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Solvability of the functional integro-differential equation with self-reference and state-dependence

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Abstract

The existence of solutions of a functional integro-differential equation with self-reference and state-dependence will be studied. The continuous dependence of the solution on the delay $\phi(t)$, the functional g and initial data x_0 will be proved.

Keywords: Functional equations, existence of solutions, continuous dependence, state-dependence, self-reference.

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1. Introduction

The differential and integral equations with deviating arguments that appear in recent literature, the deviation of the argument usually involves only the time itself, see [1, 2]. However, another case, in which the deviating arguments depend on both the state variable x and the time t, is of importance in theory and practice. Several papers have appeared recently that are devoted to such kind of differential equations, see for example [3–6, 9–12] and the references cited therein.

The first papers studying this class of functional equations with self-reference, Eder [5], studied the existence of the unique solution for the differential equation

$$x'(t) = x(x(t)), \quad x(0) = x_0, \quad t \in B \subset \mathbb{R}.$$

Wang [11], studied the strong solution and maximal strong solution for the equation

$$x'(t) = f(x(x(t))),$$

where $f : \mathbb{R} \to \mathbb{R}$ is continuous and monotone, and $f(0) = 0$.

Feckan [6], studied the existence solution

$$x'(t) = f(x(x(t))), \quad x(0) = 0, \quad \text{where} \quad f \in C^1(\mathbb{R}, \mathbb{R}).$$

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Buica, studied the existence and continuous dependence of solutions, on x_0 of the functional-differential equation [4]

$$x'(t) = f(t, x(x(t))), \quad x(t_0) = x_0, \quad t \in [a, b],$$

where $t_0, x_0 \in [a, b]$ and $f \in C([a, b], [a, b])$.

Stanek [9], studied global properties of decreasing solution of the equation

$$x'(t) = x(x(t)) + x(t).$$

Stanek [10], studied global properties of solution of functional differential equation

$$x(t)x'(t) = kx(x(t)), \quad 0 < |k| < 1.$$

Here we are concerning with the initial value problem of functional integro-differential equation of self-reference $(\phi(t) = t)$ and state-dependence $(\phi(t) \leqslant t)$

$$\frac{dx}{dt} = f(t, x(\int_0^{\phi(t)} g(s, x(s)) ds)), \quad t \in (0, T], \tag{1.1}$$

with the condition

$$x(0) = x_0, \tag{1.2}$$

The existence of solutions $x \in C[0, T]$ will be studied under assumptions on the functional f and g. The uniqueness of the solution can be deduce, when f and g are Lipschitz continuous. The continuous dependence of the solution on the delay $\phi(t)$, the functional g and initial data x_0 will be proved.

2. Integral representation

Consider the initial value problem of functional integro-differential equation (1.1) with the assumptions:

(1) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition, i.e., f is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0, T]$. There exist a function $c(t)$ that is measurable bounded and there is a positive constant $b > 0$, such that

$$|f(t, x)| \leqslant c(t) + b|x|, \quad |c(t)| \leqslant M;$$

(2) $g : [0, T] \times \mathbb{R} \to \mathbb{R}^+$ satisfies Caratheodory condition, i.e., g is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0, T]$, such that $|g(t, x)| \leqslant 1, (t, x) \in [0, T] \times \mathbb{R}$;

(3) $\phi : [0, T] \to [0, T]$ is continuous;

(4) $bT < 1$.

Definition 2.1. By a solution of the integral equation (2.1) we mean a function $x \in C[0, 1]$ that satisfies (2.1).

Lemma 2.2. *Let the solution of the initial value problem* (1.1)-(1.2) *exists, then it can be represented by the integral equation*

$$x(t) = x_0 + \int_0^t f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta)) ds, \quad t \in [0, T]. \tag{2.1}$$

Proof. Let x be a solution of the initial value problem of functional differential equation (1.1)-(1.2). Integrating both sides of (1.1) we obtain

$$x(t) = x_0 + \int_0^t f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta)) ds, \quad t \in [0, T]. \qquad \square$$

3. Existence of solution

Theorem 3.1. *Let the assumptions* (1)-(4) *be satisfied, then the integral equation* (2.1) *has at least one solution.*

Proof. Define the operator A associated with the integral equation (2.1) by

$$Ax(t) = x_0 + \int_0^t f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds.$$

Let $S_r = \{x \in \mathbb{R} : |x(t) - x(s)| \leqslant L(t-s), L > 0, \|x\| \leqslant r, \forall t, s \in [0, T]\}$, where $r = LT + |x_0|$ and $L = \frac{M + b|x_0|}{1 - bT}$. Then we have, for $x \in S_r$,

$$|Ax(t)| = |x_0 + \int_0^t f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds|$$

$$\leqslant |x_0| + \int_0^t |f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))|ds$$

$$\leqslant |x_0| + \int_0^t (c(s) + b|x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - x(0)| + b|x(0)|)ds$$

$$\leqslant |x_0| + \int_0^t (c(s) + bL \int_0^{\phi(s)} |g(\theta, x(\theta))|d\theta + b|x(0)|)ds$$

$$\leqslant |x_0| + TM + bLT^2 + bT|x_0| \leqslant |x_0| + LT.$$

Now, let $t_1, t_2 \in (0, T]$ such that $|t_2 - t_1| < \delta$, then

$$|Ax(t_2) - Ax(t_1)| = |x_0 + \int_0^{t_2} f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds - x_0 - \int_0^{t_1} f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds|$$

$$\leqslant \int_{t_1}^{t_2} |f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))|ds$$

$$\leqslant \int_{t_1}^{t_2} (c(s) + b|x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - x(0)| + b|x_0|)ds$$

$$\leqslant \int_{t_1}^{t_2} (c(s) + bL \int_0^{\phi(s)} |g(\theta, x(\theta))|d\theta + b|x_0|)ds$$

$$\leqslant (t_2 - t_1)M + (t_2 - t_1)bLT + (t_2 - t_1)b|x_0| = (t_2 - t_1)(M + bLT + b|x_0|) = (t_2 - t_1)L.$$

This proves that $A : S_r \to S_r$ and the class of functions $\{Ax\}$ is uniformly bounded And equi-continuous in S_r.

Let $x_n \in S_r$, $x_n \to x(n \to \infty)$, then from the continuity of the functions f and g, we obtain $f(t, x_n(t)) \to f(t, x(t))$ and $g(t, x_n(t)) \to g(t, x(t))$ as $n \to \infty$. Also

$$\lim_{n \to \infty} Ax_n(t) = \lim_{n \to \infty} [x_0 + \int_0^t f(s, x_n(\int_0^{\phi(s)} g(\theta, x_n(\theta))d\theta))ds].$$

Now

$$|x_n(\int_0^{\phi(s)} g(\theta, x_n(\theta))d\theta) - x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta)| \leqslant |x_n(\int_0^{\phi(s)} g(\theta, x_n(\theta))d\theta) - x_n(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta)|$$

$$+ |x_n(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta)|$$

$$\leqslant L \int_0^{\phi(s)} |g(\theta, x_n(\theta)) - g(\theta, x(\theta))|d\theta + \frac{\epsilon}{2} \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Using assumptions (1)-(2) and Lebesgue dominated convergence theorem [8] we obtain

$$\lim_{n \to \infty} Ax_n(t) = [x_0 + \int_0^t f(s, \lim_{n \to \infty} x_n(\int_0^{\phi(s)} g(\theta, x_n(\theta))d\theta))ds] = Ax(t).$$

Then $Ax_n \to Ax$ as $n \to \infty$. This mean that the operator A is continuous.

Hence by Schauder fixed point Theorem [7] there exist at least one solution $x \in C[0, T]$ of the integral equation (2.1).

Also, differentiating (2.1), we obtain

$$\frac{dx}{dt} = \frac{d}{dt}\left\{ x_0 + \int_0^t f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds \right\} = f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta)).$$

And at $t = 0$ we get

$$x(0) = x_0,$$

then there exist at least one solution $x \in C[0, T]$ of the initial value problem of functional differential equation (1.1)-(1.2). $\square$

4. Uniqueness of the solution

Let f and g satisfy the following assumptions:

(5) $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and satisfies the Lipschitz condition

$$|f(t, x) - f(t, u)| \leqslant b|x - u|;$$

(6) $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and satisfies the Lipschitz condition

$$|g(t, x) - g(t, u)| \leqslant b_1|x - u|;$$

(7) $\phi : [0, T] \to [0, T]$ is continuous;
(8) $(bb_1 LT^2 + bT) < 1$.

Theorem 4.1. *Let the assumptions* (5)-(8) *be satisfied, then the solution of the integral equation* (2.1) *is unique.*

Proof. From assumption (5) we have f is measurable in t for any $x, y \in \mathbb{R}$ and satisfies the Lipschitz condition, then it is continuous in $x \in \mathbb{R}$ $\forall t \in [0, T]$, and

$$|f(t, x)| \leqslant b|x| + |f(t, 0)|.$$

Then condition (1) is satisfied. Also by the same way we can show that assumption (2) satisfied by assumption (5). Now, from Theorem (3.1) the solution of the integral equation (2.1) exists.

Let x, y be two the solutions of (2.1), then

$$|x(t) - y(t)| = |x_0 + \int_0^t f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds - x_0 - \int_0^t f(s, y(\int_0^{\phi(s)} g(\theta, y(\theta))d\theta))ds|$$

$$\leqslant \int_0^t |f(s, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta)) - f(s, y(\int_0^{\phi(s)} g(\theta, y(\theta))d\theta))|ds$$

$$\leqslant b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - y(\int_0^{\phi(s)} g(\theta, y(\theta))d\theta)|ds$$

$$\leqslant b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - x(\int_0^{\phi(s)} g(\theta, y(\theta))d\theta)|ds$$

$$+ b \int_0^t |x(\int_0^{\phi(s)} g(\theta, y(\theta))d\theta) - y(\int_0^{\phi(s)} g(\theta, y(\theta))d\theta)|ds$$

$$\leqslant bL \int_0^t \int_0^{\phi(s)} |g(\theta, x(\theta)) - g(\theta, y(\theta))|d\theta ds + bT\|x - y\|$$

$$\leqslant bb_1 L \int_0^t \int_0^{\phi(s)} |x(\theta) - y(\theta)|d\theta ds + bT\|x - y\|$$

$$\leqslant bb_1 LT^2\|x - y\| + bT\|x - y\| = (bb_1 LT^2 + bT)\|x - y\|.$$

Hence

$$(1 - (bb_1 LT^2 + bT))\|x - y\| \leqslant 0.$$

Since $(bb_1 LT^2 + bT) < 1$, then $x(t) = y(t)$ and the solution of the integral equation (2.1) is unique. $\qquad\square$

5. Continuous dependence

5.1. Continuous dependence on initial data x_0

Definition 5.1. The solution $x \in C[0, T]$ of the integral equation (2.1) depends continuously on initial data x_0, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad |x_0 - x_0^*| < \delta \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the integral equation

$$x^*(t) = x_0^* + \int_0^t f(t, x^*(\int_0^{\phi(s)} g(\theta, x^*(\theta))d\theta))ds, \quad t \in [0, T]. \tag{5.1}$$

Theorem 5.2. *Let the assumptions of Theorem 4.1 be satisfied, then the solution of the integral equation (2.1) depends continuously on initial data x_0.*

Proof. Let x and x^* be two solutions of the integral equation (2.1) and (5.1), respectively, then

$$|x(t) - x^*(t)| = |x_0 + \int_0^t f(t, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta))ds - x_0^* - \int_0^t f(t, x^*(\int_0^{\phi(s)} g(\theta, x^*(\theta))d\theta))ds|$$

$$\leqslant |x_0 - x_0^*| + \int_0^t |f(t, x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta)) - f(t, x^*(\int_0^{\phi(s)} g(\theta, x^*(\theta))d\theta))|ds$$

$$\leqslant \delta + b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - x^*(\int_0^{\phi(s)} g(\theta, x^*(\theta))d\theta)|ds$$

$$\leqslant \delta + b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta))d\theta) - x(\int_0^{\phi(s)} g(\theta, x^*(\theta))d\theta)|ds$$

$$+ b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x^*(\theta)d\theta)) - x^*(\int_0^{\phi(s)} g(\theta, x^*(\theta)d\theta))|ds$$

$$\leqslant \delta + bL \int_0^t |\int_0^{\phi(s)} g(\theta, x(\theta))d\theta - \int_0^{\phi(s)} g(\theta, x^*(\theta))d\theta|ds + b \int_0^t \|x - x^*\|ds$$

$$\leqslant \delta + bL \int_0^t \int_0^{\phi(s)} |g(\theta, x(\theta)) - g(\theta, x^*(\theta))|d\theta ds + b \int_0^t \|x - x^*\|ds$$

$$\leqslant \delta + bLb_1 \int_0^t \int_0^{\phi(s)} |x(\theta) - x^*(\theta)|d\theta ds + b \int_0^t \|x - x^*\|ds$$

$$\leqslant \delta + bLb_1 T^2\|x - x^*\| + bT\|x - x^*\|.$$

Hence

$$\|x - x^*\| \leqslant \frac{\delta}{1 - (bLb_1 T^2 + bT)} = \epsilon.$$

This mean that the solution of the integral equation (2.1) depends continuously on x_0. The proof is completed. $\qquad\square$

5.2. Continuous dependence on the delay $\phi(t)$

Definition 5.3. The solution $x \in C[0, T]$ of the integral equation (2.1) depends continuously on the delay $\phi(t)$, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad \text{s.t} \quad |\phi(t) - \phi^*(t)| < \delta \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the integral equation

$$x^*(t) = x_0 + \int_0^t f(t, x^*(\int_0^{\phi^*(t)} g(\theta, x^*(\theta)) d\theta)) ds, \quad t \in [0, T]. \tag{5.2}$$

Theorem 5.4. *Let the assumptions of Theorem 4.1 be satisfied, then the solution of the integral equation (2.1) depends continuously on the delay $\phi(t)$.*

Proof. Let x and x^* be two solutions of the integral equation (2.1) and (5.2), respectively, then

$$|x(t) - x^*(t)| = |x_0 + \int_0^t f(t, x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta)) ds - x_0 - \int_0^t f(t, x^*(\int_0^{\phi^*(s)} g(\theta, x^*(\theta)) d\theta)) ds|$$

$$\leqslant \int_0^t |f(t, x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta)) - f(t, x^*(\int_0^{\phi^*(s)} g(\theta, x^*(\theta)) d\theta))| ds$$

$$\leqslant b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta) - x^*(\int_0^{\phi^*(s)} g(\theta, x^*(\theta) d\theta))| ds$$

$$\leqslant b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta) - x(\int_0^{\phi^*(s)} g(\theta, x^*(\theta) d\theta))| ds$$

$$+ b \int_0^t |x(\int_0^{\phi^*(s)} g(\theta, x^*(\theta) d\theta)) - x^*(\int_0^{\phi^*(s)} g(\theta, x^*(\theta) d\theta))| ds$$

$$\leqslant bL \int_0^t |\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta - \int_0^{\phi^*(s)} g(\theta, x^*(\theta)) d\theta| ds + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bL \int_0^t |\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta - \int_0^{\phi^*(s)} g(\theta, x(\theta)) d\theta$$

$$+ \int_0^{\phi^*(s)} g(\theta, x(\theta)) d\theta - \int_0^{\phi^*(s)} g(\theta, x^*(\theta)) d\theta| ds + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bL \int_0^t (\int_0^{\phi^*(s)} |g(\theta, x(\theta)) - g(\theta, x^*(\theta))| d\theta + \int_{\phi^*(s)}^{\phi(s)} |g(\theta, x(\theta))| d\theta) ds + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bL \int_0^t (b_1 \int_0^{\phi^*(s)} |x(\theta) - x^*(\theta) d\theta| + \int_{\phi^*(s)}^{\phi(s)} |g(\theta, x(\theta)) d\theta|) ds + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bLb_1 T^2 \|x - x^*\| + bLT\delta + bT\|x - x^*\|.$$

Hence

$$\|x - x^*\| \leqslant \frac{bLT\delta}{1 - (bLb_1 T^2 + bT)} = \epsilon.$$

This mean that the solution of the integral equation (2.1) depends continuously on delay $\phi(t)$. The proof is completed. □

5.3. Continuous dependence on the functional g

Definition 5.5. The solution $x \in C[0, T]$ of the integral equation (2.1) depends continuously on the functional g, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad \text{s.t} \quad |g - g^*| < \delta \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the integral equation

$$x^*(t) = x_0 + \int_0^t f(t, x^*(\int_0^{\phi(s)} g^*(\theta, x^*(\theta)) d\theta)) ds, \quad t \in [0, T]. \tag{5.3}$$

Theorem 5.6. *Let the assumptions of Theorem 4.1 be satisfied, then the solution of the integral equation (2.1) depends continuously on the functional g.*

Proof. Let x and x^* be two solutions of the integral equation (2.1) and (5.3), respectively, then

$$|x(t) - x^*(t)| = |x_0 + \int_0^t f(t, x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta)) ds - x_0 - \int_0^t f(t, x^*(\int_0^{\phi(s)} g^*(\theta, x^*(\theta)) d\theta)) ds|$$

$$\leqslant \int_0^t |f(t, x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta)) - f(t, x^*(\int_0^{\phi(s)} g^*(\theta, x^*(\theta)) d\theta))| ds$$

$$\leqslant b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta) - x^*(\int_0^{\phi(s)} g^*(\theta, x^*(\theta)) d\theta)| ds$$

$$\leqslant b \int_0^t |x(\int_0^{\phi(s)} g(\theta, x(\theta)) d\theta) - x(\int_0^{\phi(s)} g^*(\theta, x^*(\theta)) d\theta)| ds$$

$$+ b \int_0^t |x(\int_0^{\phi(s)} g^*(\theta, x^*(\theta) d\theta)) - x^*(\int_0^{\phi(s)} g^*(\theta, x^*(\theta) d\theta))| ds$$

$$\leqslant bL \int_0^t \int_0^{\phi(s)} |g(\theta, x(\theta)) - g^*(\theta, x^*(\theta))| d\theta ds + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bL \int_0^t \int_0^{\phi(s)} [|g(\theta, x(\theta)) - g(\theta, x^*(\theta))| + |g(\theta, x^*(\theta)) - g^*(\theta, x^*(\theta))|] d\theta ds + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bLb_1 \int_0^t \int_0^{\phi(s)} |x(\theta) - x^*(\theta)| d\theta ds + bLT^2 \delta + b \int_0^t \|x - x^*\| ds$$

$$\leqslant bLb_1 T^2 \|x - x^*\| + bT \|x - x^*\| + bLT^2 \delta.$$

Hence

$$\|x - x^*\| \leqslant \frac{bLT^2 \delta}{[1 - (bLb_1 T^2 + bT)]} = \epsilon.$$

This mean that the solution of the integral equation (2.1) depends continuously on the functional g. The proof is completed. □

5.4. Examples

In this section we offer some examples to illustrate our results.

Example 5.1. Consider the following nonlinear differential equation

$$\frac{dx}{dt} = \frac{t+2}{t+5} + \frac{1}{4} x(\int_0^{\alpha t^\beta} \frac{\sin^2(x(s))}{s + e^{|x(s)|}} ds), \quad \beta \in [1, B], \alpha, t \in (0, 1], \tag{5.4}$$

with initial condition

$$x(0) = 0. \tag{5.5}$$

Set

$$f\left(t, x\left(\int_0^{\phi(t)} g(s, x(s)) ds\right)\right) = \frac{t+2}{t+5} + \frac{1}{4} x\left(\int_0^{\alpha t^\beta} \frac{\cos^2(x(s))}{s + e^{|x(s)|}} ds\right).$$

Then

$$\left| f\left(t, x\left(\int_0^{\phi(t)} g(s, x(s)) ds \right) \right) \right| \leqslant \frac{t+2}{t+5} + \frac{1}{4}(|x|),$$

and also

$$|g(s, x(s))| \leqslant 1, \quad \alpha t^\beta \leqslant t.$$

It is clear that the assumptions (1)-(4) of Theorem 3.1 are satisfied with $|c(t)| = |\frac{t+2}{t+5}| \leqslant \frac{1}{2}$ that is measurable bounded, $b = \frac{1}{4}$, $bT = \frac{1}{4} < 1$, and $L = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$. Therefore, by applying to Theorem 3.1, the given initial value problem (5.4)-(5.5) has a continuous solution.

Example 5.2. Consider the following nonlinear differential equation

$$\frac{dx}{dt} = \frac{1}{5}(t+1) + \frac{x\left(\int_0^{\beta t} \frac{s}{\cos^2(s) + e^{|x(s)|}} \right)}{\sqrt{t+4}}, \quad \beta, t \in (0, 1], \tag{5.6}$$

with initial condition

$$x(0) = 0. \tag{5.7}$$

Set

$$f\left(t, x\left(\int_0^{\phi(t)} g(s, x(s)) ds \right) \right) = \frac{1}{5}(t+1) + \frac{x\left(\int_0^t \frac{\cos^2(s)}{s + e^{|x(s)|}} \right)}{\sqrt{t+4}}$$

Then

$$\left| f\left(t, x\left(\int_0^{\phi(t)} g(s, x(s)) ds0 \right) \right) \right| \leqslant \frac{1}{5}(t+1) + \frac{1}{2}|x|,$$

and also

$$|g(s, x(s))| \leqslant 1, \quad \beta \leqslant t.$$

It is clear that the assumptions (1)-(4) of Theorem 3.1 are satisfied with $|c(t)| = |\frac{1}{5}(t+1)| \leqslant \frac{2}{5}$ that is measurable bounded, $b = \frac{1}{3}$, $bT = \frac{1}{2} < 1$, and $L = \frac{\frac{2}{5}}{1-\frac{1}{2}} = \frac{4}{5}$. Therefore, by applying to Theorem 3.1, the given non-local problem (5.6)-(5.7) has a continuous solution.

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