Stability analysis of a tritrophic model with stage structure in the prey population

Gamaliel Blé\textsuperscript{a,*}, Miguel Angel Dela-Rosa\textsuperscript{b}, Iván Loreto-Hernández\textsuperscript{b}

\textsuperscript{a}División Académica de Ciencias Básicas, UJAT, Km 1, Carretera Cunduacán-Jalpa de Méndez, Cunduacán, Tabasco, c.p. 86690, México.
\textsuperscript{b}División Académica de Ciencias Básicas, CONACyT-UJAT, Km 1, Carretera Cunduacán-Jalpa de Méndez, Cunduacán, Tabasco, c.p. 86690, México.

Abstract

We analyze the role of the age structure of a prey in the dynamics of a tritrophic model. We study the effect of predation on a non-reproductive prey class, when the reproductive class of the prey has a defense mechanism. We consider two cases accordingly to the interaction between predator and reproductive class of the prey. In the first case, the functional response is Holling type II and it is possible to show up to two positive equilibria. When we consider a defense mechanism the functional response is Holling type IV. In both cases, we show sufficient parameter conditions to have a stable limit cycle obtained by a supercritical Hopf bifurcation. Some numerical simulations are carried out.

Keywords: Hopf’s Bifurcation, tritrophic model, coexistence of species, prey age structure.


1. Introduction

The mathematical modelling has become a very useful tool in ecology because it can be used to answer general or specific questions about an ecosystem. One of the interactions between species that has been most studied is the predator-prey type. Its study began with the Lotka-Volterra model, from which other models have been obtained considering more variables and more parameters in order to be closer to reality, [13]. For example, it has been included the carrying capacity, the handling time, the interference among predators and defense mechanism, among others. Since the age specific fecundity or fertility rate of a population is one of the most fundamental parameter in both the theory and practice of populations dynamics, [3], the study of age structured models is a topic of ecological interest.

The analysis of prey-predator interaction with age structure has been approached with different models. At first, the structure of ages was considered in the predator population, such is the case of Beddington et al. work, who studied a difference equations system dividing the predator population into the

\textsuperscript{*}Corresponding author

Email addresses: gble@ujat.mx (Gamaliel Blé), madelarosaca@conacyt.mx (Miguel Angel Dela-Rosa), iloretohe@conacyt.mx (Iván Loreto-Hernández)

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young and adult class and assuming that each one has a different attack rate. They showed that stable coexistence is possible, [1]. Hasting et al. analyzed a differential equations system with age structure in the predator. They proved the existence of stable equilibria and determined the effect of age structure, [6]. Cushing et al. studied an integro differential equations system derived from the McKendrick model. They considered an age structure in the predator and proved the parameter conditions to have the coexistence, [3]. Toth analyzed the effects of age structure on the predator prey resource chemostat model, he proved the coexistence equilibrium and the parameter condition to have a Hopf bifurcation, [19]. Xi et al. analyzed a delayed tritrophic food chain model with stage structure in predator and superpredator populations. They determined sufficient conditions to have positivity and permanence in the solution of the system, [20].

On the other hand, in nature it has also been found predators that eat only adults, or immature prey, it is the cicada case, which is preyed only in adult stage, or some species of perch which feed on immature prey, [9–11]. Hence, it is important to study the model with age structure in the prey population. Zhang et al. considered a predator prey model with two stage structure in the prey (immature and mature). They supposed that predator feeds only on the immature class with Lotka-Volterra functional response and they obtained necessary and sufficient conditions for the coexistence or extinction. Falconi, analyzed a predator prey model, dividing the prey population in the reproductive and non reproductive class. He considered three different functional responses to predation on the nonreproductive class and he showed the conditions to have the coexistence, [5]. More recently, it has been considered the analysis of predator prey systems with age structure in the population prey or in the population predator. Tang et al. analyzed a predator prey model with age structure in the predator population. They showed the existence and uniqueness of positive equilibrium and exhibit a Hopf bifurcation, [18]. Promrak et al. considered a predator prey model with age structure in the prey population. They showed the stability solutions and the bifurcation diagrams of the system, [15].

In this paper we analyze a tritrophic model focusing on two classes (reproductive and nonreproductive) in the prey population. Reproductive population will be denoted as \( w \), the non-reproductive by \( x \), the predator and superpredator populations by \( y \) and \( z \), respectively. We will assume that the predator population attacks in a different way to the two classes in the prey. The non-reproductive class contains the oldest organisms and its interaction with the predator is of a Lotka-Volterra type. The interaction between the reproductive population and predator is modeled by a functional response Holling type II or IV, this last functional response considers a defense mechanism in the reproductive class. Explicitly, we have the following differential equations system

\[
\begin{align*}
\frac{dw}{dt} &= w\rho \left(1 - \frac{w + x}{R} \right) - f_1(w, x)y - \nu w, \\
\frac{dx}{dt} &= \nu w - d_1x - a_3xy, \\
\frac{dy}{dt} &= c_1y f_1(w, x) + c_2a_3xy - d_2y - \frac{a_2y}{y + b_2}z, \\
\frac{dz}{dt} &= c_3 - \frac{a_2y}{y + b_2}z - d_3z.
\end{align*}
\] (1.1)

We consider two cases, which are obtained by considering that the functional response \( f_1(w, x) = \frac{a_1w}{w + x + b_1} \) or \( f_1(w, x) = \frac{a_2w}{w^2 + x + b_1} \). Even though these functional responses are not precisely of the classical Holling type (see for instance [4, 17]), through this paper we will call them of Holling type II and IV, respectively. We assume that the birth rate of the non reproductive population is proportional to the reproductive one with proportionality constant \( \nu \), hence the non reproductive population does not extinguish unless the reproductive class does.

For ecological reasons, all the parameters are positive and we restrict our analysis to the positive set \( \Gamma := \{(w, x, y, z) \in \mathbb{R}^4 : w > 0, \ x > 0, \ y > 0, \ z > 0\} \).

2. Criteria for stability and Hopf bifurcation

In next lemma we characterize under some hypothesis the Routh-Hurwitz test, [16], and the necessary condition to have a Hopf bifurcation.
Let \( \text{pol}(\lambda) = \lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 \) be the characteristic polynomial for the linear approximation of the system (1.1) at some equilibrium point \( P \). Set
\[
\text{EQ} := A_1^2 A_4 - A_1 A_2 A_3 + A_3^2. \tag{2.1}
\]

On the other hand, to have a Hopf bifurcation of the differential system (1.1) at \( P \) it is needed that the characteristic polynomial \( \text{pol}(\lambda) \) at \( P \) factorizes as
\[
(\lambda - \lambda_3)(\lambda - \lambda_4)(\lambda^2 + \omega^2), \quad \omega > 0. \tag{2.2}
\]

In this notation we state the following criteria which will be used later.

**Lemma 2.1.** If \( A_i > 0, \, i = 1, \ldots, 4 \), the following statements hold.

(i) The equilibrium point \( P \) is locally asymptotically stable if and only if \( \text{EQ} < 0 \).

(ii) Assume that \( \text{pol}(\lambda) \) factorizes as in (2.2). Then, its roots are
\[
\omega = \pm i \sqrt{\frac{A_3}{A_1}} \quad \text{and} \quad \lambda_{3,4} = \frac{-A_1 \pm \sqrt{A_1^2 - 4A_2 + 4\omega^2}}{2}
\]
if and only if \( \text{EQ} = 0 \).

**Proof.** We prove (i). Under the hypothesis, the claim follows directly observing that the Hurwitz determinants for \( \text{pol}(\lambda) \) are given by
\[
\text{det}_1 = A_1, \quad \text{det}_2 = A_1 A_2 - A_3, \quad \text{det}_3 = -A_2^2 + A_1^2 (-A_4 + A_1 A_2 A_3), \quad \text{det}_4 = -A_4 (-A_1 A_2 A_3 + A_3^2 + A_2^2 A_4)
\]
and all are positive if and only if \( A_2 > \frac{A_1^3 A_4 + A_3^2}{A_1 A_3} \). We now prove (ii). Assume that \( \text{pol}(\lambda) \) is as in the hypothesis. Then it factorizes as
\[
P(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)(\lambda^2 + \omega^2),
\]
where \( \omega > 0 \) if and only if
\[
A_1 = -(\alpha_1 + \alpha_2), \quad A_2 = (\omega^2 + \alpha_1 \alpha_2), \quad A_3 = -(\alpha_1 + \alpha_2) \omega^2, \quad A_4 = \alpha_1 \alpha_2 \omega^2.
\]
Equivalently,
\[
A_2 = (\omega^2 + \alpha_1 \alpha_2), \quad A_3 = A_1 \omega^2, \quad A_4 = \alpha_1 \alpha_2 \omega^2.
\]
That is,
\[
\alpha_1 \alpha_2 = A_2 - \omega^2, \quad A_3 = A_1 \omega^2, \quad A_4 = \alpha_1 \alpha_2 \omega^2.
\]
Equivalently,
\[
(-A_1 - \alpha_2) \alpha_2 = A_2 - \frac{A_3}{A_1} \quad \text{and} \quad A_4 = \left( A_2 - \frac{A_3}{A_1} \right) \frac{A_3}{A_1},
\]
that is,
\[
\alpha_2^2 + \alpha_1 \alpha_2 + A_2 A_1 - A_3 = 0 \quad \text{and} \quad A_1^2 A_4 - A_1 A_2 A_3 + A_3^2 = 0.
\]
This completes the proof. \( \square \)

Along this paper, the transversality condition to have a Hopf bifurcation will be computed by means of the following result that appeared as an exercise in [7, p. 189].

**Proposition 2.2.** Let \( M(\tau) \) be a parameter-dependent real \( (n \times n) \)-matrix which has a simple pair of complex eigenvalues \( \xi(\tau) \pm i\omega(\tau) \) such that \( \xi(\tau_0) = 0 \) and \( \omega(\tau_0) := \omega_0 > 0 \). Then, the derivative of the real part of the complex eigenvalues is given by
\[
\frac{d\xi}{d\tau}(\tau_0) = \text{Re} \left( \bar{p}^\text{tr} \left( \frac{dM}{d\tau}(\tau_0) \cdot q \right) \right),
\]
where \( p, q \in \mathbb{C}^n \) are eigenvectors satisfying the normalization conditions
\[
M(\tau_0) q = i \omega_0 q, \quad M^{\text{tr}}(\tau_0) p = -i \omega_0 p, \quad \text{and} \quad \bar{p}^\text{tr} \cdot q = 1.
\]
3. Case $f_1(w, x)$ Holling II

**Lemma 3.1.** The differential system (1.1) has:

(i) A unique equilibrium point in $\Gamma$ if

$$a_2 = \frac{d_3 + k_1}{c_3}, \quad d_3 = \frac{a_{1c} k_1 k_2 (a_{1b} b_2 d_3 + d_1 k_1)}{(a_{1b} b_2 d_3 + k_1 (d_1 + \nu))(a_{1b} b_2 d_3 + k_1 (b_1 \nu + k_2))}, \quad \rho = \frac{a_1 b_2 d_3}{b_1 k_1}, \quad R = \frac{b_1 (a_1 b_2 d_3 + k_1 (b_1 \nu + k_2))}{a_1 b_2 d_3},$$

where $k_1$ and $k_2$ are positive real numbers. The equilibrium is given by

$$P_0 = \left( \frac{b_1 k_1 k_2 (a_{1b} b_2 d_3 + d_1 k_1)}{(a_{1b} b_2 d_3 + k_1 (d_1 + \nu))(a_{1b} b_2 d_3 + k_1 (b_1 \nu + k_2))}, \frac{b_1 k_1 k_2 y}{(a_{1b} b_2 d_3 + k_1 (d_1 + \nu))(a_{1b} b_2 d_3 + k_1 (b_1 \nu + k_2))}, \frac{b_1 d_3}{a_{1b} b_2 d_3 + k_1 (d_1 + \nu)), \frac{a_1 b_1 b_2 c_3 k_1 k_2 y}{a_{1b} b_2 d_3 + k_1 (b_1 \nu + k_2))} \right).$$

(ii) Two equilibrium points in $\Gamma$ if

$$a_1 = k_2 \left( \frac{k_2 (w_0 + x_0)}{4w_0 y_0 (2d_1 x_0 + 2k_1 + k_2)} + \frac{w_0 + x_0}{2w_0 y_0} \right), \quad a_2 = \frac{d_3 (b_2 + y_0)}{c_3 y_0}, \quad a_3 = \frac{k_1}{x_0 y_0},$$

$$b_1 = \frac{k_2 (w_0 + x_0)}{2(2d_1 x_0 + 2k_1 + k_2)}, \quad c_1 = \frac{2(2c_2 c_3 k_1 (4d_1 x_0 + 4k_1 + k_2) + k_8)}{c_3 k_2 (4d_1 x_0 + 4k_1 + 3k_2)}, \quad c_2 = \frac{c_2 k_1}{y_0},$$

$$R = 2(w_0 + x_0) \text{ and } \rho = \frac{2d_1 x_0 + 2k_1 + k_2}{w_0},$$

where $k_1, k_2, k_8, w_0, y_0$ and $x_0$ are positive real numbers. The equilibrium points are given by

$$P_0 = \left( w_0, x_0, y_0, \frac{2c_2 c_3 k_1 (4d_1 x_0 + 4k_1 + k_2) + k_8}{d_3 (4d_1 x_0 + 4k_1 + 3k_2)} \right),$$

$$P_1 = \left( \frac{w_0 x_0 y_0}{k_2 w_0 + k_2 x_0 + k_2 y_0}, \frac{w_0}{4d_1 x_0 + 4k_1 + 2k_2}, \frac{y_0}{4d_1 x_0 + 4k_1 + 2k_2}, \frac{k_8}{8d_1 x_0 + 8d_3 k_3 + 4d_3 k_2} \right).$$

Proof. In any case, since all the parameters are positive and the points of interest are in $\Gamma$, the equilibrium points for the differential system (1.1) must satisfy the system

$$a_1 R y + (b_1 + w + x) (R (\nu - \rho) + \rho (w + x)) = 0,$$

$$\nu w - x (a_3 y + d_1) = 0,$$

$$(b_2 + y) ((b_1 + w + x) (d_2 - a_3 c_2 x) - a_1 c_1 w) + a_2 z (b_1 + w + x) = 0,$$

$$-a_2 c_3 y + b_2 d_3 + d_3 y = 0. \quad (3.1)$$

Assume that $P_0 = (w_0, x_0, y_0, z_0) \in \Gamma$ is an equilibrium point satisfying system (3.1). From fourth equation we have that $y_0 = \frac{b_1 d_3}{k_1}$ where $k_1 = a_2 c_3 - d_3 > 0$, that is, $a_2 = \frac{k_1 d_3}{c_2}$. Substituting $y_0$ in the other equations, the system (3.1) at $P_0$ simplifies to

$$a_1 b_2 d_3 R + k_1 (b_1 + w_0 + z_0) (R (\nu - \rho) + \rho (w_0 + x_0)) = 0,$$

$$\nu k_1 w_0 - x_0 (a_3 b_2 d_3 + d_1 k_1) = 0,$$

$$c_3 b_2 (b_1 + w_0 + x_0) (d_2 - a_3 c_2 x_0) - a_1 c_1 w_0 + k_1 z_0 (b_1 + w_0 + x_0) = 0.$$

Solving second equation for $w_0$, we have $w_0 = \frac{x_0 (a_3 b_2 d_3 + d_1 k_1)}{k_1 v}$. Substituting $w_0$ in the other equations, the system (3.1) at $P_0$ becomes

$$\nu^2 a_1 b_2 d_3 k_1 R + (a_3 b_2 d_3 x_0 + k_1 v (b_1 + x_0) + d_1 k_1 x_0) (a_3 b_2 d_3 R x_0 + k_1 (\rho x_0 (d_1 + \nu) + \nu R (\nu - \rho))) = 0,$$

$$((a_3 b_2 d_3 x_0 + k_1 v (b_1 + x_0) + d_1 k_1 x_0) (b_2 c_3 (d_2 - a_3 c_2 x_0) + k_1 z_0) - a_1 b_2 c_1 c_3 x_0 (a_3 b_2 d_3 + d_1 k_1)) = 0. \quad (3.2)$$

Solving second equation in (3.2) for $z_0$ we have

$$z_0 = \frac{b_2 c_3}{k_1} \left( \frac{a_1 c_1 x_0 (a_3 b_2 d_3 + d_1 k_1)}{a_3 b_2 d_3 x_0 + k_1 v (b_1 + x_0) + d_1 k_1 x_0} + a_3 c_2 x_0 - d_2 \right). \quad (3.3)$$
Substituting \( z_0 \) in first equation in (3.2) we have that the system (3.1) simplifies to
\[
R \left( \frac{a_1 b_2 d_3}{k_1} + b_1 (\nu - \rho) \right) + \frac{x_0 (b_1 \rho + R (\nu - \rho)) (a_3 b_2 d_3 + k_1 (d_1 + \nu))}{k_1 \nu} + \frac{\rho x_0^2 (a_3 b_2 d_3 + k_1 (d_1 + \nu))^2}{k_1^2 \nu^2} = 0.
\]

Now, assume that \( R (\nu - \rho) + b_1 \rho < 0 \). Then there is a positive real number \( k_2 \) such that \( R (\nu - \rho) + b_1 \rho = -k_2 \). Set \( \rho = k_3 + \nu \) for some \( k_3 > 0 \) and \( k_3 = \frac{a_1 b_1 d_3}{b_1 k_1} \). Then, \( R = \frac{b_1 (a_1 b_2 d_3 + k_1 (b_1 \nu + k_2))}{a_1 b_2 d_3} \). Substituting \( R \) and \( \rho \) in the above quadratic equation with respect to \( x_0 \) it has a positive root \( x_0 \) given by
\[
x_0 = \frac{b_1 k_3 k_2 \nu}{(a_1 b_2 d_3 + b_1 k_1 \nu) (a_3 b_2 d_3 + k_1 (d_1 + \nu))}.
\]

Finally, from (3.3) we have that \( z_0 > 0 \) if we take \( d_2 = \frac{a_1 c_1 x_0 (a_2 b_2 d_3 + d_1 k_1)}{a_3 b_2 d_3 + k_1 \nu (b_1 x_0 + d_1 k_1 x_0)} \). Hence we have proved the claim (i).

We now prove claim (ii). We suppose that \( P_0 = (w_0, x_0, y_0, z_0) \in \Gamma \) satisfies (3.1). Solving first, second, third, and fourth equation in the variables \( a_1, a_2, a_3, \) and \( c_1 \), respectively, one gets
\[
a_1 = \frac{b_1 (b_1 + w_0 + x_0)}{R y_0}, \quad a_2 = \frac{d_3 (b_2 + y_0)}{c_3 y_0}, \quad a_3 = \frac{\nu w_0 - d_1 x_0}{x_0 y_0}, \quad \text{and} \quad c_1 = \frac{2 d_3 z_0}{c_3 k_2}.
\]
Taking \( \nu = \frac{d_1 x_0 + k_1}{y_0} \), \( d_2 = \frac{c_2 k_1}{y_0} \), \( R = 2 (w_0 + x_0) \), and \( \rho = \frac{2 d_1 x_0 + 2 k_1 + k_2}{w_0} \), where \( k_1, k_2 > 0 \) we have that
\[
a_1 = \frac{k_2 (b_1 + w_0 + x_0)}{2 w_0 y_0}, \quad a_2 = \frac{d_3 (b_2 + y_0)}{c_3 y_0}, \quad a_3 = \frac{k_1}{x_0 y_0}, \quad \text{and} \quad c_1 = \frac{2 d_3 z_0}{c_3 k_2}.
\]
and \( P_0 \in \Gamma \) is an equilibrium point for the differential system (1.1). Hence under these conditions for the parameters \( a_1, a_2, a_3, \) and \( c_1 \) the system (3.1) takes the form
\[
y_0 (b_1 + w + x) (2 d_1 x_0 (w - w_0 + x - x_0) + 2 k_1 (w - w_0 + x - x_0) + k_2 (w - 2 w_0 + x - 2 x_0))
\]
\[
\quad + k_2 y (w_0 + x_0) (b_1 + w_0 + x_0) = 0,
\]
\[
x_0 y_0 w (d_1 x_0 + k_1) - w_0 x (x_0 y_0 d_1 + k_1 y_0) = 0,
\]
\[
\quad (b_1 + w + x) (d_3 x_0 z (b_2 + y_0) - c_2 c_3 k_1 (b_2 + y_0) (x - x_0)) - \frac{d_3 w_0 (b_2 + y) (b_1 + w_0 + x_0)}{x_0} = 0,
\]
\[
\quad \frac{w_0}{b_2 d_3 (y - y_0)} = 0.
\]

Assume that \( P_1 = (w_1, x_1, y_1, z_1) \in \Gamma \setminus \{ P_0 \} \) satisfies (3.4). We will give conditions to find expressions for \( P_1 \). Since all the parameters are positive, from fourth equation we have \( y_1 = y_0 \) and substituting \( y_1 \) in the other equations we have that system (3.4) simplifies to
\[
(w_1 - w_0 + x_1 - x_0) (2 k_1 (w_1 + x_1) + k_2 (w_1 - w_0 + x_1))
\]
\[
\quad + (-k_2 + 2 d_1 (w_1 + x_1) x_0 + b_1 (2 k_1 + k_2 + 2 d_1 x_0)) = 0,
\]
\[
w_1 x_0 - w_1 x_1 = 0,
\]
\[
-w_0 (b_1 + w_1 + x_1) (c_2 c_3 k_1 (x_1 - x_0) - d_3 x_0 z_1) - d_3 w_1 x_0 z_0 (b_1 + w_0 + x_0) = 0.
\]

Therefore, from second equation \( w_1 = \frac{w_0 x_0}{x_0} \). Substituting \( w_1 \) in first equation and solving the resulting equation for \( x_1 \), we have \( x_1 = \frac{x_0 (k_2 (w_0 + x_0) - b_1 (2 d_1 x_0 + 2 k_1 + k_2))}{w_0 x_0 + 2 d_1 x_0 + 2 k_1 + k_2} \). Since \( x_1 \) must be positive, we have \( k_2 (w_0 + x_0) - b_1 (2 d_1 x_0 + 2 k_1 + k_2) = k_0 \) for some \( k_0 > 0 \). Take \( k_0 = \frac{1}{2} k_2 (w_0 + x_0) \). Then, substituting \( w_1 \) and \( x_1 \) in third equation we have \( z_1 = \frac{d_3 x_0 (d_1 x_0 + 4 k_1 + 4 k_2) - 2 c_2 c_3 k_1 (4 d_1 x_0 + 4 k_1 + 4 k_2)}{4 d_3 (2 d_1 x_0 + 2 k_1 + k_2)} \). Since \( z_1 \) must be positive, we take \( k_8 > 0 \) such that \( -2 c_2 c_3 k_1 (4 k_1 + 2 k_2 + 4 d_1 x_0) + d_3 (4 k_1 + 3 k_2 + 4 d_1 x_0) z_0 = k_8 \). These conditions guarantee that \( P_1 \) and \( P_0 \) are as claim (ii) states. \( \square \)
3.1. Dynamics of one equilibrium point

**Lemma 3.2.** If the hypothesis of Lemma 3.1 (i) are satisfied and

\[
\begin{align*}
k_1 &= d_3, \quad a_1 = \frac{b_1 \nu}{b_2}, \quad a_2 = 2d_3, \quad a_3 = \frac{3\nu}{b_2}, \quad d_1 = 2\nu, \quad k_2 = 35b_1\nu, \quad c_1 = \frac{2738c_2}{395}, \\
d_2 &= \frac{1295b_1 c_2 \nu}{237b_2}, \quad b_{20} := b_1 c_2 \left(\frac{27983097981 d_3}{16779436846 \nu} + \frac{28749}{37756}\right), \text{ and } \nu \neq \frac{30421101094081d_3}{16443848109080},
\end{align*}
\]

then the eigenvalues of the linear approximation \( M_0(b_{20}) \) of system (1.1) at \( P_0 \) are

\[
\begin{align*}
\lambda_{1,2} &= \pm i \sqrt{\frac{41948592112 \nu}{239}}, \\
\lambda_{3,4} &= -\frac{10961510700d_3 \nu}{7038482959d_3 \pm \sqrt{745681 \nu d_3(30421101094081d_3 - 16443848109080 \nu)}}.
\end{align*}
\]

**Proof.** Assume that the hypothesis in Lemma 3.1 (i) are valid. To simplify the analysis we set \( k_1 = d_3, \ a_1 = \frac{b_1 \nu}{b_2} \) and \( a_3 = \frac{d_1 \nu}{b_2} \). Then the characteristic polynomial for the linear approximation \( M(P_0) \) of system (1.1) at \( P_0 \) is \( \text{pol}_1(\nu, b_2) = \lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 \), where

\[
\begin{align*}
A_1 &= \frac{4b_2(2d_1 + \nu)}{8b_2(d_1 + \nu)(2b_1 \nu + k_2)^2}, \\
A_2 &= \frac{k_2(4\nu(2d_1 + \nu)E_1 + c_2(d_1 + \nu)(2b_1 \nu + k_2)E_2)}{16b_2(d_1 + \nu)^2(2b_1 \nu + k_2)^3}, \\
A_3 &= \frac{k_2(2d_1 + \nu)(4b_1 c_1 \nu(8b_2 \nu(2d_1 + \nu) - k_2^2) + c_2(2b_1 \nu + k_2)E_3)}{16b_2(d_1 + \nu)(2b_1 \nu + k_2)^3}, \\
A_4 &= \frac{c_2 d_3 k_3 \nu(2d_1 + \nu)}{8b_2(2b_1 \nu + k_2)^2}, \\
E_0 &= \nu(8b_1^2 \nu^2 + 8b_1 k_2 \nu + 3k_2^2) + 2d_1(2b_1 \nu + k_2)^2, \\
E_1 &= b_1 c_1 \nu(8b_2^2 \nu^3 - d_1(k_2 - 2b_1 \nu)(4b_1 \nu + k_2) - k_2^2 \nu) + 4b_2 k_2(d_1 + \nu)^2(2b_1 \nu + k_2), \\
E_2 &= \nu(8b_2^2 \nu^2 + 8b_1 k_2 \nu + 3k_2^2) + 2d_1(2b_1 \nu + k_2)^2 + 2d_1(d_3(2b_1 \nu + k_2)^2 + 4b_1 \nu^2(2b_1 \nu + k_2)), \\
E_3 &= \nu(8b_1^2 \nu^3 + 8b_1 \nu^2(b_1 d_3 + k_2) + 8b_1 d_3 k_2 \nu + 3d_3 k_2^2) + 2d_1(2b_1 \nu + k_2)(2b_1 d_3 \nu + 4b_1 \nu^2 + d_3 k_2^2).
\end{align*}
\]

Therefore, setting \( d_1 = 2\nu \) expression (2.1) becomes the function, in the variable \( b_2 \),

\[
\text{EQ}(b_2) = \frac{5k_2^2}{165888b_2^2(2b_1 \nu + k_2)^6}
\]

where

\[
\begin{align*}
G_0 &= 4b_1 c_1 \nu^3(40b_2^2 \nu^2 - k_2) + c_2 \nu^2(2b_1 \nu + k_2)(48b_1^2 \nu^3 + 24b_1 \nu^2(b_1 d_3 + k_2) + 24b_1 d_3 k_2 \nu + 7d_3 k_2^2), \\
G_1 &= 2b_2 \nu(24b_1^2 \nu^2 + 24b_1 k_2 \nu + 7k_2^2) - 3c_2 k_2(2b_1 \nu + k_2)^2, \\
G_2 &= 4b_1 c_1 \nu^3(40b_2^2 \nu^2 - k_2^2) + c_2 \nu^2(2b_1 \nu + k_2)(48b_1^2 \nu^3 + 24b_1 \nu^2(b_1 d_3 + k_2) + 24b_1 d_3 k_2 \nu + 7d_3 k_2^2), \\
G_3 &= 20b_2 \nu^3(b_1 c_1 (24b_1^2 \nu^2 - 4b_1 k_2 \nu - 3k_2^2) + 36b_2 k_2(2b_1 \nu + k_2)) \\
&\quad + 3c_2 \nu^2(2b_1 \nu + k_2)^2(42b_2^2 \nu^2 + 24b_1 k_2 \nu + k_2^2) + 6d_3(2b_1 \nu + k_2)^2), \\
G_4 &= 3c_2 k_2(2b_1 \nu + k_2)^2 - 20b_2 \nu(24b_1^2 \nu^2 + 24b_1 k_2 \nu + 7k_2^2).
\end{align*}
\]

Now, if \( k_2 = 35b_1 \nu \), then
\[
\text{EQ}(b_2) = \frac{30625b_1\nu^4(12\nu^2(2738c_2 - 395c_1)G_5 + 9154363015180278b_3^2d_3^2d_3^2 + G_6)}{582678191652046848b_2^3},
\]

where

\[
G_5 = -28749b_1^2c_2(75820c_1 - 231879c_2) + 4b_1b_2(598861900c_1 + 5322323571c_2) - 35203694400b_2^2,
\]
\[
G_6 = 37b_1c_2d_3\nu(37756b_2(598861900c_1 - 8080460229c_2) - 28749b_1c_2(832468060c_1 - 9699735333c_2)).
\]

Taking \(c_1 = \frac{2738c_2}{395}\), we have

\[
A_1 = \frac{105577244385d_3\nu}{18375071202d_3 + 8389718423\nu},
\]
\[
A_2 = \frac{5\nu^2(41707912505993669046d_3 + 70387375217225606929\nu)}{15990341652(18375071202d_3 + 8389718423\nu)},
\]
\[
A_3 = \frac{67469796(18375071202d_3 + 8389718423\nu)}{18686990554143624575d_3\nu^3},
\]
\[
A_4 = \frac{214375d_3\nu^3}{10952\left(\frac{27983097891d_1}{1679436846\nu} + \frac{28749}{37756}\right)}.
\]

Hence, (2.1) simplifies to

\[
\text{EQ}(b_2) = \frac{30625b_1c_2d_3\nu(8389718423\nu(28749b_1c_2 - 37756b_2) + 528264921986298b_1c_2d_3)}{336242345803591668b_2^3}
\]

and \(\text{EQ}(b_{20}) = 0\), where \(b_{20} = b_1c_2\left(\frac{27983097891d_1}{1679436846\nu} + \frac{28749}{37756}\right)\). The proof follows from Lemma 2.1 (ii). \(\Box\)

**Lemma 3.3.** If the hypothesis of Lemma 3.2 are valid and \(d_3 = 10\), \(c_2 = \frac{1}{10}\), \(c_3 = \frac{1}{2}\), \(b_1 = 1\), then there exist \(\nu_0, \nu_1\) positive real numbers such that the first Lyapunov coefficient for the system (1.1) at \(P_0\) is positive if either \(0 < \nu < \nu_0\) or \(\nu > \nu_1\), and it is negative if \(\nu_0 < \nu < \nu_1\). Moreover, the transversality condition holds:

\[
\frac{d\text{Re}(\lambda_{12})}{db_2}(b_{20}) \neq 0.
\]

**Proof.** If the hypothesis are valid, using Proposition 2.2 for the transversality condition and the Kuznetsov formulae (see [2, 7, 8]) for the first Lyapunov coefficient, the Mathematica software allows to get by a direct calculation:

\[
\frac{d\text{Re}(\lambda_{12})}{db_2}(b_{20}) = -\frac{22070466729033979537997184429118254339826180728050\nu^3}{s_1(\nu)} \neq 0,
\]

where

\[
s_1(\nu) = 118072143(495438258997050556215207055672412811041\nu^2
\]
\[-153100675464069678846765525147766678141320\nu
\]
\[+31089982892973169931537646413833578802713600).
\]

And that \(\ell_1(P_0, b_{20}) = -\frac{N_0}{N_1}\), where

\[
N_0 = 351936876086128034645 \sqrt{\frac{3313938777085}{3} \nu}
\]
\[\times \left(66010555705229180470125262120743126878526217240129008744752929505585827275106939261562682335\nu^5
\]
\[-597617197657733210660613592683799516740679120588348665075617698567694548604919209508115326404\nu^4
\]
\[-76596224628557684030051225997470099342333475309263857165708452193625218362629656244443950086020\nu^3
\]

\]
A numeric calculation shows that the positive roots of \( \ell_1(P_0, b_{20}) \) are \( \nu_0 \approx 0.223694 \) and \( \nu_1 \approx 108.971 \) which satisfy the desired properties (see Fig. 1).

**Theorem 3.4** \((f_1\) Holling II, one equilibrium point). Assume that the parameters of system (1.1) satisfy the hypothesis of Lemma 3.3. Then, the system exhibits a Hopf bifurcation at \( P_0 \) with respect to the parameter \( b_2 \) and its bifurcation value is \( b_{20} \). This bifurcation is supercritical if \( 0 < \nu < \nu_0 \) or \( \nu > \nu_1 \), and it is subcritical if \( \nu_0 < \nu < \nu_1 \).

**Proof.** It follows from Lemma 3.2 and the Andronov-Hopf Theorem [8, 12, 14].

**3.2. Dynamics of two equilibria**

**Lemma 3.5.** If the hypothesis of Lemma 3.1 (ii) are satisfied and

\[
\begin{align*}
c_2 &= \frac{365y_0}{44w_0}, \quad d_1 = \frac{2920d_3}{83}, \quad k_1 = \frac{365d_3w_0}{87}, \quad k_2 = \frac{2920d_3w_0}{87}, \\
k_8 &= \frac{97254250b_2c_2d_3^3w_0}{7569}, \quad \text{and} \quad b_{20} := \frac{2361971095625y_0}{131632971184},
\end{align*}
\]

then the eigenvalues of the linear approximation \( M_0(b_{20}) \) of system (1.1) at \( P_0 \) are

\[
\lambda_{1,2} = \pm i \frac{365}{\sqrt{1668424897}} d_3, \quad \lambda_{3,4} = -\frac{73}{22968} \left( 1056 \pm i \sqrt{677949239} \right) d_3.
\]
Moreover, the transversality condition holds: $d\text{Re}\lambda x$ for the linear approximation of system (1.1) at Lemma 3.6.

Proof. The proof is similar to the given for Lemma 3.1 (i). Assume that the hypothesis in Lemma 3.1 (ii) hold. In order to simplify the calculations set $x_0 = w_0$,

$$k_8 = \frac{b_2c_3}{4w_0} (24d_1^2w_0^3 + 18d_1k_2w_0 + 24k_1^2 + 18k_1k_2 + k_2^2)$$

and $c_2 = \frac{y_0}{8} (24d_1^2w_0^3 + 18d_1k_2w_0 + 24k_1^2 + 18k_1k_2 + k_2^2)$.

Now making $k_1 = \frac{k_2}{N}$ and $k_2 = d_1w_0$ we have that the linear approximation at $P_0$ for the system (1.1) has characteristic polynomial

$$\text{pol}_0(\lambda, b_2) = \lambda^4 + A_{10}\lambda^3 + A_{20}\lambda^2 + A_{30}\lambda + A_{40},$$

where

$$A_{10} = \frac{d_1}{5}, \quad A_{20} = \frac{d_1(803b_2(11d_1 + 48d_3) - 20233d_1y_0)}{2534490},$$

$$A_{30} = \frac{73d_1^2(22b_2(d_1 + 3304d_3) - 38261d_1y_0)}{20275200}, \quad \text{and} \quad A_{40} = \frac{10439b_2d_1^3d_3}{5120y_0}.$$

In this case, expression (2.1) is $\text{EQ} = \frac{73d_1^4T_1}{2055418675200y_0^2}$, where

$$T_1 = -176660b_2^2(d_1 + 3304d_3)(87d_1 - 2920d_3) + 572b_2d_1y_0(46255055d_1 - 3097232904d_3) + 472394219125d_1^2y_0^2.$$

Taking $d_1 = \frac{2920d_3}{b_2}$, we have that all the coefficients of $\text{pol}_0(\lambda, b_2)$ are positive and $\text{EQ}(b_2) = 0$, where

$$b_2 = \frac{2361971095625y_0}{131632971184}.$$

The proof concludes using Lemma 2.1 (ii). \qed

**Lemma 3.6.** Under the hypothesis of Lemma 3.5 the first Lyapunov coefficient for the system (1.1) at $P_0$ is negative. Moreover, the transversality condition holds: $\frac{d\text{Re}\{\lambda_{1,2}\}}{db_2}(b_2) \neq 0$.

Proof. Similarly to the proof of Lemma 3.3, a calculation proves that

$$\frac{d\text{Re}\{\lambda_{1,2}\}}{db_2}(b_2) = \frac{-9287262564940698723330882878656d_3}{413625650595546268863797036236319y_0} \neq 0$$

and that the first Lyapunov coefficient is $\ell_1(P_0) = -\frac{N_v}{N_1}$, where

$$N_v = 85073656100438044717374616484388706047917265654519261322685394268640524528695997$$

$$329625616819212905094934773841920 \cdot \sqrt{110116043202} \neq 38261,$$

$$N_1 = 493017549442976135855639050879301699402603777598726258708502056302804492108705495457$$

$$\times (488073431380199929 \left(77703138380692461328125c_3^2 + 186770227571152422912\right) y_0^5$$

$$+ 335372560661434388185520232467456w_0^2).$$ \qed

Under the hypothesis of Lemma 3.1 (ii), a direct calculation shows that the characteristic polynomial for the linear approximation of system (1.1) at $P_1$ has constant term

$$A_{41} = -\frac{b_2d_1k_8(d_1x_0 + k_1)(4d_1x_0 + 4k_1 + k_2)}{32c_3w_0x_0y_0(b_2 + y_0)(2d_1x_0 + 2k_1 + k_2)},$$

which is negative since all the parameters are positive. Hence from the Routh-Hurwitz test we have that the equilibrium point $P_1$ is locally unstable. In summary, we have proved next result, which follows from Lemma 3.6 and the Andronov-Hopf Theorem.

**Theorem 3.7** ($f_1$ Holling II, two equilibrium points). Assume that the parameters of system (1.1) satisfy the hypothesis of Lemma 3.5. Then, $P_1$ is locally unstable and the system (1.1) exhibits a supercritical Hopf bifurcation at $P_0$ with respect to the parameter $b_2$ and its bifurcation value is $b_2$. 

4. Case $f_1(w, x)$ Holling IV

**Lemma 4.1.** The differential system (1.1) has:

i) An equilibrium point in $\Gamma$ if

\[
\begin{align*}
    a_2 &= \frac{d_3 + k_3}{c_3}, \\
    b_1 &= \frac{k_1^2 + k_4}{(a_3b_2d_3 + d_1k_1)^2}, \\
    d_2 &= \frac{a_1c_1k_1k_4\nu(a_3b_2d_3 + d_1k_1)}{(k_1^2 + k_4)^2}, \\
    \rho &= \frac{a_1b_2d_3(a_3b_2d_3 + d_1k_1)^2}{k_1(k_1^2 + k_4)} + \nu \\
    R &= \left(\frac{k_1^2 + k_4}{a_3b_2d_3 + d_1k_1}\right)\left(\frac{k_1^2 + k_4 + k_1(d_1 + \nu)}{a_3b_2d_3 + d_1k_1}\right)^2 + \nu.
\end{align*}
\]

where $k_1, k_4$ are positive real numbers. The equilibrium is given by

\[
P_0 = \left(\frac{k_4}{a_3b_2d_3k_1\nu + d_1k_1^2k_4}, \frac{k_4}{a_3b_2d_3k_1\nu + d_1k_1^2k_4}, \frac{b_2d_3}{k_1}, \frac{a_3b_2c_3k_4}{k_1(a_3b_2d_3 + d_1k_1)^2}\right).
\]

ii) Two equilibrium points in $\Gamma$ if

\[
\begin{align*}
    a_1 &= \frac{(d_1w_0^2 + 2k_1w_0^2)^2}{32w_0^2y_0(d_1w_0^2 + 2k_1w_0^2)^2}, \\
    a_2 &= \frac{d_3(b_2 + y_0)}{c_3y_0}, \\
    a_3 &= \frac{2k_1}{w_0^2y_0}, \\
    b_1 &= \frac{1}{16}w_0^2\left(\frac{k_1(12d_1w_0^4 + 24k_1w_0^2 + 13k_0)}{(d_1w_0^2 + 2k_1w_0^2 + k_0)} + 12\right), \\
    d_2 &= \frac{c_2k_1}{y_0}, \\
    R &= w_0^2(w_0 + 2), \\
    c_1 &= \frac{w_0^2(2c_2c_3k_1(d_1w_0^2 + 2k_1w_0^2 + 2k_3)(4d_1w_0^4 + 8k_1w_0^2 + 3k_3) + k_5)}{(c_3k_3(\frac{d_1w_0^2}{2} + k_1w_0^2 + k_3))}, \quad and \quad \rho = \frac{2(d_1w_0^4 + 2k_1w_0^2 + k_3)}{w_0^2}.
\end{align*}
\]

where $w_0, y_0, k_1, k_3$ and $k_5$ are positive real numbers. The equilibrium points are given by

\[
P_0 = \left(\frac{w_0^2}{2}, y_0, \frac{2c_2c_3k_1(d_1w_0^2 + 2k_1w_0^2 + 2k_3)(4d_1w_0^4 + 8k_1w_0^2 + 3k_3) + k_5}{d_3k_3(6d_1w_0^4 + 12k_1w_0^2 + 7k_3)}, \right), \quad P_1 = \left(\frac{k_3w_0^2}{4d_1w_0^4 + 8k_1w_0^2 + 4k_5}, \frac{w_0^2}{2}, \frac{k_5}{8d_3(d_1w_0^4 + 2k_1w_0^2 + k_3)(d_1w_0^4 + 2k_1w_0^2 + 2k_3)}\right).
\]

iii) Three equilibrium points in $\Gamma$ if

\[
\begin{align*}
    a_1 &= \frac{3}{128w_0y_0}\left(\frac{1}{(d_1w_0^2 + 2k_1w_0^2 + k_3)}(12d_1w_0^4 + 24k_1w_0^2 + 13k_3)\right), \\
    a_2 &= \frac{d_3(b_2 + y_0)}{c_3y_0}, \\
    a_3 &= \frac{2k_1}{w_0^2y_0}, \\
    b_1 &= \frac{3}{64}w_0^2\left(\frac{k_1(16d_1w_0^4 + 32k_1w_0^2 + 17k_3)}{(d_1w_0^2 + 2k_1w_0^2 + k_3)} + 16\right), \\
    c_1 &= \frac{w_0^2(4c_2c_3k_1(d_1w_0^2 + 2k_1w_0^2 + 2k_3)(8d_1w_0^4 + 16k_1w_0^2 + 7k_3) + k_6)}{3c_3k_3(\frac{d_1w_0^2}{2} + k_1w_0^2 + k_3)}(4d_1w_0^4 + 8k_1w_0^2 + 5k_3), \quad c_2 = \frac{c_2k_1}{y_0}, \\
    R &= w_0^2(w_0 + 2), \quad and \quad \rho = \frac{2(d_1w_0^4 + 2k_1w_0^2 + k_3)}{w_0^2}.
\end{align*}
\]

where $w_0, y_0, k_1, k_3,$ and $k_5$ are positive real numbers. The equilibrium points are

\[
P_0 = \left(\frac{w_0^2}{2}, y_0, \frac{4c_2c_3k_1(d_1w_0^2 + 2k_1w_0^2 + 2k_3)(8d_1w_0^4 + 16k_1w_0^2 + 7k_3) + k_6}{3d_3k_3(4d_1w_0^4 + 8k_1w_0^2 + 5k_3)}\right), \quad P_1 = \left(\frac{k_3w_0^2}{8(d_1w_0^4 + 2k_1w_0^2 + k_3)}, \frac{w_0^2}{2}, \frac{k_5}{16(d_1w_0^4 + 2k_1w_0^2 + 2k_3)}, \frac{y_0}{8(d_1w_0^4 + 2k_1w_0^2 + k_3)}\right), \quad P_2 = \left(\frac{3k_3w_0^2}{8(d_1w_0^4 + 2k_1w_0^2 + k_3)}, \frac{w_0^2}{2}, \frac{3k_3w_0^2}{16(d_1w_0^4 + 2k_1w_0^2 + k_3)}, \frac{y_0}{8(d_1w_0^4 + 2k_1w_0^2 + k_3)}\right).
\]

where

\[
\begin{align*}
    z_2 &= \frac{3(8c_2c_3k_1k_2 + k_5)}{k_1(d_1w_0^2 + 2k_1w_0^2 + k_3)} + \frac{32(k_6 - 9c_2c_3k_1k_2)}{k_1(4d_1w_0^4 + 8k_1w_0^2 + 5k_3)} + 192c_2c_3k_1 - \frac{11k_6}{k_1(d_1w_0^4 + 2k_1w_0^2 + 2k_3)} > 0.
\end{align*}
\]
Proof. The proof of i) and ii) is analogous to the given for Lemma 3.1. We only will prove claim iii). In the present case, since all the parameters are positive and the points of interest are in $\Gamma$, the equilibrium points for the differential system (1.1) must satisfy the system

$$\begin{align*}
a_1 R y + (b_1 + w^2 + x)(R(v - \rho) + (w + x)\rho) &= 0, \\
-x(d_1 + a_3 y) + w v &= 0, \\
(-a_1 c_1 w + (b_1 + w^2 + x)(d_2 - a_3 c_2 x))(b_2 + y) + a_2 (b_1 + w^2 + x) z &= 0, \\
-a_2 c_3 y + d_3 (b_2 + y) &= 0.
\end{align*}
$$

(4.1)

Assume that $P_0 = (w_0, x_0, y_0, z_0) \in \Gamma$ is an equilibrium point satisfying system (4.1). Solving first, second, third, and fourth equation in the variables $a_1, a_2, a_3$, and $c_1$, respectively, one gets

$$\begin{align*}
a_1 &= -\frac{(b_1 + w_0^2 + x_0)}{R y_0} (R(v - \rho) + \rho(w_0 + x_0)), \\
da_2 &= \frac{d_3 (b_2 + y_0)}{c_3 y_0}, \\
da_3 &= \frac{y w_0 - d_1 x_0}{x_0 y_0}, \\
c_1 &= -\frac{R(c_2 c_3 (d_1 x_0 - w v_0) + c_3 d_2 y_0 + d_3 z_0)}{c_3 w_0 (v R + \rho(-R + w_0 + x_0))}.
\end{align*}$$

Taking $v = \frac{d_1 x_0 + k_1}{w_0}, d_2 = \frac{c_2 k_1}{y_0}, R = 2(w_0 + x_0)$ and $\rho = \frac{2d_1 x_0 + 2k_1 + k_2}{w_0}$, where $k_1, k_2 > 0$ we have that

$$\begin{align*}
a_1 &= \frac{k_2 (b_1 + w_0^2 + x_0)}{2 w_0 y_0}, \\
da_2 &= \frac{d_3 (b_2 + y_0)}{c_3 y_0}, \\
da_3 &= \frac{k_1}{x_0 y_0}, \\
c_1 &= \frac{2d_3 z_0}{c_3 k_2},
\end{align*}$$

and $P_0 \in \Gamma$ is an equilibrium point for the differential system (1.1). Hence under these conditions for the parameters $a_1, a_2, a_3$, and $c_1$ the system (4.1) takes the form

$$\begin{align*}
(b_1 + w^2 + x) &((k_2 (w - 2w_0 + x - 2x_0) + 2k_1 (w - w_0 + x - x_0) + 2d_1 (w - w_0 + x - x_0) x_0) \\
+ (1/y_0) k_2 (w_0 + x_0)(b_1 + w_0^2 + x_0) y_0 = 0, \\
-k_1 w_0 x y + x_0 (k_1 w - d_1 w_0 x + d_1 w_0 y_0) y_0 = 0, \\
\end{align*}$$

(4.2)

Assume that $P_1 = (w_1, x_1, y_1, z_1) \in \Gamma \setminus \{P_0\}$ satisfies (4.2). We will give conditions to find expressions for $P_1$. Since all the parameters are positive, from fourth equation we have $y_1 = y_0$ and substituting $y_1$ in the other equations we have that system (4.2) simplifies to

$$\begin{align*}
k_2 (w_0 + x_0) &((b_1 + w_0^2 + x_0) + (b_1 + w_0^2 + x_1)(k_2 (w_1 - 2w_0 + x_1 - 2x_0) \\
+ 2k_1 (w_1 - w_0 + x_1 - x_0) + 2d_1 (w_1 - w_0 + x_1 - x_0) x_0) x_1) = 0, \\
w_1 x_0 - w_0 x_1 &= 0, \\
-w_0 (b_1 + w_0^2 + x_1)(c_2 c_3 k_1 (x_1 - x_0) - d_3 x_0 z_1) - d_3 w x_0 (b_1 + w_0^2 + x_0) (b_2 + y_0) z_0 &= 0.
\end{align*}$$

(4.3)

From the third equation in (4.3) and solving for $z_1$ we have an expression in terms of $x_1$

$$\begin{align*}z_1 &= \frac{w_1 z_0 (b_1 + w_0^2 + x_0)}{w_0 (b_1 + w_0^2 + x_1)} + \frac{c_2 c_3 k_1 (x_1 - x_0)}{d_3 x_0}.
\end{align*}$$

(4.4)

Now, from second equation $w_1 = \frac{w_0 x_1}{x_0}$. Substituting $w_1$ in first equation we have the equation

$$\begin{align*}1 \\
\frac{w_0 x_0}{x_0^2} (x_1 - x_0) (w_0 + x_0) S_0 &= 0,
\end{align*}$$

(4.5)

where $S_0$ is a quadratic polynomial in the variable $x_1$:

$$\begin{align*}S_0 &= w_0^2 (2k_1 + k_2 + 2d_1 x_0)x_1^2 + x_0 (k_2 (-w_0^2 + x_0) + 2x_0 (k_1 + d_1 x_0)) x_1.
\end{align*}$$

$$\frac{d}{dx}z = \frac{w}{2}.$$  

On the other hand, from (4.4) we have two values for $k_4$, that is, $b_1 = \frac{3d_1w_0^6 + 6k_1w_0^6 + 6k_3w_0^6 + 4k_4}{4d_1w_0^6 + 8k_1w_0^6 + 4k_3w_0^6}$ and equation (4.5) becomes the condition

$$S_0 = (k_4/4) - (1/2)k_3w_0^6 x_1 + 2(k_3 + 2k_1w_0^2 + d_1w_0^4) x_2^2 = 0.$$  

Solving $S_0 = 0$ for $x_1$ we have two roots which without loss of generality will be labeled as

$$x_{1,2} = \frac{k_4}{2k_3w_0^2 \pm 2\sqrt{w_0^4 (k_3^2 - 4d_1k_4) - 8k_1k_4w_0^2 - 4k_3k_4}}.$$  

Let $k_5 > 0$ such that $w_0^2 (k_3^2 - 4d_1k_4) - 8k_1k_4w_0^2 - 4k_3k_4 = k_5^2$. Taking $k_5 = k_3w_0^2/2$ we have $k_4 = \frac{3k_2w_0^6}{16(d_1w_0^4 + 2k_1w_0^2 + k_3)}$ and

$$x_1 = \frac{k_3w_0^2}{16(d_1w_0^4 + 2k_1w_0^2 + k_3)}, \quad x_2 = \frac{3k_3w_0^2}{16(d_1w_0^4 + 2k_1w_0^2 + k_3)}.$$  

Since $w_1 = \frac{w_0x_1}{80}$, we have that $w_1$ takes two values at $x_{1,2}$ given by $w_1 = \frac{k_3w_0^2}{8(d_1w_0^4 + 2k_1w_0^2 + k_3)}$ and $w_2 = \frac{3k_3w_0^2}{8(d_1w_0^4 + 2k_1w_0^2 + k_3)}$.

On the other hand, from (4.4) we have two values for $z_1$ given by

$$z_1 = \frac{3d_3k_3z_0 (4d_1w_0^4 + 8k_1w_0^2 + 5k_3) - 4c_2c_3k_1 (d_1w_0^4 + 2k_1w_0^2 + 2k_3)}{32d_3 (d_1w_0^4 + 2k_1w_0^2 + k_3)} (d_1w_0^4 + 16k_1w_0^2 + 7k_3),$$

$$z_2 = \frac{3d_3k_3z_0 (12d_1w_0^4 + 24k_1w_0^2 + 13k_3) - 4c_2c_3k_1 (d_1w_0^4 + 2k_1w_0^2 + 2k_3)}{32d_3 (d_1w_0^4 + 2k_1w_0^2 + k_3)} (d_1w_0^4 + 16k_1w_0^2 + 5k_3).$$  

Since $z_1$ must be positive, let $k_6 > 0$ such that

$$3d_3k_3z_0 (4d_1w_0^4 + 8k_1w_0^2 + 5k_3) - 4c_2c_3k_1 (d_1w_0^4 + 2k_1w_0^2 + 2k_3) (8d_1w_0^4 + 16k_1w_0^2 + 7k_3) = k_6$$  

and hence solving this equation for $z_0$ we have

$$z_0 = \frac{4c_2c_3k_1 (d_1w_0^4 + 2k_1w_0^2 + 2k_3)}{3d_3k_3 (4d_1w_0^4 + 8k_1w_0^2 + 5k_3)} (8d_1w_0^4 + 16k_1w_0^2 + 7k_3) + k_6.$$  

Since $y_1 = y_0$ we have obtained two points $P_1, P_2 \in \Gamma \setminus \{P_0\}$ that are equilibrium points for system (1.1) and hence $P_0, P_1$ and $P_2$ satisfy the conditions as in claim iii) which completes the proof.

4.1. Dynamics of one equilibrium point

In this case we will consider that the hypothesis in Lemma 4.1 i) are valid. And that the linear approximation of the differential system (1.1) at $P_0$ is a one parameter matrix with respect to $d_3$, which we denote as $M_0(d_3)$. We will guarantee that a Hopf bifurcation takes place.
Lemma 4.2. If the hypothesis of Lemma 4.1 i) are satisfied and

\[
\begin{align*}
a_1 &= \frac{(m+7)\nu}{4b_2}, \quad a_2 = \frac{2d_3}{c_3}, \quad a_3 = \frac{\nu}{2b_2}, \quad b_1 = 2, \quad c_1 = b_2, \quad c_2 = \frac{b_2}{2}, \\
d_1 &= \frac{\nu}{2}, \quad d_2 = \frac{1}{16}(m+7)\nu, \quad R = \frac{32}{m+7} + 4, \quad \rho = \frac{m+15}{8}, \text{ and} \\
d_{31} &= \frac{(m+7)(29m+7\left(\sqrt{m(25m-354)} + 4489 - 67\right))}{1024},
\end{align*}
\]

where \( m > 0 \), then the eigenvalues of the linear approximation \( M_0(d_{31}) \) of system (1.1) at \( P_0 \) are

\[
\lambda_{1,2} = \pm i\sqrt{R_0}\nu \quad \text{and} \quad \lambda_{3,4} = -\frac{1}{128}\left(\pm\sqrt{2}\sqrt{R_1} + 56\right)\nu,
\]

where

\[
\begin{align*}
R_0 &= \frac{(m+7)\left(5m + \sqrt{m(25m-354)} + 4489 - 45\right)}{1024} > 0, \\
R_1 &= 7\left(\sqrt{m(25m-354)} + 4489 + 157\right) + m\left(-5m + \sqrt{m(25m-354)} + 4489 - 102\right).
\end{align*}
\]

We have that

\[
R_1 > 0 \text{ if } m < m_0 := 14.7403; \quad R_1 = 0 \text{ if } m = m_0; \quad \text{and} \quad R_1 < 0 \text{ if } m > m_0.
\]

![Figure 2: Graphic of \( R_1(m) \).](image)

**Proof.** We assume the hypothesis as in Lemma 4.1 i) are valid. In this occasion we set \( a_3 = \frac{d_1k_1}{b_2d_5}, \quad k_4 = 2k_1^2\nu^2, \quad d_1 = \frac{\nu}{2} \) and \( k_1 = d_3 \). Then we have that the linear approximation \( M_0(d_3) \) has characteristic polynomial

\[
\text{pol}_0(\lambda, d_3) = \lambda^4 + A_{10}\lambda^3 + A_{20}\lambda^2 + A_{30}\lambda + A_{40},
\]

where

\[
\begin{align*}
A_{10} &= \nu - c_2\nu, \quad A_{20} = \frac{a_1^2b_2^2c_1 + a_1b_2\nu(2b_2 - c_1) + 8c_2d_3\nu}{32b_2}, \\
A_{30} &= \nu \left(3a_1^2b_2^2c_1 + 28a_1b_2c_2\nu + 64c_2d_3\nu\right) + 256b_2 \\
A_{40} &= \frac{1}{64}a_1c_2d_3\nu^2.
\end{align*}
\]

In this case, we have that (2.1) is

\[
E_Q = -\frac{\nu^2F_0}{65536b_2^2},
\]

where

\[
F_0 = b_2\left(3a_1^2b_2^2c_1 + 28a_1b_2c_2\nu + 64c_2d_3\nu\right)^2 - 2(4b_2 - c_2) \times \left(a_1^2b_2^2c_1 + a_1b_2\nu(2b_2 - c_1) + 8c_2d_3\nu\right) \\
\times \left(3a_1^2b_2^2c_1 + 28a_1b_2c_2\nu + 64c_2d_3\nu\right) + 64a_1b_2c_2d_3\nu^2(c_2 - 4b_2)^2.
\]
Lemma 4.5. If the hypothesis of Lemma 4.1 ii) hold and \(k_1 = d_1 w_0^2\) and \(k_3 = d_1 w_0^4\), then the linear approximation for the system (1.1) at \(P_0\) is locally unstable. Hence, from the Routh-Hurwitz test we have that in these conditions \(P_1\) is locally unstable.

Now, we will consider the linear approximation of system (1.1) at \(P_0\) as a one parameter matrix with respect to \(b_2\), which we denote as \(M_0(b_2)\).

Lemma 4.5. If the hypothesis of Lemma 4.1 ii) are satisfied and

\[
\begin{align*}
\alpha_1 &= \frac{92015625d_3}{246208y_0}, \\
\alpha_2 &= \frac{d_3(b_2 + y_0)}{c_3y_0}, \\
\alpha_3 &= \frac{58890d_3}{3847y_0}, \\
\beta_1 &= \frac{241}{64}, \\
\beta_2 &= \frac{13689d_3(b_2 + y_0)}{7694y_0},
\end{align*}
\]

Then the eigenvalues of the linear approximation \(M_0(b_2)\) of system (1.1) at \(P_0\) are

\[
\lambda_{1,2} = \pm i 353340 \sqrt{\frac{21305}{63059587200009}} d_3 \quad \text{and} \quad \lambda_{3,4} = \frac{-117 \left(94375 \mp i \sqrt{194109621935}\right) d_3}{1923500}.
\]

Proof. Suppose that the parameters of the system (1.1) satisfy the hypothesis as in Lemma 4.1 ii). As was made in the other cases, setting \(k_1 = d_1 w_0^2\) and \(k_3 = d_1 w_0^4\), the linear approximation \(M_0(b_2)\) has

\[
\begin{align*}
\alpha_1 &= \frac{92015625d_3}{246208y_0}, \\
\alpha_2 &= \frac{d_3(b_2 + y_0)}{c_3y_0}, \\
\alpha_3 &= \frac{58890d_3}{3847y_0}, \\
\beta_1 &= \frac{241}{64}, \\
\beta_2 &= \frac{13689d_3(b_2 + y_0)}{7694y_0},
\end{align*}
\]
characteristic polynomial $\text{pol}_0(\lambda, b_2) = \lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4$, such that

$$B_1 = \frac{1}{125}d_1 \left( -\frac{755c_2w_0^2}{b_2 + y_0} - 256w_0 + \frac{125(11w_0 + 6)}{w_0 + 2} \right), \quad B_2 = \frac{d_1w_0(\xi_0)}{3125(w_0 + 2)y_0(b_2 + y_0)},$$

$$B_3 = -\frac{c_2d_2w_0^2(\xi_0)}{15625(w_0 + 2)y_0(b_2 + y_0)}, \quad B_4 = \frac{16308b_2c_2d_2^2d_3w_0^3}{625y_0(b_2 + y_0)},$$

where

$$\xi_0 = b_2(w_0 + 2)(c_2w_0(85315d_1w_0 - 14812d_1 + 188750d_3 + 135000d_1y_0) + d_1y_0(c_2w_0(471875w_0 - 1147312) - 1662124) + 135000(w_0 + 2)y_0),$$

$$\xi_1 = b_2(16d_1w_0(229w_0 - 12902) + 755d_3(w_0(256w_0 - 863) - 750)) + 4d_1w_0(102841w_0 + 152242)y_0.$$

Setting $w_0 = 2$, $c_2 = \frac{351(b_2 + y_0)}{6040}$ and $d_1 = \frac{29445d_1}{3847}$, we have that (2.1) is

$$\text{EQ} = \frac{1092211792811596337270411d_1(95480385736y_0 - 16391886457b_2)}{252346667490523417913281250y_0}.$$

Moreover, all the coefficients of $\text{pol}_0(\lambda, b_2)$ are positive and $\text{EQ}(b_2) = 0$, where $b_2 = \frac{95480385736y_0}{16391886457}$. Therefore, the proof is concluded using Lemma 2.1 ii).

**Lemma 4.6.** If the hypothesis of Lemma 4.5 are satisfied, then the first Lyapunov coefficient $\ell_1(P_0, b_2)$ for the system (1.1) at $P_0$ is negative. Moreover, the transversality condition holds: $\frac{\partial \ell_1(P_2, b_2)}{\partial b_2}(b_2) \neq 0$.

**Proof.** Under the hypothesis of Lemma 4.5 we have that

$$\frac{\partial \text{Re}(\lambda_{1,2})}{\partial b_2}(b_2) = \frac{-1180156939356166931213599722872813535d_3}{19478350806921655804860578959172649472y_0} \neq 0$$

and that the first Lyapunov coefficient is $\ell_1(P_0, b_2) = -\frac{M_0}{M_1} < 0$, where

$$M_0 = 21767642320156754358766855527011380942175340698814119572466,$$

$$9764374472301079117588114055366058034338623801083047117711125 \sqrt{\frac{31529793600395}{4261}},$$

$$M_1 = 607150697028745575459137144567343623577578601721847016911101682580953931726210786136 \times (240939830344563547654) (457623738857437854641c_2^2 + 492653025006171875y_0^2 + 2223872637203087455431709728114420048675).$$

Next theorem follows from Lemma 4.6 and the Andronov-Hopf Theorem.

**Theorem 4.7** ($f_1$ Holling IV, two equilibrium points). Assume that the parameters of system (1.1) satisfy the hypothesis of Lemma 4.6. Then, the system exhibits a supercritical Hopf bifurcation at $P_0$ with respect to the parameter $b_2$ and its bifurcation value is $b_2 = 0$.

### 4.3. Dynamics of three equilibrium points

If the hypothesis of Lemma 4.1 iii) hold and $k_1 = d_1w_0^2$ and $k_3 = d_1w_0^3$, then a direct calculation shows that the linear approximation for the system (1.1) at $P_2$ has a characteristic polynomial with a negative constant term. Hence, from Routh-Hurwitz test we have that under these conditions the equilibrium point $P_2$ is locally unstable.

This time we will consider the linear approximations of system (1.1) at $P_0$ and $P_1$ as one parameter matrices with respect to $b_2$, which we denote as $M_0(b_2)$ and $M_1(b_2)$, respectively.
Lemma 4.8. Assume that the hypothesis of Lemma 4.1 iii) are satisfied.

a) If
\[
\begin{align*}
\lambda_{1,2} &= \pm 1393714 \left( \frac{19316813}{104530613389866407261} \right) d_3 \quad \text{and} \quad \lambda_{3,4} = -\frac{951 (57477 + i \sqrt{31910155004111})}{23911364960}.
\end{align*}
\]

b) If
\[
\begin{align*}
\lambda_{1,2} &= \pm 1761782000 \left( \frac{1473121548605}{181860826304309343134135160609495671} \right) d_3, \\
\lambda_{3,4} &= \frac{272065 \left( 210492409575823459 + 4418937814818135688902413019247481 \right)}{462833171939453734903948632} d_3.
\end{align*}
\]

Proof. Suppose that the parameters of the system (1.1) satisfy the hypothesis as in Lemma 4.1 iii). To simplify the calculations set \( k_1 = d_1 w_0^0 \) and \( k_3 = d_1 w_0^4 \).

We prove claim a).

The linear approximation \( M_0(b_2) \) has characteristic polynomial \( p_{0}(\lambda,b_2) = \lambda^4 + B_0 \lambda^3 + B_0^2 \lambda^2 + B_0 \lambda + B_0^4 \), such that
\[
\begin{align*}
B_0 &= d_1 \left( -\frac{512 w_0}{2499} - \frac{16}{w_0 + 2} + 11 \right) - 620 c_3 c_5 d_1^2 w_0^0 + k_6, \\
B_0 &= \frac{\lambda_0}{254898 c_3 d_1 w_0^0 (w_0 + 2) y_0 (b_2 + y_0)^2}, \\
B_0 &= \frac{\lambda_0}{254898 c_3 d_1 w_0^0 (w_0 + 2) y_0 (b_2 + y_0)^2}, \\
B_0 &= \frac{\lambda_0}{254898 c_3 d_1 w_0^0 (w_0 + 2) y_0 (b_2 + y_0)^2},
\end{align*}
\]
\[
\begin{align*}
S_1 &= -d_1 w_0 \left( c_2 c_3 d_1 \left( 999965 w_0 - 868998 w_0^0 + k_7 \left( 3011 w_0 + 9794 \right) \right) \right) \\
S_1 &= -d_1 w_0 \left( c_2 c_3 d_1 \left( 999965 w_0 - 868998 w_0^0 + k_7 \left( 3011 w_0 + 9794 \right) \right) \right) \\
S_1 &= -d_1 w_0 \left( c_2 c_3 d_1 \left( 999965 w_0 - 868998 w_0^0 + k_7 \left( 3011 w_0 + 9794 \right) \right) \right)
\end{align*}
\]
setting \( k_6 = c_2 c_3 d_1^2 w_0^1, w_0 = 4 \), \( k_7 = \frac{1}{16}, c_2 = \frac{317(b_2 + y_0)}{16259880} \), and \( d_1 = \frac{5619d_3}{1435326} \), it follows that (2.1) for \( P_0 \) becomes
\[
\begin{align*}
EQ_{P_0} = \frac{590603773376433329361449973d_0 (1316348622408722y_0 - 582645121928723b_2)}{52609750859820970832269202211232925353948243072000y_0}.
\end{align*}
\]
Moreover, all the coefficients of $\text{pol}_1(\lambda, b_2)$ are positive and $\text{EQ}_{P_0}(b_20) = 0$, where $b_{20} = \frac{13416348622408722\lambda_0}{58845121928723}$. Therefore, the proof of the claim for $P_0$ is concluded using Lemma 2.1 (ii). Now, we prove the claim for $P_1$. Under these assingnations as above a calculation shows that the characteristic polynomial $\text{pol}_1(\lambda)$ of $M_1(b_2)$ has positive coefficients and that (2.1) for $P_1$ is

$$\text{EQ}_{P_1} = -\frac{22195448555703486537d^3_5}{3698779993755905624919937039498286002816482719825920000000y_0^2} < 0,$$

where

$$d_0 = 17776155316167905958812163517b_2^2 + 1095685583423619310437058086372b_2y_0 + 1077907578968001990869491705056y_0^2.$$

Hence, from Lemma 2.1 (i) $P_1$ is locally asymptotically stable. We now prove claim b).

The linear approximation $M_1(b_2)$ has characteristic polynomial $\text{pol}_1(\lambda, b_2) = \lambda^4 + B_{11}\lambda^3 + B_{12}\lambda^2 + B_{13}\lambda + B_{14}$, such that

$$B_{11} = d_1 \left( \frac{51w_0}{3920} - \frac{1}{2(w_0 + 2)} + \frac{13}{4} \right) - \frac{k_6}{640c_3d_1w_0^0(b_2 + y_0)}, \quad B_{12} = \frac{d_2}{2508800c_3d_1w_0^0(b_2 + y_0)}.$$

$$B_{13} = \frac{B_1}{2508800c_3d_1w_0^0(b_2 + y_0)} \text{ and } B_{14} = \frac{93b_2d_3k_6}{2508800c_3w_0^0(b_2 + y_0)}.$$

where

$$D_0 = b_2(w_0 + 2)(20c_5d_1^2w_0^0(c_2w_0(773109w_0 + 3872) + 2976y_0)) + d_1k_6(24939w_0 - 128) + 3920d_3k_6 + 2d_1y_0 \left(10c_3d_1^2(w_0 + 2)w_0^0(c_2w_0(773109w_0 + 3872) + 2976y_0) + k_6(w_0(12495w_0 + 18556) - 11888)\right),$$

$$D_1 = b_2 \left(d_1w_0(20c_5d_1^2(2351229w_0 + 4764062)w_0^0 + k_6(73431w_0 + 148786)) + d_3k_6(w_0(12638 - 51w_0) + 23520)\right) + 2d_1w_0y_0 \left(10c_5d_1^2(2351229w_0 + 4764062)w_0^0 + k_6(36669w_0 + 74300)\right).$$

Setting $k_6 = c_2c_3d_3^3w_0^{10}$, $w_0 = \frac{28\lambda_0}{107}$, $k_7 = \frac{2}{10}$, $c_2 = \frac{13603250b_2 + y_0}{4047675}$ and $d_1 = \frac{38089100000d_5}{4585189928341}$, we have that equation (2.1) is

$$\text{EQ}_{P_1} = \frac{\partial \text{Re}(\lambda_{12})}{\partial b_2}(b_{20}) < 0.$$
Proof. Proof of claim i): Under the hypothesis of Lemma 4.8 a) we have that
\[
\frac{\partial \text{Re}(\lambda_{1,2})}{\partial b_2}(b_2) = -\frac{8396632279002461439070218375488228526848666556509d_3}{563065662081392491481014219333781446381362348589949294y_0} \neq 0
\]
and that the first Lyapunov coefficient is \(\ell_1(P_0, b_2) = -\frac{\gamma_0}{\gamma_1} < 0\), where
\[
\gamma_0 = 6510810322491891436389306962780\sqrt{758333390904722989759548164019310703020583084463339583499808327}
\]
\[
75214558813445359338101343750281232337477446401515794984655051823072
\]
\[
942461984640386690386235067731641129296215210484026129430957410007533
\]
\[
2702430128458655829651805999913476399240810745150112683711440889205
\]
\[
645543731058950957067802577242101193168130.
\]
\[
\gamma_1 = 135750239588722107527224073922273432429534300327621012960131383508759631752974912752548439494
\]
\[
10328263837635323980675799\sqrt{1490142207788331497498101237169439934607221821852591354209219
\]
\[
+70372176673647636659077623(147323813582352947581870707654400 + 233834673402738187928644543c_3^2y_0^2).
\]

Proof of claim ii): Under the hypothesis of Lemma 4.8 b) we have that
\[
\frac{\partial \text{Re}(\lambda_{1,2})}{\partial b_2}(b_2) = -\frac{T_0}{T_1} \neq 0, \text{ where}
\]
\[
T_0 = 4761642942994499792207408743366686735021383216432164461985487593993107207084166106549269\ d_3,
\]
\[
T_1 = 8660506070812536285392147610758201852338558609440083677973579296208919851350789498468218223447310664565380 y_0,
\]
and that the first Lyapunov coefficient is \(\ell_1(P_1, b_2) = -\frac{\zeta_0}{\zeta_1} < 0\), where
\[
\zeta_0 = 65037223551813270578986766038515916230112374732312107160516265784528904
\]
\[
315332088678279217761444295462057299992837090314159291751234486162303263
\]
\[
465649013516076244599951229579159910216834713481092680382200019744055
\]
\[
7432486826762358084594640342601582397043771383526920963463168628907547
\]
\[
10694769143578848166613121625306834124000000 \times \sqrt{63651289206432596947303621332351635/42089187103},
\]
\[
\zeta_1 = 12386434639232289120446759327190503463631164158700044380545912113846936
\]
\[
0971269184611196395815957410247489562144022760515809253707052505638169
\]
\[
7947297683537158372905436825837200512339727096499853148746760661906624
\]
\[
7656923(43455538436471872346235422851807062198737736310095857262301925
\]
\[
11103499714992152313760596948821588845514436143324311469
\]
\[
+ 84144203552867157412129081289671638443958865502261810000000 \times (1452875053862333705729342718653742307939018202766506772538931652
\]
\[
+ 5725838075229805631753520594723314242971281687069580978125 c_3^2y_0^2).
\]

Next theorem follows from Lemma 4.9 and the Andronov-Hopf Theorem.

**Theorem 4.10** (\(f_1\) Holling IV, three equilibrium points).

i) Assume that the parameters of system (1.1) satisfy the hypothesis of Lemma 4.8 a). Then, \(P_1\) is locally stable and the system exhibits a supercritical Hopf bifurcation at \(P_0\) with respect to the parameter \(b_2\) and its bifurcation value is \(b_{20}\).

ii) Assume that the parameters of system (1.1) satisfy the hypothesis of Lemma 4.8 b). Then, \(P_0\) is locally unstable and the system exhibits a supercritical Hopf bifurcation at \(P_1\) with respect to the parameter \(b_2\) and its bifurcation value is \(b_{21}\).
5. Numerical examples

In all cases, the coexistence of the three species takes place due to the existence of a supercritical Hopf bifurcation with respect to the corresponding parameters. A direct calculation shows that the hypothesis of the proved theorems for the differential system (1.1) are valid for parameters values with ecological sense, we show this in the following numerical examples.

5.1. $f_1$ Holling II

In the following two examples the functional response $f_1$ in the system (1.1) is Holling type II.

**Example 5.1.** From the hypothesis in Theorem 3.4 the differential system (1.1) takes the form

\[
\begin{align*}
\dot{w} &= \frac{1}{37} \nu w \left( -\frac{37y}{b_2(w+x+1)} - 2w - 2x + 37 \right), \\
\dot{y} &= \nu \left( \frac{8214w}{w+x+1} + 3555x - 6475 \right) - \frac{40z}{11850b_2}, \\
\dot{x} &= \nu \left( x \left( -\frac{3y}{b_2} - 2 \right) + w \right), \\
\dot{z} &= \frac{10z(y - b_2)}{b_2 + y}.
\end{align*}
\]

Therefore, the equilibrium point is

\[
P_0 = \left( \frac{175}{12}, \frac{35}{72}, b_2, \frac{7\nu}{160} \right)
\]

and

\[
b_20 = \frac{1}{10} \left( \frac{26729}{325900} + \frac{130915489455}{83897160256} \nu \right).
\]

Taking $\nu = 110$ we have that $b_20 \approx 0.0913051$ and $P_0$ is $(14.5833, 2.91667, b_2, 4.8125)$. The first Lyapunov coefficient is $\ell_1(P_0, b_20) \approx -0.00118615$. Hence the system exhibits a supercritical Hopf bifurcation at $P_0$ with respect to $b_2$. Setting $b_2 = b_20 - 10^{-3} \approx 0.0903051$, we have in Figure 3 a projection to the $(x, y, z)$ space of an orbit with initial condition $Q_0 = (14.5933, 2.92667, 0.100305, 4.8225)$ which tends to the stable limit cycle. Figure 4 shows the corresponding time series.

**Remark 5.2.** The bifurcation parameter value $b_20$ depends directly on $\nu$ and is bounded below. Near the bifurcation value, the third coordinate of $P_0$ (predator density) is approximately $b_20$. The fourth coordinate of $P_0$ (superpredator density) is directly proportional to $\nu$.
Example 5.3. From the hypothesis in Theorem 3.7 the differential system (1.1) takes the form

\[
\dot{w} = \frac{365d_3w}{1131} \left( -\frac{1560w_0y}{13wy_0 + 4w_0y_0 + 13xy_0} - \frac{169(w+x)}{2w_0} + 221 \right),
\]

\[
\dot{x} = \frac{365d_3(9wy_0 - x(y + 8y_0))}{87y_0},
\]

\[
\dot{y} = \frac{d_3y}{3828} \left( \frac{133225(44b_2w + y_0(31w - 4w_0 - 13x))}{(13w + 4w_0 + 13x)y_0} - \frac{3828z(b_2 + y_0)}{y_0c_3(b_2 + y)} + \frac{133225x}{w_0} \right),
\]

\[
\dot{z} = \frac{b_2d_3z(y - y_0)}{y_0(b + y)}.
\]

Therefore, \( b_{20} = \frac{2361971095625y_0}{13163977184} \),

\[
P_0 = \left( w_0, w_0, y_0, \frac{26645}{522}c_3(b_2 + y_0) \right) \quad \text{and} \quad P_1 = \left( \frac{2w_0}{13}, \frac{2w_0}{13}, y_0, \frac{133225b_2c_3}{4524} \right).
\]

Taking \( c_3 = \frac{1}{2} \), \( d_3 = 1 \), \( w_0 = 58 \) and \( y_0 = 1 \) we have that \( b_{20} \approx 17.9436 \) and the first Lyapunov coefficient at \( P_0 \) is \( \ell_1(P_0, b_{20}) \approx -0.0000736782 \). Hence the system exhibits a supercritical Hopf bifurcation at \( P_0 \) with respect to \( b_2 \). Setting \( b_2 = b_{20} - 10^{-1} \approx 17.8436 \), we have in Figure 5 a projection to the \((x, y, z)\) space of an orbit with initial condition \( Q_0 = (58.01, 58.01, 1.01, 480.937) \) which tends to the stable limit cycle. Figure 6 shows the corresponding time series.
Figure 5: Projection of limit cycle with respect to $P_0$ (two equilibria, $f_1$ Holling II).

Figure 6: Time series with respect to $P_0$ (two equilibria, $f_1$ Holling II).

Remark 5.4. When the system (1.1) has two equilibrium points and the parameter $b_2$ is near to the bifurcation value $b_{20}$, the third and fourth coordinates of $P_0$ are directly proportional to $y_0$. 
5.2. \( f_1 \) Holling IV

In the following three examples the functional response \( f_1 \) in the system (1.1) is Holling type IV.

**Example 5.5.** From the hypothesis in Theorem 4.4 the differential system (1.1) takes the form

\[
\begin{align*}
\dot{w} &= \frac{529}{800}yw \left( -\frac{8y}{b_2(w^2 + x + 2)} - \frac{w - x + 2}{b_2} \right), \\
\dot{y} &= \frac{1}{400}y \left( -\frac{529}{w^2 + x + 2} + 100x \right) - \frac{800d_3z}{c_3(b_2 + y)}, \\
\dot{z} &= \frac{d_3(y - b_2)}{b_2 + y}.
\end{align*}
\]

Therefore, if \( y = 110, b_2 = \frac{1}{2}, c_3 = \frac{1}{2} \), then \( d_3 = \frac{529}{800} \approx 933.404 \) and the positive equilibrium point is \( P_0 = \left(1, 1, \frac{529}{800}\right) \).

The first Lyapunov coefficient is \( \ell_1(P_0, d_3) \approx -0.222841 \). Hence the system exhibits a supercritical Hopf bifurcation at \( P_0 \) with respect to \( d_3 \). Setting \( d_3 = b_3 + 10^{-1} \approx 933.504 \), we have in Figure 7 a projection to the \((x, y, z)\) space of an orbit with initial condition \( Q_0 = (1.00001, 1.00001, 0.50001, 0.00737472) \) which tends to the stable limit cycle. Figure 8 shows the corresponding time series.

![Figure 7: Projection of limit cycle with respect to \( P_0 \) (one equilibrium, \( f_1 \) Holling IV).](image)

**Example 5.6.** From the hypothesis in Theorem 4.7 the differential system (1.1) takes the form

\[
\begin{align*}
\dot{w} &= \frac{29445d_3w}{246208} \left( -\frac{3125y}{y_0(w^2 + x + \frac{241}{64})} - 128w - 128x + 832 \right), \\
\dot{y} &= \frac{d_3y(b_2 + y_0)}{984832y_0} \left( -\frac{984832z}{c_3(b_2 + y)} + \frac{51675975w}{w^2 + x + \frac{241}{64}} + 876096x - 1752192 \right), \\
\dot{z} &= \frac{b_2d_3z(y - y_0)}{y_0(b_2 + y)}.
\end{align*}
\]

Therefore, \( b_{20} = \frac{95480385376y_0}{15391886457} \) and

\[
P_0 = \left(2, 2, y_0, \frac{2067039c_3(b_2 + y_0)}{192350} \right), \quad P_1 = \left(\frac{1}{8}, \frac{1}{8}, \frac{13689c_3(b_2 + y_0)}{1231040} \right).
\]

Taking \( y_0 = 1, d_3 = 1, c_3 = \frac{1}{2} \), we have that \( b_{20} \approx 5.82486 \) and the first Lyapunov coefficient is \( \ell_1(P_0, b_{20}) \approx -1.91318 \). Hence the system exhibits a supercritical Hopf bifurcation at \( P_0 \) with respect to \( b_2 \). Setting \( b_2 = b_{20} - 10^{-1} \approx 5.72486 \), we have in Figure 9 a projection to the \((x, y, z)\) space of an orbit with initial condition \( Q_0 = (2.01, 2.01, 1.01, 36.1435) \) which tends to the stable limit cycle. Figure 10 shows the corresponding time series.
Figure 8: Time series with respect to $P_0$ (one equilibrium, $f_1$ Holling IV).

Figure 9: Projection of limit cycle with respect to $P_0$ (two equilibria, $f_1$ Holling IV).

Remark 5.7. When the system (1.1) has two equilibrium points and the parameter $b_2$ is near to the bifurcation value $b_{20}$, the third and fourth coordinates of $P_0$ are directly proportional to $y_0$. 
Example 5.8. From the hypothesis in Theorem 4.10 i) the differential system (1.1) takes the form

\[
\dot{w} = \frac{21873d_3w}{45928192} \left( -\frac{37485y}{y_0(w^2 + x + \frac{963}{64})} - 128w - 128x + 2496 \right),
\]

\[
\dot{x} = \frac{65619d_3(6wy_0 - x(2y + y_0))}{1435256y_0},
\]

\[
\dot{y} = \frac{d_3y(b_2 + y_0)}{180038512640y_0} \left( 256 \left( \frac{703275440z}{c_3(b_2 + y)} - 100489 \right) + \frac{305779781w}{w^2 + x + \frac{963}{64}} + 3215648x \right),
\]

\[
\dot{z} = \frac{b_2d_3z(y - y_0)}{y_0(b_2 + y)}.
\]

We have that \(b_{20} = \frac{1341634862408722y_0}{8926451219258323} \) and the positive equilibrium points are

\[P_0 = \left( 4, 8, y_0, \frac{20801223(b_2 + y_0)}{23911364960} \right), \quad P_1 = \left( \frac{1}{8}, \frac{1}{4}, y_0, \frac{100489(b_2 + y_0)}{90019256320} \right), \quad P_2 = \left( \frac{3}{8}, \frac{3}{4}, y_0, \frac{2066958241(b_2 + y_0)}{15303273574400} \right).\]

Taking \(d_3 = 1, c_3 = \frac{1}{2}\) and \(y_0 = 40\), the first Lyapunov coefficient is \(\ell_1(P_0, b_{20}) \approx -0.0134592\). Hence the system exhibits a supercritical Hopf bifurcation at \(P_0\) with respect to \(b_2\) and \(P_1\) is locally asymptotically stable. Setting \(b_2 = b_{20} - 10^{-2} \approx 921.055\), we have in Figure 11 a projection to the \((x, y, z)\) space of an orbit with initial condition \(Q_0 = (4.01, 8.01, 40.01, 0.846051)\) which tends to the stable limit cycle. Figure 12 shows the corresponding time series.

Remark 5.9. When the system (1.1) has three equilibrium points and the parameter \(b_2\) is near to the bifurcation value \(b_{20}\), the third and fourth coordinates of \(P_0\) are directly proportional to \(y_0\).
6. Conclusions

The dynamics of the tritrophic chain model with age structure in the prey given by the differential system (1.1) is determined under sufficient conditions on the parameter space. We considered two different types of interaction between predator and reproductive population prey: through a Holling type II or IV functional response $f_1$.

If $f_1$ is Holling type II it is possible to have one or two positive equilibria $P_0$ and $P_1$. There are parameters conditions such that the differential system (1.1) has a Hopf bifurcation at $P_0$ with respect to the parameter $b_2$ representing the handling time. If there is only one equilibrium point, the bifurcation could be sub- or super-critical, whereas if there are two equilibria, it is only supercritical and the other
equilibrium point is locally unstable.

When \( f_1 \) is Holling type IV, there are sufficient conditions to have one, two or three positive equilibrium points, \( P_0, P_1, P_2 \). In all cases the differential system (1.1) exhibits a supercritical Hopf bifurcation at \( P_0 \). In the first case, the bifurcation is with respect to \( d_3 \), which represents the mortality superpredator rate growth. In the second case, the bifurcation is with respect to \( b_2 \) and \( P_1 \) is locally unstable. In the third case, there is a non-simultaneous Hopf bifurcation at \( P_0 \) and \( P_1 \) with respect to \( b_2 \), and \( P_2 \) is locally unstable. From Theorems 3.4, 3.7, 4.7, and 4.10 we have that given a predator density there are parameters values that guarantee the coexistence coming from a supercritical Hopf bifurcation whose bifurcation value is approximately the predator density. Finally, we emphasize that the differential system (1.1) may presents bistability (see Theorem 4.10 i)) when there is defense in the prey.

References


