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Fixed point theorems for Θ-contractions in left K-complete T_1-quasi metric space

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Abstract

The aim of this paper is to define $\Theta_\beta^u = \left\{v \in \mathcal{J}u : \Theta(\varrho(u,v)) \leq [\Theta(\varrho(u,\mathcal{J}u))]^\beta\right\}$ and establish some new fixed point theorems in the setting of left K-complete T_1-quasi metric space. Our theorems generalize, extend, and unify several results of literature.

Keywords: Θ-contractions, property P, property Q, fixed points.

2010 MSC: 47H10, 54H25.

1. Introduction and preliminaries

Definition 1.1 ([7]). Let $\mathcal{E} \neq \varnothing$. A function $\varrho : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ is said to be a quasi-pseudo metric so that $\forall u, v, w \in \mathcal{E}$:

- $\varrho(u,v) = 0$;
- $\varrho(u,v) \leq \varrho(u,w) + \varrho(w,v)$.

If it satisfies:

- $\varrho(u,v) = \varrho(v,u) = 0 \Rightarrow u = v$,

then ϱ is called T_1-quasi metric.

Remark 1.2 ([7]).

(i) Each metric is a T_1-quasi metric.
(ii) Each T_1-quasi metric is a quasi metric.
(iii) Each quasi metric is a quasi-pseudo metric.

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Definition 1.3 ([7])**.** Assume that a quasi-pseudo metric space $(\mathcal{E}, \varrho)$. Given $u_0 \in \mathcal{E}$ as centre and $\epsilon > 0$ as radius, then

$$B_\varrho(u_0, \epsilon) = \{v \in \mathcal{E} : \varrho(u_0, r) < \epsilon\}$$

denotes the open ball and

$$\bar{B}_\varrho(u_0, \epsilon) = \{v \in \mathcal{E} : \varrho(u_0, r) \leq \epsilon\}$$

denotes the closed ball.

Every quasi-pseudo metric ϱ on $\mathcal{E}$ originate a topology τ_ϱ on $\mathcal{E}$. If ϱ is a quasi metric on $\mathcal{E}$, then the originated topology τ_ϱ must be T_0. If ϱ is a T_1-quasi metric, then the generated topology τ_ϱ is a T_1.

If ϱ is a quasi-pseudo metric on $\mathcal{E}$, then define ϱ^{-1}, ϱ^s, and ϱ_+ as

$$\varrho^{-1}(u, v) = \varrho(v, u), \quad \varrho^s(u, v) = \max\{\varrho(u, v), \varrho^{-1}(u, v)\}, \quad \text{and} \quad \varrho_+(u, v) = \varrho(u, v) + \varrho^{-1}(u, v).$$

All these metrics are also quasi-pseudo metrics on $\mathcal{E}$. Moreover, if ϱ satisfies

$$u \neq v \Longrightarrow \varrho(u, v) + \varrho^{-1}(u, v) > 0,$$

then ϱ_+ (and also ϱ^s) is a metric on $\mathcal{E}$. Here $\mathrm{cl}_\varrho(A)$, $\mathrm{cl}_{\varrho^{-1}}(A)$, and $\mathrm{cl}_{\varrho^s}(A)$ denote the closure of A in $\mathcal{E}$ with respect to τ_ϱ ,$\tau_{\varrho^{-1}}$, and τ_{ϱ^s}, respectively.

We give the following examples in which the mapping $\varrho : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ is a quasi metric but not a T_1-quasi metric.

Example 1.4 ([7])**.**

(i) Let $\mathcal{E} = \mathbb{R}$. Define $\varrho : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ as follows

$$\varrho(u, v) = \max\{v - u, 0)\}$$

$\forall u, v \in \mathcal{E}$.

(ii) Let $\mathcal{E} = \mathbb{R}$ and

$$\varrho(u, v) = \begin{cases} 0, & u = v, \\ |v|, & u \neq v, \end{cases}$$

$\forall u, v \in \mathcal{E}$.

We give the following example in which the mapping $\varrho : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ is a T_1-quasi metric although not a metric on $\mathcal{E}$.

Example 1.5 ([7])**.** Let $\mathcal{E} = \mathbb{R}$ and

$$\varrho(u, v) = \begin{cases} v - u, & u \leq v, \\ 1, & u > v, \end{cases}$$

$\forall u, v \in \mathcal{E}$.

Let $(\mathcal{E}, \varrho)$ be a quasi metric space, B a nonempty subset of $\mathcal{E}$, and $u \in \mathcal{E}$. Then

$$u \in \mathrm{cl}_\varrho(A) \Longleftrightarrow \varrho(u, B) = \inf\{\varrho(u, a) : a \in B\} = 0.$$

Likewise,

$$u \in \mathrm{cl}_{\varrho^{-1}}(A) \Longleftrightarrow \varrho(B, u) = \inf\{\varrho(a, u) : a \in B\} = 0.$$

If B is compact subset of $\mathcal{E}$ and $(\mathcal{E}, \varrho)$ is a metric space, then for each $u \in \mathcal{E}$, $\exists\, a \in B$ so that

$$\varrho(u, a) = \varrho(u, B).$$

But this property does not hold in a quasi metric space $(\mathcal{E}, \varrho)$.

However, if $(\mathcal{E}, \varrho)$ is a quasi metric space and A is a $\tau_{\varrho^{-1}}$-compact subset of $\mathcal{E}$, then this property holds. Let $(\mathcal{E}, \varrho)$ be a quasi metric space and $u \in \mathcal{E}$. A sequence $\{u_n\}$ converges to u regarding τ_ϱ is said to be ϱ-convergence and denoted by $u_n \underset{\rightarrow}{\varrho} u$ and is defined by

$$\varrho(u, u_n) \to 0,$$

as $n \to \infty$. Similarly, the convergence of $\{u_n\}$ to u regarding $\tau_{\varrho^{-1}}$ is said to be ϱ^{-1}-convergence and denoted by $u_n \underset{\longrightarrow}{\varrho^{-1}} u$ and is defined by

$$\varrho^{-1}(u_n, u) \to 0,$$

as $n \to \infty$. Finally, the convergence of $\{u_n\}$ to u regarding τ_{ϱ^s} is said to be ϱ^s-convergence and denoted by $u_n \underset{\rightarrow}{\varrho^s} u$ and is defined by

$$\varrho^s(u_n, u) \to 0,$$

as $n \to \infty$. It is clear that $u_n \underset{\rightarrow}{\varrho^s} u \Longleftrightarrow u_n \underset{\rightarrow}{\varrho} u$ and $u_n \underset{\longrightarrow}{\varrho^{-1}} u$.

Definition 1.6 ([7])**.** Assume that $(\mathcal{E}, \varrho)$ is a quasi metric space (QMS).

(i) If $\forall \epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ so that
$$\forall n, k,\ n \geq k \geq n_0,\ \varrho(u_k, u_n) < \epsilon,$$
then $\{u_n\}$ in $\mathcal{E}$ is called a left K-Cauchy.

(ii) If $\forall \epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ so that
$$\forall n, k,\ n \geq k \geq n_0,\ \varrho(u_n, u_k) < \epsilon,$$
then $\{u_n\}$ in $\mathcal{E}$ is called a right K-Cauchy.

(iii) If $\forall \epsilon > 0$, $\exists\ n_0 \in \mathbb{N}$ so that
$$\forall n, k \geq n_0,\ \varrho(u_n, u_k) < \epsilon,$$
then $\{u_n\}$ in $\mathcal{E}$ is said to be ϱ^s-Cauchy.

Definition 1.7 ([7])**.** Assume that $(\mathcal{E}, \varrho)$ be a QMS.

- If each left (right) K-Cauchy sequence is ϱ-convergent then $(\mathcal{E}, \varrho)$ is called a left (right) K-complete.
- If each left (right) K-Cauchy sequence is ϱ^{-1}-convergent then $(\mathcal{E}, \varrho)$ is called a left (right) M-complete.
- If each left (right) K-Cauchy sequence is ϱ^s-convergent then $(\mathcal{E}, \varrho)$ is called a left (right) Smyth complete.

Currently, Jleli and Samet [12] initiated a contemporary kind of contraction and proved a new result for this contraction in the framework of generalized metric spaces.

Definition 1.8. Let $\Theta : (0, \infty) \to (1, \infty)$ be a function satisfying:

(Θ_1) Θ is nondecreasing;
(Θ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \Theta(\alpha_n) = 1 \Longleftrightarrow \lim_{n\to\infty}(\alpha_n) = 0$;
(Θ_3) $\exists\ 0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha \to 0^+} \frac{\Theta(\alpha)-1}{\alpha^k} = l$.

A mapping $\mathcal{J} : \mathcal{E} \to \mathcal{E}$ is said to be Θ-contraction if there exist the function Θ satisfying (Θ_1)-(Θ_3) and $\alpha \in (0, 1)$ so that $\forall u, v \in \mathcal{E}$,

$$\varrho(\mathcal{J}u, \mathcal{J}v) \neq 0 \Longrightarrow \Theta(\varrho(\mathcal{J}u, \mathcal{J}v)) \leq [\Theta(\varrho(u, v))]^\alpha.$$

Theorem 1.9 ([12]). *If $\mathcal{J}$ be a Θ-contraction on be a complete metric space $(\mathcal{E}, \varrho)$, then $u^* = \mathcal{J}u^*$.*

To be consistent with Samet et al. [12], we denote by Ω the set of all functions $\Theta : (0, \infty) \to (1, \infty)$ satisfying the above conditions.

Many researchers [1–6, 9–11, 13–16] have generalized various theorems on metric space by taking the class Ω.

Subsequently Hancer et al. [8] extended the above definition and added one more condition in this way.

(Θ_4) $\Theta(\inf A) = \inf \Theta(A)$, $\forall A \subset (0, \infty)$ with $\inf A > 0$.

We denote by Ω^* the set of all functions Θ satisfying (Θ_1)-(Θ_4).

The purpose of this manuscript is to define a new family Θ^u_β for a multivalued mapping and obtain some fixed point theorems.

2. Main Result

Let $(\mathcal{E}, \varrho)$ be a quasi metric space, $\mathcal{J} : \mathcal{E} \to P(\mathcal{E})$, $\Theta \in \Omega$, and $\beta \geq 0$. For $u \in \mathcal{E}$ with $\varrho(u, \mathcal{J}u) > 0$, define the set $\Theta^u_\beta \subseteq \mathcal{E}$ as

$$\Theta^u_\beta = \left\{ v \in \mathcal{J}u : \Theta(\varrho(u, v)) \leq [\Theta(\varrho(u, \mathcal{J}u))]^\beta \right\}.$$

It is obvious that, if $\beta_1 \leq \beta_2$, then $\Theta^u_{\beta_1} \subseteq \Theta^u_{\beta_2}$. Now, we explore these cases for Θ^u_β.

If $\mathcal{J} : \mathcal{E} \to A_\varrho(\mathcal{E})$, then it is clear that $\Theta^u_\beta \neq \emptyset$, $\forall \beta \geq 0$ and $u \in \mathcal{E}$ with $\varrho(u, \mathcal{J}u) > 0$.

In this section, we defined Θ-contraction with respect to a self mapping and establish a common fixed point theorem using the concept of dominating and dominated mappings.

Theorem 2.1. *Let $(\mathcal{E}, \varrho)$ be a left K-complete T_1-quasi metric space, $\Theta \in \Omega$ and $\mathcal{J} : \mathcal{E} \to A_\varrho(\mathcal{E})$. If $\exists\, \alpha \in (0, 1)$ such that for any $u \in \mathcal{E}$ with $\varrho(u, \mathcal{J}u) > 0$ and $v \in \Theta^u_\beta$ satisfying*

$$\Theta(\varrho(v, \mathcal{J}v)) \leq [\Theta(\varrho(u, v))]^\alpha,$$

then $u^* \in \mathcal{J}u^*$ provided that $\alpha < \beta$ and $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding τ_ϱ.

Proof. Let $u^* \notin \mathcal{J}u^*$. Now, $\forall u \in \mathcal{E}$ we get $\varrho(u, \mathcal{J}u) > 0$. (Note that if $\varrho(u, \mathcal{J}u) = 0$, then since $\mathcal{J}u \in A_\varrho(\mathcal{E})$, there exists $a \in \mathcal{J}u$ such that

$$\varrho(u, a) = \varrho(u, \mathcal{J}u) = 0.$$

So, $a = u \in \mathcal{J}u$ because ϱ is a T_1-quasi metric). Now, since $\mathcal{J}u \in A_\varrho(\mathcal{E})$ for every $u \in \mathcal{E}$, so the set Θ^u_β is nonempty. Let $u_0 \in \mathcal{E}$, be an arbitrary initial point, then $\exists\, u_1 \in \Theta^{u_0}_\beta$ so that

$$\Theta(\varrho(u_1, \mathcal{J}u_1)) \leq [\Theta(\varrho(u_0, u_1))]^\alpha,$$

and for $u_1 \in \mathcal{E}$, $\exists\, u_2 \in \Theta^{u_1}_\beta$ satisfying

$$\Theta(\varrho(u_2, \mathcal{J}u_2)) \leq [\Theta(\varrho(u_1, u_2))]^\alpha.$$

Pursuing in this way, we have a sequence $\{u_n\}$, where $u_{n+1} \in \Theta^{u_n}_\beta$ and

$$\Theta(\varrho(u_{n+1}, \mathcal{J}u_{n+1})) \leq [\Theta(\varrho(u_n, u_{n+1}))]^\alpha. \tag{2.1}$$

Now, we will prove that $\{u_n\}$ is a left K-Cauchy sequence. As $u_{n+1} \in \Theta^{u_n}_\beta$, we have

$$\Theta(\varrho(u_n, u_{n+1})) \leq [\Theta(\varrho(u_n, \mathcal{J}u_n))]^\beta. \tag{2.2}$$

From (2.1) and (2.2), we have

$$\Theta(\varrho(u_{n+1},\mathcal{J}u_{n+1}))\leq[\Theta(\varrho(u_n,\mathcal{J}u_n))]^{\alpha\beta},$$

and

$$\Theta(\varrho(u_{n+1},u_{n+2}))\leq[\Theta(\varrho(u_n,u_{n+1}))]^{\alpha\beta}.$$

By this way we can obtain

$$\Theta(\varrho(u_n,u_{n+1}))\leq[\Theta(\varrho(u_0,u_1))]^{(\alpha\beta)^n}, \tag{2.3}$$

and

$$\Theta(\varrho(u_n,\mathcal{J}u_n))\leq[\Theta(\varrho(u_0,\mathcal{J}u_0))]^{(\alpha\beta)^n}. \tag{2.4}$$

By $n\to\infty$ in (2.3), we have

$$\lim_{n\to\infty}\Theta(\varrho(u_n,u_{n+1}))=1,$$

which implies that

$$\lim_{n\to\infty}\varrho(u_n,u_{n+1})=0,$$

by (Θ_2). From the condition (Θ_3), $\exists\ 0<k<1$ and $l\in(0,\infty]$ so that

$$\lim_{n\to\infty}\frac{\Theta(\varrho(u_n,u_{n+1}))-1}{\varrho(u_n,u_{n+1})^k}=l.$$

Let $l<\infty$ and $\lambda_1=\frac{l}{2}>0$. By definition of the limit, $\exists\ n_1\in\mathbb{N}$ so that

$$\left|\frac{\Theta(\varrho(u_n,u_{n+1}))-1}{\varrho(u_n,u_{n+1})^k}-l\right|\leq\lambda_1$$

$\forall n>n_1$, which implies that

$$\frac{\Theta(\varrho(u_n,u_{n+1}))-1}{\varrho(u_n,u_{n+1})^k}\geq l-\lambda_1=\frac{l}{2}=\lambda_1$$

$\forall n>n_1$. Then

$$n\varrho(u_n,u_{n+1})^k\leq\lambda_2 n[\Theta(\varrho(u_n,u_{n+1}))-1],$$

$\forall n>n_1$, where $\lambda_2=\frac{1}{\lambda_1}$. Now we suppose that $l=\infty$. Let $\lambda_1>0$. By the definition of the limit, $\exists\ n_1\in\mathbb{N}$ so that

$$\lambda_1\leq\frac{\Theta(\varrho(u_n,u_{n+1}))-1}{\varrho(u_n,u_{n+1})^k}$$

$\forall n>n_1$, which implies that

$$n\varrho(u_n,u_{n+1})^k\leq\lambda_2 n[\Theta(\varrho(u_n,u_{n+1}))-1]$$

$\forall n>n_1$, where $\lambda_2=\frac{1}{\lambda_1}$. Hence, in any case, $\exists\ \lambda_2>0$ and $n_1\in\mathbb{N}$ so that

$$n\varrho(u_n,u_{n+1})^k\leq\lambda_2 n[\Theta(\varrho(u_n,u_{n+1}))-1] \tag{2.5}$$

$\forall n>n_1$. Thus by (2.3) and (2.5), we get

$$n\varrho(u_n,u_{n+1})^k\leq\lambda_2 n([\Theta(\varrho(u_0,u_1))]^{(\alpha\beta)^n}-1).$$

Taking $n\to\infty$, we obtain

$$\lim_{n\to\infty}n\varrho(u_n,u_{n+1})^k=0.$$

Hence $\exists\ n_2\in\mathbb{N}$ so that

$$\varrho(u_n,u_{n+1})\leq\frac{1}{n^{1/k}}$$

$\forall n > n_2$. Now for $m > n > n_2$ we get

$$\begin{aligned}\varrho(u_n, u_m) &\leq \varrho(u_n, u_{n+1}) + \varrho(u_{n+1}, u_{n+2}) + \cdots + \varrho(u_{m-1}, u_m) \\ &= \sum_{i=n}^{m-1} \varrho(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} \varrho(u_i, u_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.\end{aligned}$$

Since, $0 < k < 1$ and $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent. So taking $n \to \infty$, we have $\varrho(u_n, u_m) \to 0$. Hence $\{u_n\}$ is left K-Cauchy in $(\mathcal{E}, \varrho)$. As $(\mathcal{E}, \varrho)$ is a left K-complete, so $\exists\, u^* \in \mathcal{E}$ such that $\{u_n\}$ is ϱ-convergent to u^*, that is, $\varrho(u^*, u_n) \to 0$ as $n \to \infty$. □

On the other hand, from (2.4) and (Θ_2) we have

$$\lim_{n\to\infty} \varrho(u_n, \mathcal{J}u_n) = 0.$$

Since $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding τ_ϱ, then

$$0 < \varrho(u^*, \mathcal{J}u^*) \leq \lim_{n\to\infty} \inf \varrho(u_n, \mathcal{J}u_n) = 0,$$

which contradicts to the supposition. Thus $u^* \in \mathcal{J}u^*$.

Remark 2.2. As $K_{\varrho^{-1}}(\mathcal{E}) \subseteq A_\varrho(\mathcal{E})$, we can take $\mathcal{J}u \in K_{\varrho^{-1}}(\mathcal{E})$, $\forall u \in \mathcal{E}$ in the overhead result.

Theorem 2.3. *Let $(\mathcal{E}, \varrho)$ be a left M-complete T_1-quasi metric space, $\Theta \in \Omega$, and $\mathcal{J} : \mathcal{E} \to A_\varrho(\mathcal{E})$. If $\exists\, \alpha \in (0, 1)$ so that for any $u \in \mathcal{E}$ with $\varrho(u, \mathcal{J}u) > 0$, $\exists\, v \in \Theta_\beta^u$ satisfying*

$$\Theta(\varrho(v, \mathcal{J}v)) \leq [\Theta(\varrho(u, v))]^\alpha,$$

then $u^* \in \mathcal{J}u^*$ provided that $\alpha < \beta$ and $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding $\tau_{\varrho^{-1}}$.

Proof. Let $u^* \notin \mathcal{J}u^*$. Then $\varrho(u^*, \mathcal{J}u^*) > 0$. By Theorem 2.1, there exists left K-Cauchy sequence $\{u_n\}$. By the left M-completeness of $(\mathcal{E}, \varrho)$, $\exists\, u^* \in \mathcal{E}$ so that, $\varrho(u_n, u^*) \to 0$ as $n \to \infty$. As

$$\lim_{n\to\infty} \varrho(u_n, \mathcal{J}u_n) = 0,$$

and $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding $\tau_{\varrho^{-1}}$, then

$$0 < \varrho(u^*, \mathcal{J}u^*) \leq \lim_{n\to\infty} \inf \varrho(u_n, \mathcal{J}u_n) = 0,$$

which contradicts the supposition. Thus $u^* \in \mathcal{J}u^*$. □

If $C_\varrho(\mathcal{E})$ is considered on the place of $A_\varrho(\mathcal{E})$ in the overhead results with the given conditions, then $\mathcal{J}$ may not have a fixed point. But, if we consider Ω^* on the place of Ω, then the fixed point of J must exists.

Theorem 2.4. *Let $(\mathcal{E}, \varrho)$ be a left K-complete quasi metric space, $\Theta \in \Omega^*$, and $\mathcal{J} : \mathcal{E} \to C_\varrho(\mathcal{E})$. If $\exists\, \alpha \in (0, 1)$ so that for any $u \in \mathcal{E}$ with $\varrho(u, \mathcal{J}u) > 0$ and $v \in \Theta_\beta^u$ satisfying*

$$\Theta(\varrho(v, \mathcal{J}v)) \leq [\Theta(\varrho(u, v))]^\alpha,$$

then $u^* \in \mathcal{J}u^*$ provided that $\alpha < \beta$ and $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding τ_ϱ.

Proof. Let $\mathcal{J}$ has no fixed point. Then, $\forall u \in \mathcal{E}$ we get $\varrho(u, \mathcal{J}u) > 0$. (But if $\varrho(u, \mathcal{J}u) = 0$, then $u \in C_\varrho(\mathcal{E}) = \mathcal{J}u$). Since $\Theta \in \Omega^*$, for every $u \in \mathcal{E}$, so the set Θ_β^u is nonempty. Let $u_0 \in \mathcal{E}$, be an arbitrary initial point, then $\exists\, u_1 \in \Theta_\beta^{u_0}$ so that

$$\Theta(\varrho(u_1, \mathcal{J}u_1)) \leq [\Theta(\varrho(u_0, u_1))]^\alpha.$$

Considering the condition (Θ_4), we can write

$$\Theta(\varrho(u_1, \mathcal{J}u_1)) = \inf_{v \in \mathcal{J}u_1} \Theta(\varrho(u_1, v)).$$

Thus from

$$\Theta(\varrho(u_1, \mathcal{J}u_1)) \leq [\Theta(\varrho(u_0, u_1))]^{\alpha},$$

we have

$$\inf_{v \in \mathcal{J}u_1} \Theta(\varrho(u_1, v)) \leq [\Theta(\varrho(u_0, u_1))]^{\alpha} < [\Theta(\varrho(u_0, u_1))]^{\gamma},$$

where $0 < \alpha < \gamma < 1$. Therefore, $\exists\, u_2 \in \mathcal{J}u_1$ so that

$$\Theta(\varrho(u_1, u_2)) \leq [\Theta(\varrho(u_0, u_1))]^{\gamma}.$$

Doing the same as the proof of Theorem 2.1 by considering the $\mathcal{J}u^* \in C_{\varrho}(\mathcal{E})$. □

Following theorem generalized the Feng-Liu's fixed point theorem.

Theorem 2.5. *Let $(\mathcal{E}, \varrho)$ be a left M-complete quasi metric space, $\Theta \in \Omega^*$, and $\mathcal{J} : \mathcal{E} \to C_{\varrho}(\mathcal{E})$. If $\exists\, \alpha \in (0,1)$ such that for any $u \in \mathcal{E}$ with $\varrho(u, \mathcal{J}u) > 0$ and $v \in \Theta^u_{\beta}$ satisfying*

$$\Theta(\varrho(v, \mathcal{J}v)) \leq [\Theta(\varrho(u, v))]^{\alpha},$$

then $u^* \in \mathcal{J}u^*$ provided that $\alpha < \beta$ and $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding $\tau_{\varrho^{-1}}$.

Proof. Let there does not exist $u^* \in \mathcal{E}$ such that $u^* \in \mathcal{J}\, u^*$. From Theorem 2.4, there exists $\{u_n\}$ which is left K-Cauchy. As $(\mathcal{E}, \varrho)$ is left M-complete, so $\exists\, u^* \in \mathcal{E}$ so that, $\varrho(u_n, u^*) \to 0$ as $n \to \infty$. Now as

$$\lim_{n \to \infty} \varrho(u_n, \mathcal{J}u_n) = 0.$$

Since $u \to \varrho(u, \mathcal{J}u)$ is lower semi-continuous regarding $\tau_{\varrho^{-1}}$, then

$$0 < \varrho(u^*, \mathcal{J}u^*) \leq \lim_{n \to \infty} \inf \varrho(u_n, \mathcal{J}u_n) = 0,$$

which contradicts the supposition. Thus $u^* \in \mathcal{J}u^*$. □

References

[1] M. Abbas, B. Ali, S. Romaguera, *Fixed and periodic points of generalized contractions in metric spaces*, Fixed Point Theory Appl., **2013** (2013), 11 pages. 1

[2] J. Ahmad, A. Al-Rawashdeh, A. Azam, *New fixed point theorems for generalized F-contractions in complete metric spaces*, Fixed Point Theory Appl., **2015** (2015), 18 pages.

[3] A. Al-Rawashdeh, J. Ahmad, *Common Fixed Point Theorems for JS-Contractions*, Bull. Math. Anal. Appl., 8 (2016), 12–22.

[4] I. Altun, B. Damjanović, D. Djorić, *Fixed point and common fixed point theorems on ordered cone metric spaces*, Appl. Math. Lett., **23** (2010), 310–316.

[5] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fundam. Math., **3** (1922), 133–181.

[6] I. Beg, M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl., **2006** (2006), 7 pages. 1

[7] H. Dağ, G. Minak, I. Altun, *Some fixed point results for multivalued F–contractions on quasi metric spaces*, Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM, **111** (2017), 177–187. 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7

[8] H. A. Hançer, G. Minak, I. Altun, *On a broad category of multivalued weakly Picard operators*, Fixed Point Theory, **18** (2017), 229–236. 1

[9] N. Hussain, J. Ahmad, L. Ćirić, A. Azam, *Coincidence point theorems for generalized contractions with application to integral equations*, Fixed Point Theory Appl., **2015** (2015), 13 pages. 1

[10] N. Hussain, V. Parvaneh, B. Samet, C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2015** (2015), 17 pages.

[11] G. S. Jeong, B. E. Rhoades, *Maps for which $F(T) = F(T^n)$*, In: Fixed Point Theory and Applications, Nova Science Publishers, **2007** (2007), 71–105. 1

[12] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl., **2014** (2014), 8 pages. 1, 1.9, 1

[13] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, **83** (1976), 261–263. 1

[14] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci., **9** (1986), 771–779.

[15] B. Samet, C. Vetro, P. Vetro, *Fixed point theorem for $\alpha-\psi$ contractive type mappings*, Nonlinear Anal., **75** (2012), 2154–2165.

[16] F. Vetro, *A generalization of Nadler fixed point theorem*, Carpathian J. Math., **31** (2015), 403–410. 1