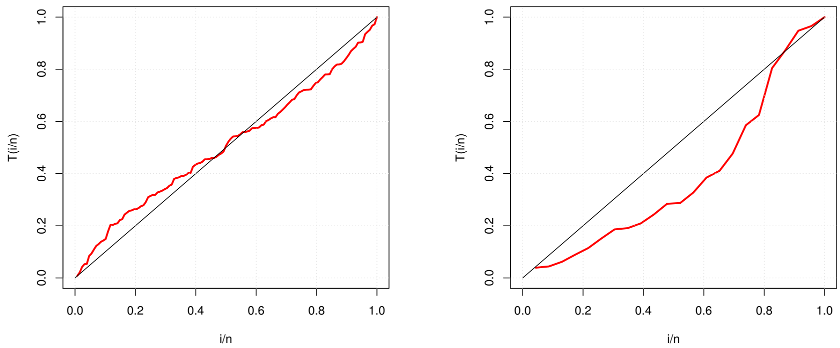

Figure 4: TTT-plot for the first dataset (left Fig.) and for the second dataset (right Fig.)

It is clear from Tables 3 and 4 and Figures 3 and 4 that the MOEG-L model provides the best fits to both data sets.

7. Conclusions

There has been a great deal of interest among the statisticians, specialists and practitioners to generate new extended families from classic ones. In this article, we present a new class of distributions called Marshall-Olkin exponentiated generalized G family of distributions, which extends the Marshall-Olkin-G family. The mathematical properties of this new family including explicit expansions for the ordinary and incomplete moments, generating function, mean deviations, order statistics, and probability weighted moments are provided. Characterizations based on two truncated moments and as well as based on reverse hazard function are presented. The model parameters are estimated by the maximum likelihood estimation method and the observed information matrix is determined. It is shown, by means of two real data sets, that special cases of the MOEG-G class can provide better fit than other models generated by well-known families.

Appendix A

Theorem Appendix .1.

$$E\left[q_2(X) \mid X \geqslant x\right] = E\left[q_1(X) \mid X \geqslant x\right] \xi(x), \quad x \in H,$$

is defined with some real function η. *Assume that* $q_1, q_2 \in C^1(H)$, $\xi \in C^2(H)$ *and* F *is twice continuously differentiable and strictly monotone function on the set* H. *Finally, assume that the equation* $\xi q_1 = q_2$ *has no real solution in the interior of* H. *Then* F *is uniquely determined by the functions* q_1, q_2, *and* ξ, *particularly*

$$F(x) = \int_a^x C \left| \frac{\xi'(u)}{\xi(u) q_1(u) - q_2(u)} \right| \exp\left(-s(u)\right) du,$$

where the function s *is a solution of the differential equation* $s' = \frac{\xi' q_1}{\xi q_1 - q_2}$ *and* C *is the normalization constant, such that* $\int_H dF = 1$.

Appendix B

$$U_a = \frac{n}{a} + \sum_{i=1}^n \log \overline{G}(x_i; \boldsymbol{\phi}) + (b-1) \sum_{i=1}^n \frac{u_i}{p_i} - 2\sum_{i=1}^n \frac{m_i}{z_i},$$

$$U_b = \frac{n}{b} + \sum_{i=1}^{n} \log p_i - 2\sum_{i=1}^{n} \frac{w_i}{z_i}, U_\theta = \frac{n}{\theta} - 2\sum_{i=1}^{n} \frac{q_i}{z_i},$$

and (for $r = 1, 2, \dots, q$)

$$U_{\phi_r} = \sum_{i=1}^{n} \frac{g_r'(x_i;\boldsymbol{\phi})}{g(x_i;\boldsymbol{\phi})} - (a-1)\sum_{i=1}^{n} \frac{G_r'(x_i;\boldsymbol{\phi})}{\overline{G}(x_i;\boldsymbol{\phi})} + (b-1)\sum_{i=1}^{n} \frac{d_{i,r}}{p_i} - 2\sum_{i=1}^{n} \frac{t_{i,r}}{z_i},$$

where

$$u_i = \frac{-\log \overline{G}(x_i;\boldsymbol{\phi})}{\overline{G}(x_i;\boldsymbol{\phi})^{-a}}, \qquad m_i = \frac{-b(1-\theta)\log \overline{G}(x_i;\boldsymbol{\phi})}{\overline{G}(x_i;\boldsymbol{\phi})^{-a}\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{1-b}}, \qquad g_r'(x_i;\boldsymbol{\phi}) = \frac{\partial g(x_i;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}_r},$$

$$w_i = \frac{(1-\theta)\log\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}}{\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{-b}}, \qquad q_i = 1-\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}, \qquad G_r'(x_i;\boldsymbol{\phi}) = \frac{\partial G(x_i;\boldsymbol{\phi})}{\partial \boldsymbol{\phi}_r},$$

$$d_{i,r} = \frac{aG_r'(x_i;\boldsymbol{\phi})}{\overline{G}(x_i;\boldsymbol{\phi})^{1-a}}, \qquad \text{and } t_{i,r} = \frac{ab(1-\theta)G_r'(x_i;\boldsymbol{\phi})}{\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{1-b}\overline{G}(x_i;\boldsymbol{\phi})^{1-a}}.$$

Appendix C

$$\begin{aligned}
\frac{\partial \ell_n(\mathbf{P})}{\partial \theta} &= \frac{n_0}{\theta} - 2\sum_{i=1}^{n_0} \frac{1-\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}} \\
&\quad + \sum_{i=1}^{n_1} \frac{[A(s_i)-A(s_{i-1})]}{\left\{\frac{\left[1-\overline{G}(s_i;\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^{a}\right]^{b}} - \frac{\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^{a}\right]^{b}}\right\}} \\
&\quad + \sum_{i=1}^{n_2} \frac{\left(\frac{\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}\left\{1-\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}\right\}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}\right\}^{2}}\right)}{\left\{1-\frac{\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}}\right\}}, \\
\frac{\partial \ell_n(\mathbf{P})}{\partial a} &= \frac{n_0}{a} + \sum_{i=1}^{n_0} \log \overline{G}(x_i;\boldsymbol{\phi}) + (b-1)\sum_{i=1}^{n_0} \frac{-\overline{G}(x_i;\boldsymbol{\phi})^{a}\log \overline{G}(x_i;\boldsymbol{\phi})}{1-\overline{G}(x_i;\boldsymbol{\phi})^{a}} \\
&\quad - 2\sum_{i=1}^{n_0} \frac{-b(1-\theta)\overline{G}(x_i;\boldsymbol{\phi})^{a}\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b-1}\log \overline{G}(x_i;\boldsymbol{\phi})}{\theta+(1-\theta)\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}} \\
&\quad + \sum_{i=1}^{n_1} \frac{[B(s_i)-B(s_{i-1})]}{\left\{\frac{\left[1-\overline{G}(s_i;\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^{a}\right]^{b}} - \frac{\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^{a}\right]^{b}}\right\}} \\
&\quad + \sum_{i=1}^{n_2} \frac{\left(\frac{\theta b\overline{G}(r_i;\boldsymbol{\phi})^{a}\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b-1}\log \overline{G}(r_i;\boldsymbol{\phi})}{\left\{\theta+(1-\theta)\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}\right\}^{2}}\right)}{\left\{1-\frac{\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}}{\theta+(1-\theta)\left[1-\overline{G}(r_i;\boldsymbol{\phi})^{a}\right]^{b}}\right\}}, \\
\frac{\partial \ell_n(\mathrm{P})}{\partial b} &= \frac{n_0}{b} + \sum_{i=1}^{n_0} \log\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right] - 2\sum_{i=1}^{n_0} \frac{(1-\theta)\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}\log\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]}{\theta+(1-\theta)\left[1-\overline{G}(x_i;\boldsymbol{\phi})^{a}\right]^{b}}
\end{aligned}$$

$$
+\sum_{i=1}^{n_1}\frac{[C(s_i)-C(s_{i-1})]}{\left\{\frac{[1-\overline{G}(s_i;\boldsymbol{\phi})^a]^b}{\theta+(1-\theta)[1-\overline{G}(s_i;\boldsymbol{\phi})^a]^b}-\frac{[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a]^b}{\theta+(1-\theta)[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a]^b}\right\}}
+\sum_{i=1}^{n_2}\frac{\left(\frac{-\log[1-\overline{G}(r_i;\boldsymbol{\phi})^a][1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b\left\{\theta+2(1-\theta)[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b\right\}}{\left\{\theta+(1-\theta)[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b\right\}^2}\right)}{\left\{1-\frac{[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b}{\theta+(1-\theta)[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b}\right\}}
$$

and (for $m=1,2,\dots,q$)

$$
\begin{aligned}
\frac{\partial\ell_n(\mathbf{P})}{\partial\boldsymbol{\phi}}=&+\sum_{i=1}^{n_0}\frac{g_m'(x_i;\boldsymbol{\phi})}{g(x_i;\boldsymbol{\phi})}-(a-1)\sum_{i=1}^{n_0}\frac{G_m'(x_i;\boldsymbol{\phi})}{\overline{G}(x_i;\boldsymbol{\phi})}-a(b-1)\sum_{i=1}^{n_0}\frac{G_m'(x_i;\boldsymbol{\phi})\,\overline{G}(x_i;\boldsymbol{\phi})^{a-1}}{1-\overline{G}(x_i;\boldsymbol{\phi})^a}\\
&+2ab(1-\theta)\sum_{i=1}^{n_0}\frac{G_m'(x_i;\boldsymbol{\phi})\,\overline{G}(x_i;\boldsymbol{\phi})^{a-1}\left[1-\overline{G}(x_i;\boldsymbol{\phi})^a\right]^{b-1}}{\theta+(1-\theta)\left[1-\overline{G}(x_i;\boldsymbol{\phi})^a\right]^b}\\
&+\sum_{i=1}^{n_1}\frac{[D(s_i)-D(s_{i-1})]}{\left\{\frac{[1-\overline{G}(s_i;\boldsymbol{\phi})^a]^b}{\theta+(1-\theta)[1-\overline{G}(s_i;\boldsymbol{\phi})^a]^b}-\frac{[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a]^b}{\theta+(1-\theta)[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a]^b}\right\}}\\
&+\sum_{i=1}^{n_2}\frac{\left(\frac{ab\theta G_m'(r_i;\boldsymbol{\phi})\overline{G}(s_i;\boldsymbol{\phi})^{a-1}[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^{b-1}}{\left\{\theta+(1-\theta)[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b\right\}^2}\right)}{\left\{1-\frac{[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b}{\theta+(1-\theta)[1-\overline{G}(r_i;\boldsymbol{\phi})^a]^b}\right\}},
\end{aligned}
$$

where

$$
\begin{aligned}
A(s_i)&=\frac{-\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\left\{1-\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\right\}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\right\}^2},\\
A(s_{i-1})&=\frac{-\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\left\{1-\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\right\}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\right\}^2},\\
B(s_i)&=\frac{-\theta b\overline{G}(s_i;\boldsymbol{\phi})^a\log\overline{G}(s_i;\boldsymbol{\phi})\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^{b-1}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\right\}^2},\\
B(s_{i-1})&=\frac{-\theta b\overline{G}(s_{i-1};\boldsymbol{\phi})^a\log\overline{G}(s_{i-1};\boldsymbol{\phi})\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^{b-1}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\right\}^2},\\
C(s_i)&=\frac{\log\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\left\{\theta+2(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\right\}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\right\}^2},\\
C(s_{i-1})&=\frac{\log\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\left\{\theta+2(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\right\}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^a\right]^b\right\}^2},\\
D(s_i)&=\frac{-ab\theta G_m'(s_i;\boldsymbol{\phi})\,\overline{G}(s_i;\boldsymbol{\phi})^{a-1}\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^{b-1}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_i;\boldsymbol{\phi})^a\right]^b\right\}^2},
\end{aligned}
$$

$$D(s_{i-1}) = \frac{-ab\theta G'_m(s_{i-1};\boldsymbol{\phi})\,\overline{G}(s_{i-1};\boldsymbol{\phi})^{a-1}\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^{a}\right]^{b-1}}{\left\{\theta+(1-\theta)\left[1-\overline{G}(s_{i-1};\boldsymbol{\phi})^{a}\right]^{b}\right\}^{2}},$$

$$g'_m(x_i;\boldsymbol{\phi}) = \frac{\partial g(x_i;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}} \quad \text{and} \quad G'_m(\bullet;\boldsymbol{\phi}) = \frac{\partial G(\bullet;\boldsymbol{\phi})}{\partial\boldsymbol{\phi}}.$$

Appendix D

Plot of Biased and MSE of MLE

```
rMOEGW=function(n,a,b,theta,lambda,c){
G=rep(0,0)
u=runif(n,0,1)
v=1-((1-(((theta*u)/(1-((1-theta)*u)))^(1/b)))^(1/a))
G=qweibull(v, shape=lambda, scale = c)
return(G)
}
MOEGWd=function(x,a,b,theta,lambda,c){
G=rep(0,0)
for(i in 1:length(x)){
G[i]=(a*b*theta*dweibull(x[i], shape=lambda, scale = c)*((1-pweibull(x[i], shape=lambda, scale = c))^(a-1))*(1-((1-pweibull(x[i], shape=lambda, scale = c))^a))^(b-1))/((theta+(1-theta)*(1-((1-pweibull(x[i], shape= lambda, scale = c))^a))^b)^2)
}
return(G)
}
MOEGWlikelihood=function(par){
a=par[1]
b=par[2]
theta=par[3]
lambda=par[4]
c=par[5]
G=-sum(log(MOEGWd(x,a,b,theta,lambda,c)))
return(G)
} M=150
MSE=matrix(c(0),ncol=5,nrow=200)
biased=matrix(c(0),ncol=5,nrow=200)
real value
a=0.9
b=0.25
theta=2.5
lambda=3
c=1.5
size=seq(30,500,5)
for (i in 1:length(size)){
estimate=matrix(c(0),ncol=5,nrow=10000)
count1=0
repeat {
if (length(estimate[,1][estimate[,1]!=0])==M) break
count1 = count1 + 1
#print(count)
```

```
#print(length(estimate[,1][estimate[,1]!=0]))
x=rMOEGW(size[i],a,b,theta,lambda,c)
MOEGWest=try(optim(c(a,b,theta,lambda,c),MOEGWlikelihood,method="L-BFGS-B",hessian=TRUE,
lower =c(.2,.2,.2,.2,.2), upper =c(Inf,Inf,Inf,Inf)), silent=TRUE)
if ('try-error' %in% class(MOEGWest)) next
else{
estimate[count1,]=MOEGWest$par
}
}
biased[i,1]=mean(estimate[,1][estimate[,1]!=0]-a)
biased[i,2]=mean(estimate[,2][estimate[,2]!=0]-b)
biased[i,3]=mean(estimate[,3][estimate[,3]!=0]-theta)
biased[i,4]=mean(estimate[,4][estimate[,4]!=0]-lambda)
biased[i,5]=mean(estimate[,5][estimate[,5]!=0]-c)
MSE[i,1]=mean((estimate[,1][estimate[,1]!=0]-a)^2)
MSE[i,2]=mean((estimate[,2][estimate[,2]!=0]-b)^2)
MSE[i,3]=mean((estimate[,3][estimate[,3]!=0]-theta)^2)
MSE[i,4]=mean((estimate[,4][estimate[,4]!=0]-lambda)^2)
MSE[i,5]=mean((estimate[,5][estimate[,5]!=0]-c)^2)
print(i)
}
```

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