Fixed point theorems for $\Theta$-contractions in left $K$-complete $T_1$-quasi metric space

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Abstract

The aim of this paper is to define $\Theta^\beta_u = \{ v \in \mathcal{J}(u) : \Theta(\rho(u,v)) \leq \Theta(\rho(u,\mathcal{J}(u)))^\beta \}$ and establish some new fixed point theorems in the setting of left $K$-complete $T_1$-quasi metric space. Our theorems generalize, extend, and unify several results of literature.

Keywords: $\Theta$-contractions, property P, property Q, fixed points.

2010 MSC: 47H10, 54H25.
**Definition 1.3** ([7]). Assume that a quasi-pseudo metric space \((E, \rho)\). Given \(u_0 \in E\) as centre and \(\epsilon > 0\) as radius, then
\[
B_\rho(u_0, \epsilon) = \{v \in E : \rho(u_0, v) < \epsilon\}
\]
denotes the open ball and
\[
\overline{B}_\rho(u_0, \epsilon) = \{v \in E : \rho(u_0, v) \leq \epsilon\}
\]
denotes the closed ball.

Every quasi-pseudo metric \(\rho\) on \(E\) originate a topology \(\tau_\rho\) on \(E\). If \(\rho\) is a quasi metric on \(E\), then the originated topology \(\tau_\rho\) must be \(T_0\). If \(\rho\) is a \(T_1\)-quasi metric, then the generated topology \(\tau_\rho\) is a \(T_1\).

If \(\rho\) is a quasi-pseudo metric on \(E\), then define \(\rho^-\), \(\rho^+\), and \(\rho_+\) as
\[
\rho^-(u, v) = \rho(v, u), \quad \rho^+(u, v) = \max\{\rho(u, v), \rho^{-1}(u, v)\}, \quad \text{and} \quad \rho_+(u, v) = \rho(u, v) + \rho^{-1}(u, v).
\]
All these metrics are also quasi-pseudo metrics on \(E\). Moreover, if \(\rho\) satisfies
\[
u \neq v \implies \rho(u, v) + \rho^{-1}(u, v) > 0,
\]
then \(\rho_+\) (and also \(\rho^+\)) is a metric on \(E\). Here \(\text{cl}_\rho(A), \text{cl}_{\rho^{-1}}(A), \text{cl}_{\rho^+}(A)\) denote the closure of \(A\) in \(E\) with respect to \(\tau_\rho, \tau_{\rho^{-1}}, \text{and} \tau_{\rho^+}\), respectively.

We give the following examples in which the mapping \(\rho : E \times E \rightarrow \mathbb{R}^+\) is a quasi metric but not a \(T_1\)-quasi metric.

**Example 1.4** ([7]).

(i) Let \(E = \mathbb{R}\). Define \(\rho : E \times E \rightarrow \mathbb{R}^+\) as follows
\[
\rho(u, v) = \max\{|v - u, 0|\}
\]
for all \(u, v \in E\).

(ii) Let \(E = \mathbb{R}\) and
\[
\rho(u, v) = \begin{cases} 0, & u = v, \\ |v|, & u \neq v, \end{cases}
\]
for all \(u, v \in E\).

We give the following example in which the mapping \(\rho : E \times E \rightarrow \mathbb{R}^+\) is a \(T_1\)-quasi metric although not a metric on \(E\).

**Example 1.5** ([7]). Let \(E = \mathbb{R}\) and
\[
\rho(u, v) = \begin{cases} v - u, & u \leq v, \\ 1, & u > v, \end{cases}
\]
for all \(u, v \in E\).

Let \((E, \rho)\) be a quasi metric space, \(B\) a nonempty subset of \(E\), and \(u \in E\). Then
\[
u \in \text{cl}_\rho(A) \iff \rho(u, B) = \inf\{\rho(u, a) : a \in B\} = 0.
\]
Likewise,
\[
u \in \text{cl}_{\rho^{-1}}(A) \iff \rho(B, u) = \inf\{\rho(a, u) : a \in B\} = 0.
\]
If \(B\) is compact subset of \(E\) and \((E, \rho)\) is a metric space, then for each \(u \in E\), there exists \(a \in B\) so that
\[
\rho(u, a) = \rho(u, B).
\]
But this property does not hold in a quasi metric space \((E, \rho)\).
However, if \((E, \rho)\) is a quasi metric space and \(A\) is a \(\tau_{\rho^{-1}}\)-compact subset of \(E\), then this property holds. Let \((E, \rho)\) be a quasi metric space and \(u \in E\). A sequence \(\{u_n\}\) converges to \(u\) regarding \(\tau_{\rho}\) is said to be \(\rho\)-convergence and denoted by \(u_n \xrightarrow{\rho} u\) and is defined by

\[
\rho(u, u_n) \rightarrow 0,
\]
as \(n \rightarrow \infty\). Similarly, the convergence of \(\{u_n\}\) to \(u\) regarding \(\tau_{\rho^{-1}}\) is said to be \(\rho^{-1}\)-convergence and denoted by \(u_n \xrightarrow{\rho^{-1}} u\) and is defined by

\[
\rho^{-1}(u_n, u) \rightarrow 0,
\]
as \(n \rightarrow \infty\). Finally, the convergence of \(\{u_n\}\) to \(u\) regarding \(\tau_{\rho}\) is said to be \(\rho^s\)-convergence and denoted by \(u_n \xrightarrow{\rho^s} u\) and is defined by

\[
\rho^s(u_n, u) \rightarrow 0,
\]
as \(n \rightarrow \infty\). It is clear that \(u_n \xrightarrow{\rho^s} u \iff u_n \xrightarrow{\rho} u\) and \(u_n \xrightarrow{\rho^{-1}} u\).

**Definition 1.6 ([7]).** Assume that \((E, \rho)\) is a quasi metric space (QMS).

(i) If \(\forall \epsilon > 0, \exists n_0 \in \mathbb{N}\) so that

\[
\forall n, k, \ n \geq k \geq n_0, \ \rho(u_k, u_n) < \epsilon,
\]

then \(\{u_n\}\) in \(E\) is called a left K-Cauchy.

(ii) If \(\forall \epsilon > 0, \exists n_0 \in \mathbb{N}\) so that

\[
\forall n, k, \ n \geq k \geq n_0, \ \rho(u_n, u_k) < \epsilon,
\]

then \(\{u_n\}\) in \(E\) is called a right K-Cauchy.

(iii) If \(\forall \epsilon > 0, \exists n_0 \in \mathbb{N}\) so that

\[
\forall n, k \geq n_0, \ \rho(u_n, u_k) < \epsilon,
\]

then \(\{u_n\}\) in \(E\) is said to be \(\rho^s\)-Cauchy.

**Definition 1.7 ([7]).** Assume that \((E, \rho)\) be a QMS.

- If each left (right) K-Cauchy sequence is \(\rho\)-convergent then \((E, \rho)\) is called a left (right) K-complete.
- If each left (right) K-Cauchy sequence is \(\rho^{-1}\)-convergent then \((E, \rho)\) is called a left (right) M-complete.
- If each left (right) K-Cauchy sequence is \(\rho^s\)-convergent then \((E, \rho)\) is called a left (right) Smyth complete.

Currently, Jleli and Samet [12] initiated a contemporary kind of contraction and proved a new result for this contraction in the framework of generalized metric spaces.

**Definition 1.8.** Let \(\Theta : (0, \infty) \rightarrow (1, \infty)\) be a function satisfying:

\(\Theta_1\) \(\Theta\) is nondecreasing;

\(\Theta_2\) for each sequence \(\{\alpha_n\} \subseteq \mathbb{R}^+\), \(\lim_{n \to \infty} \Theta(\alpha_n) = 1 \iff \lim_{n \to \infty} (\alpha_n) = 0\);

\(\Theta_3\) there exists \(0 < k < 1\) and \(l \in (0, \infty)\) such that \(\lim_{a \to 0^+} \frac{\Theta(a)}{\alpha^k} = l\).

A mapping \(J : E \rightarrow E\) is said to be \(\Theta\)-contraction if there exist the function \(\Theta\) satisfying \((\Theta_1)-(\Theta_3)\) and \(\alpha \in (0, 1)\) so that for all \(u, v \in E\),

\[
\rho(Ju, Jv) \neq 0 \implies \Theta(\rho(u, v)) \leq \Theta(\rho(Ju, Jv)) \leq \Theta(\rho(u, v))^{\alpha}.
\]

**Theorem 1.9 ([12]).** If \(J\) be a \(\Theta\)-contraction on a complete metric space \((E, \rho)\), then \(u^* = Ju^*\).
To be consistent with Samet et al. [12], we denote by \( \Omega \) the set of all functions \( \Theta: (0, \infty) \to (1, \infty) \) satisfying the above conditions.

Many researchers [1–6, 9–11, 13–16] have generalized various theorems on metric space by taking the class \( \Omega \).

Subsequently Hancer et al. [8] extended the above definition and added one more condition in this way.

\[(\Theta_4) \ \Theta(\inf \Lambda) = \inf \Theta(\Lambda), \text{ for all } \Lambda \subset (0, \infty) \text{ with } \inf \Lambda > 0.\]

We denote by \( \Omega^* \) the set of all functions \( \Theta \) satisfying \((\Theta_1)-(\Theta_4)\).

The purpose of this manuscript is to define a new family \( \Theta^u_n \) for a multivalued mapping and obtain some fixed point theorems.

2. Main Result

Let \( (\mathcal{E}, \rho) \) be a quasi metric space, \( \mathcal{J}: \mathcal{E} \to \mathcal{P}(\mathcal{E}) \), \( \Theta \in \Omega \), and \( \beta \geq 0 \). For \( u \in \mathcal{E} \) with \( \rho(u, \mathcal{J}u) > 0 \), define the set \( \Theta^u_\beta \subseteq \mathcal{E} \) as

\[\Theta^u_\beta = \{ v \in \mathcal{J}u : \Theta(\rho(u, v)) \leq [\Theta(\rho(u, \mathcal{J}u))]^{\beta} \} \]

It is obvious that, if \( \beta_1 \leq \beta_2 \), then \( \Theta^u_{\beta_1} \subseteq \Theta^u_{\beta_2} \). Now, we explore these cases for \( \Theta^u_{\beta} \).

If \( \mathcal{J}: \mathcal{E} \to \mathcal{A}_{\rho}(\mathcal{E}) \), then it is clear that \( \Theta^u_{\beta} \neq \emptyset \) for all \( \beta \geq 0 \) and \( u \in \mathcal{E} \) with \( \rho(u, \mathcal{J}u) > 0 \).

In this section, we defined \( \Theta \)-contraction with respect to a self mapping and establish a common fixed point theorem using the concept of dominating and dominated mappings.

**Theorem 2.1.** Let \( (\mathcal{E}, \rho) \) be a left \( K \)-complete \( T_1 \)-quasi metric space, \( \Theta \in \Omega \) and \( \mathcal{J}: \mathcal{E} \to \mathcal{A}_{\rho}(\mathcal{E}) \). If there exists \( \alpha \in (0, 1) \) such that for any \( u \in \mathcal{E} \) with \( \rho(u, \mathcal{J}u) > 0 \) and \( v \in \Theta^u_{\beta} \) satisfying\n
\[\Theta(\rho(v, v)) \leq [\Theta(\rho(u, v))]^{\alpha},\]

then \( u^* \in \mathcal{J}u^* \) provided that \( \alpha < \beta \) and \( u \to \rho(u, \mathcal{J}u) \) is lower semi-continuous regarding \( \tau_\rho \).

**Proof.** Let \( u^* \notin \mathcal{J}u^* \). Now, for all \( u \in \mathcal{E} \) we get \( \rho(u, \mathcal{J}u) > 0 \). (Note that if \( \rho(u, \mathcal{J}u) = 0 \), then since \( \mathcal{J}u \in \mathcal{A}_{\rho}(\mathcal{E}) \), there exists \( a \in \mathcal{J}u \) such that

\[\rho(u, a) = \rho(u, \mathcal{J}u) = 0.\]

So, \( a = u \in \mathcal{J}u \) because \( \rho \) is a \( T_1 \)-quasi metric. Now, since \( \mathcal{J}u \in \mathcal{A}_{\rho}(\mathcal{E}) \) for every \( u \in \mathcal{E} \), so the set \( \Theta^u_{\beta} \) is nonempty. Let \( u_0 \in \mathcal{E} \), be an arbitrary initial point, then there exists \( u_1 \in \Theta^u_{\beta} \) so that

\[\Theta(\rho(u_1, \mathcal{J}u_1)) \leq [\Theta(\rho(u_0, u_1))]^{\alpha},\]

and for \( u_1 \in \mathcal{E} \), there exists \( u_2 \in \Theta^u_{\beta} \) satisfying

\[\Theta(\rho(u_2, \mathcal{J}u_2)) \leq [\Theta(\rho(u_1, u_2))]^{\alpha}.\]

Pursuing in this way, we have a sequence \( \{u_n\} \), where \( u_{n+1} \in \Theta^u_{\beta} \) and

\[\Theta(\rho(u_{n+1}, \mathcal{J}u_{n+1})) \leq [\Theta(\rho(u_n, u_{n+1}))]^{\alpha}. \tag{2.1}\]

Now, we will prove that \( \{u_n\} \) is a left \( K \)-Cauchy sequence. As \( u_{n+1} \in \Theta^u_{\beta} \), we have

\[\Theta(\rho(u_n, u_{n+1})) \leq [\Theta(\rho(u_n, \mathcal{J}u_n))]^{\beta}. \tag{2.2}\]

From (2.1) and (2.2), we have

\[\Theta(\rho(u_{n+1}, \mathcal{J}u_{n+1})) \leq [\Theta(\rho(u_n, \mathcal{J}u_n))]^{\alpha \beta},\]
and
\[ \Theta(\rho(u_{n+1}, u_{n+2})) \leq \Theta(\rho(u_n, u_{n+1}))^{\alpha \beta}. \]

By this way we can obtain
\[ \Theta(\rho(u_n, u_{n+1})) \leq \Theta(\rho(u_0, u_1))^{(\alpha \beta)^n}, \tag{2.3} \]
and
\[ \Theta(\rho(u_n, \partial u_n)) \leq \Theta(\rho(u_0, \partial u_0))^{(\alpha \beta)^n}. \tag{2.4} \]

By \( n \to \infty \) in (2.3), we have
\[ \lim_{n \to \infty} \Theta(\rho(u_n, u_{n+1})) = 1, \]
which implies that
\[ \lim_{n \to \infty} \rho(u_n, u_{n+1}) = 0, \]
by \((\Theta_2)\). From the condition \((\Theta_3)\), there exists \( 0 < k < 1 \) and \( l \in (0, \infty) \) so that
\[ \lim_{n \to \infty} \frac{\Theta(\rho(u_n, u_{n+1})) - 1}{\rho(u_n, u_{n+1})^k} = l. \]

Let \( l < \infty \) and \( \lambda_1 = \frac{1}{2} > 0 \). By definition of the limit, there exists \( n_1 \in \mathbb{N} \) so that
\[ \left| \frac{\Theta(\rho(u_n, u_{n+1})) - 1}{\rho(u_n, u_{n+1})^k} - l \right| \leq \lambda_1 \]
for all \( n > n_1 \), which implies that
\[ \frac{\Theta(\rho(u_n, u_{n+1})) - 1}{\rho(u_n, u_{n+1})^k} \geq 1 - \lambda_1 = \frac{1}{2} = \lambda_1 \]
for all \( n > n_1 \). Then
\[ n\rho(u_n, u_{n+1})^k \leq \lambda_2 n[\Theta(\rho(u_n, u_{n+1})) - 1], \]
for all \( n > n_1 \), where \( \lambda_2 = \frac{1}{\lambda_1} \). Now we suppose that \( l = \infty \). Let \( \lambda_1 > 0 \). By the definition of the limit, there exists \( n_1 \in \mathbb{N} \) so that
\[ \lambda_1 \leq \frac{\Theta(\rho(u_n, u_{n+1})) - 1}{\rho(u_n, u_{n+1})^k} \]
for all \( n > n_1 \), which implies that
\[ n\rho(u_n, u_{n+1})^k \leq \lambda_2 n[\Theta(\rho(u_n, u_{n+1})) - 1] \]
for all \( n > n_1 \), where \( \lambda_2 = \frac{1}{\lambda_1} \). Hence, in any case, there exists \( \lambda_2 > 0 \) and \( n_1 \in \mathbb{N} \) so that
\[ n\rho(u_n, u_{n+1})^k \leq \lambda_2 n[\Theta(\rho(u_n, u_{n+1})) - 1] \tag{2.5} \]
for all \( n > n_1 \). Thus by (2.3) and (2.5), we get
\[ n\rho(u_n, u_{n+1})^k \leq \lambda_2 n[(\Theta(\rho(u_0, u_1))^{(\alpha \beta)^n} - 1). \]

Taking \( n \to \infty \), we obtain
\[ \lim_{n \to \infty} n\rho(u_n, u_{n+1})^k = 0. \]

Hence there exists \( n_2 \in \mathbb{N} \) so that
\[ \rho(u_n, u_{n+1}) \leq \frac{1}{n_1^{1/k}} \]
\[ 
\]
for all $n > n_2$. Now for $m > n > n_2$ we get
\[
\rho(u_n, u_m) \leq \rho(u_n, u_{n+1}) + \rho(u_{n+1}, u_{n+2}) + \cdots + \rho(u_{m-1}, u_m)
\]
\[
= \sum_{i=n}^{m-1} \rho(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} \rho(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^1/k}.
\]
Since, $0 < k < 1$ and $\sum_{i=1}^{\infty} \frac{1}{i^1/k}$ is convergent. So taking $n \to \infty$, we have $\rho(u_n, u_m) \to 0$. Hence $\{u_n\}$ is left $K$-Cauchy in $(\mathcal{E}, \rho)$. As $(\mathcal{E}, \rho)$ is a left $K$-complete, so there exists $u^* \in \mathcal{E}$ such that $\{u_n\}$ is $\rho$-convergent to $u^*$, that is, $\rho(u^*, u_n) \to 0$ as $n \to \infty$. \hfill \qed

On the other hand, from (2.4) and (Θ2) we have
\[
\lim_{n \to \infty} \rho(u_n, J u_n) = 0.
\]
Since $u \to \rho(u, J u)$ is lower semi-continuous regarding $\tau_\rho$, then
\[
0 < \rho(u^*, J u^*) \leq \lim_{n \to \infty} \inf \rho(u_n, J u_n) = 0,
\]
which contradicts to the supposition. Thus $u^* \in J u^*$.

**Remark 2.2.** As $K_{\rho^{-1}}(\mathcal{E}) \subseteq A_\rho(\mathcal{E})$, we can take $J u \in K_{\rho^{-1}}(\mathcal{E})$ for all $u \in \mathcal{E}$ in the overhead result.

**Theorem 2.3.** Let $(\mathcal{E}, \rho)$ be a left $M$-complete $T_1$-quasi metric space, $\Theta \in \Omega$, and $J : \mathcal{E} \to A_\rho(\mathcal{E})$. If there exists $\alpha \in (0, 1)$ so that for any $u \in \mathcal{E}$ with $\rho(u, J u) > 0$, there exists $v \in \Theta^u_\rho$ satisfying
\[
\Theta(\rho(v, J v)) \leq [\Theta(\rho(u, v))]^\alpha,
\]
then $u^* \in J u^*$ provided that $\alpha < \beta$ and $u \to \rho(u, J u)$ is lower semi-continuous regarding $\tau_{\rho^{-1}}$.

**Proof.** Let $u^* \not\in J u^*$. Then $\rho(u^*, J u^*) > 0$. By Theorem 2.1, there exists left $K$-Cauchy sequence $\{u_n\}$. By the left $M$-completeness of $(\mathcal{E}, \rho)$, there exists $u^* \in \mathcal{E}$ so that, $\rho(u_n, u^*) \to 0$ as $n \to \infty$. As
\[
\lim_{n \to \infty} \rho(u_n, J u_n) = 0,
\]
and $u \to \rho(u, J u)$ is lower semi-continuous regarding $\tau_{\rho^{-1}}$, then
\[
0 < \rho(u^*, J u^*) \leq \lim_{n \to \infty} \inf \rho(u_n, J u_n) = 0,
\]
which contradicts the supposition. Thus $u^* \in J u^*$. \hfill \qed

If $C_\rho(\mathcal{E})$ is considered on the place of $A_\rho(\mathcal{E})$ in the overhead results with the given conditions, then $\mathcal{J}$ may not have a fixed point. But, if we consider $\Omega^*$ on the place of $\Omega$, then the fixed point of $J$ must exists.

**Theorem 2.4.** Let $(\mathcal{E}, \rho)$ be a left $K$-complete quasi metric space, $\Theta \in \Omega^*$, and $J : \mathcal{E} \to C_\rho(\mathcal{E})$. If there exists $\alpha \in (0, 1)$ so that for any $u \in \mathcal{E}$ with $\rho(u, J u) > 0$ and $v \in \Theta^u_\rho$ satisfying
\[
\Theta(\rho(v, J v)) \leq [\Theta(\rho(u, v))]^\alpha,
\]
then $u^* \in J u^*$ provided that $\alpha < \beta$ and $u \to \rho(u, J u)$ is lower semi-continuous regarding $\tau_\rho$.

**Proof.** Let $J$ has no fixed point. Then, for all $u \in \mathcal{E}$ we get $\rho(u, J u) > 0$. (But if $\rho(u, J u) = 0$, then $u \in C_\rho(\mathcal{E}) = J u$). Since $\Theta \in \Omega^*$, for every $u \in \mathcal{E}$, so the set $\Theta^u_\rho$ is nonempty. Let $u_0 \in \mathcal{E}$, be an arbitrary initial point, then there exists $u_1 \in \Theta^{u_0}_\rho$ so that
\[
\Theta(\rho(u_1, J u_1)) \leq [\Theta(\rho(u_0, u_1))]^\alpha.
\]
Considering the condition \((\Theta_4)\), we can write
\[
\Theta(\rho(u_1, J u_1)) = \inf_{v \in J u_1} \Theta(\rho(u_1, v)).
\]
Thus from
\[
\Theta(\rho(u_1, J u_1)) \leq [\Theta(\rho(u_0, u_1))]^\alpha,
\]
we have
\[
\inf_{v \in J u_1} \Theta(\rho(u_1, v)) \leq [\Theta(\rho(u_0, u_1))]^\alpha < [\Theta(\rho(u_0, u_1))]^\gamma,
\]
where \(0 < \alpha < \gamma < 1\). Therefore, there exists \(u_2 \in J u_1\) so that
\[
\Theta(\rho(u_1, u_2)) \leq [\Theta(\rho(u_0, u_1))]^\gamma.
\]
Doing the same as the proof of Theorem 2.1 by considering the \(J u^* \in C_\rho(\mathcal{E})\).

Following theorem generalized the Feng-Liu’s fixed point theorem.

**Theorem 2.5.** Let \((\mathcal{E}, \rho)\) be a left M-complete quasi metric space, \(\Theta \in \Omega^\ast\), and \(J : \mathcal{E} \to C_\rho(\mathcal{E})\). If there exists \(\alpha \in (0, 1)\) such that for any \(u \in \mathcal{E}\) with \(\rho(u, J u) > 0\) and \(v \in \Theta^\beta\) satisfying
\[
\Theta(\rho(v, J v)) \leq [\Theta(\rho(u, v))]^\alpha,
\]
then \(u^* \in J u^*\) provided that \(\alpha < \beta\) and \(u \to \rho(u, J u)\) is lower semi-continuous regarding \(\tau_{\rho^{-1}}\).

**Proof.** Let there does not exist \(u^* \in \mathcal{E}\) such that \(u^* \in J u^*\). From Theorem 2.4, there exists \(\{u_n\}\) which is left K-Cauchy. As \((\mathcal{E}, \rho)\) is left M-complete, so ther exists \(u^* \in \mathcal{E}\) so that, \(\rho(u_n, u^*) \to 0\) as \(n \to \infty\). Now as
\[
\lim_{n \to \infty} \rho(u_n, J u_n) = 0.
\]
Since \(u \to \rho(u, J u)\) is lower semi-continuous regarding \(\tau_{\rho^{-1}}\), then
\[
0 < \rho(u^*, J u^*) \leq \lim_{n \to \infty} \inf \rho(u_n, J u_n) = 0,
\]
which contradicts the supposition. Thus \(u^* \in J u^*\).

**References**


