Some properties of graded 2-absorbing and graded weakly 2-absorbing submodules

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Abstract

Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper we will obtain some results concerning the graded 2-absorbing and graded weakly 2-absorbing submodules of a graded modules over a commutative graded ring.

Keywords: Graded 2-absorbing submodule, graded weakly 2-absorbing submodule, graded submodules.

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1. Introduction and preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary.

The concept of weakly prime ideals was initiated by Anderson and Smith in [8]. The concept of weakly 2-absorbing ideals was introduced in [14] as a generalization of the notion of weakly prime ideals. Badawi in [13] introduced the concept of 2-absorbing ideals of commutative rings that is a generalization of the concept of prime ideals. Later on, Anderson and Badawi in [7] generalized the concept of 2-absorbing ideals of commutative rings to the concept of n-absorbing ideals of commutative rings for every positive integer n ≥ 2. In light of [7, 13] many authors studied the concept of 2-absorbing submodules and n-absorbing submodules, (see for example, [15, 18, 24, 26, 27]).

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. One use of rings and modules with gradings is in describing certain topics in algebraic geometry. Graded prime ideals, and graded weakly prime ideals have been studied by various authors, (see for example [5, 10, 25]). The concept of graded 2-absorbing ideals and graded weakly 2-absorbing ideals, generalizations of graded prime ideals, and graded weakly prime ideals, respectively, were studied by Al-Zoubi and Abu-Dawwas, and other authors, (see [2, 19]). Graded prime submodules, and graded weakly prime submodules have been studied by various authors, (see for example [4, 6, 9, 11, 23]). The concept of...
graded 2-absorbing submodules and graded weakly 2-absorbing submodules, generalizations of graded prime submodules, and graded weakly prime submodules, respectively, were studied by Al-Zoubi and Abu-Dawwas in [1]. Later on, Hamoda and Ashour in [17] introduced the concept of graded n-absorbing submodules that is a generalization of the concept of graded prime ideals.

Here, we study several results concerning of graded 2-absorbing and graded weakly 2-absorbing submodules of graded modules over graded commutative rings.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [16, 20–22] for these basic properties and more information on graded rings and modules.

Let G be a group with identity e and R be a commutative ring with identity 1R. Then R is a G-graded ring if there exist additive subgroups Rg of R such that R = \( \bigoplus_{g \in G} R_g \) and \( R_g R_h \subseteq R_{gh} \) for all \( g, h \in G \). The elements of \( R_g \) are called to be homogeneous of degree \( g \) where the \( R_g \)'s are additive subgroups of \( R \) indexed by the elements \( g \in G \). If \( x \in R \), then \( x \) can be written uniquely as \( \sum_{g \in G} x_g \), where \( x_g \) is the component of \( x \) in \( R_g \). Moreover, \( h(R) = \bigcup_{g \in G} R_g \). Let \( I \) be an ideal of \( R \). Then \( I \) is called a graded ideal of \( (R, G) \) if \( I = \bigoplus_{g \in G} (I \cap R_g) \). Thus, if \( x \in I \), then \( x = \sum_{g \in G} x_g \) with \( x_g \in I \).

Let \( R \) be a G-graded ring and \( M \) an R-module. We say that \( M \) is a G-graded R-module (or graded R-module) if there exists a family of subgroups \( \{M_g\}_{g \in G} \) of \( M \) such that \( M = \bigoplus_{g \in G} M_g \) (as Abelian groups) and \( R_g M_h \subseteq M_{gh} \) for all \( g, h \in G \). Here, \( R_g M_h \) denotes the additive subgroup of \( M \) consisting of all finite sums of elements \( r_g s_h \) with \( r_g \in R_g \) and \( s_h \in M_h \). Also, we write \( h(M) = \bigcup_{g \in G} M_g \) and the elements of \( h(M) \) are called to be homogeneous. Let \( M = \bigoplus M_g \) be a graded R-module and \( N \) a submodule of \( M \). Then \( N \) is called a graded submodule of \( M \) if \( N = \bigoplus N_g \), where \( N_g = N \cap M_g \) for \( g \in G \). In this case, \( N_g \) is called the g-component of \( N \). Let \( R \) be a G-graded ring, \( M \) a graded R-module, and \( N \) a graded submodule of \( M \). Then \( (N :_R M) \) is defined as \( N :_R M = \{ r \in R : rM \subseteq N \} \). It is shown in [9, Lemma 2.1] that if \( N \) is a graded submodule of \( M \), then \( (N :_R M) = \{ r \in R : r \in R M \subseteq N \} \) is a graded ideal of \( R \). A proper graded submodule \( P \) of \( M \) is said to be a graded prime submodule (Resp. graded weakly prime submodule) of \( M \) if whenever \( r \in h(R) \) and \( m \in h(M) \) with \( rm \in P \) (Resp. \( 0 \neq rm \in P \)), then either \( r \in (P :_R M) \) or \( m \in P \) (see [9, 11]). A proper graded ideal \( I \) of \( R \) is said to be a graded 2-absorbing ideal (Resp. a graded weakly 2-absorbing ideal) of \( R \) if whenever \( r, s, t \in h(R) \) with \( rst \in I \) (Resp. \( 0 \neq rst \in I \)), then \( rs \in (N :_R M) \) or \( rm \in N \) or \( sm \in N \) (see [1]).

2. Graded 2-absorbing submodules

**Lemma 2.1.** Let \( R \) be a G-graded ring, \( M \) a graded R-module, and \( N \) a graded 2-absorbing submodule of \( M \). Let \( I = \bigoplus G I_g \) be a graded ideal of \( R \). Then for every \( r \in h(R) \), \( m \in h(M) \) and \( g \in G \) with \( I_g m \subseteq N \), either \( rm \in N \) or \( I_g m \subseteq N \) or \( I_g m \subseteq (N :_R M) \).

**Proof.** Let \( r \in h(R) \), \( m \in h(M) \) and \( g \in G \) such that \( I_g m \subseteq N \), \( rm \notin N \) and \( I_g \notin (N :_R M) \). We have to show that \( I_g m \subseteq N \). As \( I_g \notin (N :_R M) \), there exists \( i_g \in I_g \) such that \( I_g \notin (N :_R M) \). Since \( N \) is a graded 2-absorbing submodule, \( I_g m \subseteq N \), \( rm \notin N \) and \( I_g \notin (N :_R M) \), we have \( i_g m \notin N \). Now, let \( i' \in I_g \). By \( i_g + i' \in I_g \) it follows that \( I_g (i_g + i') \subseteq N \). Then either \( (i_g + i') \subseteq N \) or \( (i_g + i') \subseteq (N :_R M) \) as \( N \) is a graded 2-absorbing submodule. If \( (i_g + i') \subseteq N \), then we get \( I_g m \subseteq N \) since \( I_g m \subseteq N \). If \( (i_g + i') \subseteq (N :_R M) \), then we get \( I_g m \subseteq (N :_R M) \) since \( I_g \subseteq (N :_R M) \), but \( I_g m \subseteq N \), so \( i_g m \subseteq N \) since \( N \) is a graded 2-absorbing submodule, \( rm \notin N \) and \( I_g \notin (N :_R M) \). Hence \( I_g m \subseteq N \). □

**Theorem 2.2.** Let \( R \) be a G-graded ring, \( M \) a graded R-module, and \( N \) a graded 2-absorbing submodule of \( M \). Let \( I = \bigoplus G I_g \) and \( J = \bigoplus G J_g \) be a graded ideals of \( R \). Then for every \( m \in h(M) \) and \( g, h \in G \) with \( I_g J_h m \subseteq N \), either \( I_g m \subseteq N \) or \( J_h m \subseteq N \) or \( I_g J_h m \subseteq (N :_R M) \).
Proof. Let \( m \in h(M) \) and \( g, h \in G \) such that \( I_g J_h m \subseteq N \), \( I_g m \not\subseteq N \) and \( J_h m \not\subseteq N \). We have to show that \( I_g J_h \subseteq (N : R M) \). Let \( i_g \in I_g \) and \( j_h \in J_h \). As \( I_g m \not\subseteq N \) and \( J_h m \not\subseteq N \), there exist \( i'_g \in I_g \) and \( j'_h \in J_h \) such that \( i'_g m \not\subseteq N \) and \( j'_h m \not\subseteq N \). Since \( i'_g J_h m \subseteq N \), \( i'_g m \not\subseteq N \) and \( J_h m \not\subseteq N \), by Lemma 2.1, we get \( i'_g J'_h m \subseteq (N : R M) \). Also since \( j'_h I_g m \subseteq N \), \( j'_h m \not\subseteq N \) and \( I_g m \not\subseteq N \), we get \( j'_h I_g \subseteq (N : R M) \), which implies \( (I_g \setminus (N : R m)) J_h \subseteq (N : R M) \) and \( (J_h \setminus (N : R m)) I_g \subseteq (N : R M) \). Hence we have \( i'_g j'_h \in (N : R M) \), \( i'_g j'_h \in (N : R M) \) and \( i'_g j'_h \in (N : R M) \). By \( (i'_g + i'_g')(j_h + j'_h) \in J_h \) it follows that \( (i'_g + i'_g')(j_h + j'_h)m \in N \). Since \( N \) is a graded 2-absorbing submodule, we get either \( (i'_g + i'_g')(j_h + j'_h)m \in N \) or \( (i'_g + i'_g')(j_h + j'_h)m \in N \). By Theorem 2.2, we get \( i'_g j'_h \in (N : R M) \), \( i'_g j'_h \in (N : R M) \) and \( i'_g j'_h \in (N : R M) \). Thus \( I_g J_h \subseteq (N : R M) \).

\[ \square \]

**Theorem 2.3.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module, and \( N \) be a proper graded submodule of \( M \). Let \( I = \bigoplus_{g \in G} I_g \) and \( J = \bigoplus_{g \in G} J_g \) be graded ideals of \( R \) and \( U = \bigoplus_{g \in G} U_g \) be a graded submodule of \( M \). Then the following statements are equivalent:

(i) \( N \) is a graded 2-absorbing submodule of \( M \);
(ii) for every \( g, h, \lambda \in G \) with \( I_g J_h U_\lambda \subseteq N \), either \( I_g U_\lambda \subseteq N \) or \( J_h U_\lambda \subseteq N \) or \( I_g J_h \subseteq (N : R M) \).

Proof. Let \( m \in h(M) \) and \( g, h \in G \) such that \( I_g J_h U_\lambda \subseteq N \) and \( I_g J_h \not\subseteq (N : R M) \). By Theorem 2.2 for all \( \lambda \in U \), we have either \( I_g U_\lambda \subseteq N \) or \( J_h U_\lambda \subseteq N \). If \( I_g U_\lambda \subseteq N \) for all \( \lambda \in U \), then \( I_g U_\lambda \subseteq N \). Similarly, if \( J_h U_\lambda \subseteq N \) for all \( \lambda \in U \), then \( J_h U_\lambda \subseteq N \). Thus \( I_g U_\lambda \subseteq N \) or \( J_h U_\lambda \subseteq N \).

(iii)\( \Rightarrow \) (i) Assume that \( (i) \) holds. Let \( r_g, s_h \in h(R) \) and \( m_\lambda \in h(M) \) such that \( r_g s_h m_\lambda \in N \). Let \( I = r_g R \) and \( J = s_h R \) be graded ideals of \( R \) generated by \( r_g, s_h \), respectively and \( U = m_\lambda R \) be a graded submodule of \( M \) generated by \( m_\lambda \). Then \( I_g J_h U_\lambda \subseteq N \). By our assumption we obtain \( I_g U_\lambda \subseteq N \) or \( J_h U_\lambda \subseteq N \) or \( I_g J_h \subseteq (N : R M) \). Hence \( r_g m_\lambda \in N \) or \( s_h m_\lambda \in N \) or \( r_g s_h \in (N : R M) \). Therefore \( N \) is a graded 2-absorbing submodule of \( M \).

\[ \square \]

3. Graded weakly 2-absorbing submodules

Let \( N \) be a graded \( R \)-module of \( M \) and let \( g \in G \). We say that \( N_g \) is a weakly \( g \)-absorbing submodule of \( R \)-module \( M_g \), if \( N_g \neq M_g \), and whenever \( r_s \in R \), \( m \in M_g \) with \( 0 \neq r_s m \in N_g \), then either \( r_s \in N_g : R_m \) or \( r_s m \in N_g \) or \( s \in N_g \) (see [1]).

**Lemma 3.1.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module and \( N \) a graded weakly 2-absorbing submodule of \( M \), and \( g \in G \). If \( r_s e_s U \subseteq N_g \) and \( 0 \neq r_s e_s U \) for some \( r_s, e_s \in R \) and some submodule \( U \) of \( M_g \), then either \( r_s e_s U \subseteq N_g \) or \( e_s U \subseteq N_g \) or \( s \subseteq N_g \).

Proof. By [1, Lemma 3.2], \( N_g \) is a weakly 2-absorbing \( R \)-submodule of \( M_g \) for every \( g \in G \). Assume that \( r_s e_s U \subseteq N_g \). If \( 0 \neq r_s e_s U \), then \( r_s e_s U \in N_g : R_m \) for some \( r_s, e_s \in R \) and some submodule \( U \) of \( M_g \). We have to show that \( U \subseteq (N_g : R_m) r_s U \cup (N_g : R_m) e_s U \). Let \( u_g \in U \subseteq M_g \). If \( 0 \neq r_s e_s u_g \), then either \( r_s u_g \in N_g \) or \( s \subseteq u_g \in N_g \) since \( N_g \) is a weakly \( g \)-absorbing \( R \)-submodule of \( M_g \). Suppose that \( r_s e_s u_g = 0 \). Since \( 0 \neq r_s e_s U \), there exists \( u_g' \in U \subseteq M_g \) such that \( 0 \neq r_s e_s u_g' \), hence \( 0 \neq r_s e_s u_g' \in N_g \). Since \( N_g \) is a weakly \( g \)-absorbing \( R \)-submodule of \( M_g \),
we have either \( r_eu'_g \in N_g \) or \( seu'_g \in N_g \). Let \( v_g = u_g + u'_g \). Hence \( 0 \neq resev_g \in N_g \). Then \( rev_g \in N_g \) or \( sev_g \in N_g \) as \( N_g \) is a weakly 2-absorbing \( R_e \)-submodule of \( M_g \). Now, we consider three cases.

Case 1: \( reu'_g \in N_g \) and \( seu'_g \notin N_g \). On the contrary let \( reu'_g \notin N_g \). Then \( rev_g \notin N_g \) and hence \( sev_g \in N_g \). This yields that \( re(v_g + u'_g) \notin N_g \) and \( se(v_g + u'_g) \notin N_g \). So \( 0 = rese(v_g + u'_g) = 2reseu'_g \) since \( N_g \) is a weakly 2-absorbing submodule and \( rese \notin \langle N_g : R_e \ M_g \rangle \), which is a contradiction. Thus \( reu'_g \in N_g \).

Case 2: \( reu'_g \notin N_g \) and \( seu'_g \notin N_g \). The proof is similar to that of Case 1.

Case 3: \( reu'_g \in N_g \) and \( seu'_g \in N_g \). Since \( rev_g \in N_g \) or \( sev_g \in N_g \), we get \( reu'_g \in N_g \) or \( seu'_g \in N_g \). Thus \( u_g \in (N_g : R_e \ M_g) \cup (N_g : M_g se) \).

**Theorem 3.2.** Let \( R \) be a \( G \)-graded ring, \( M \) a graded \( R \)-module, and \( N \) a graded weakly 2-absorbing submodule of \( M \) and \( g \in G \). If \( reU \subseteq N_g \) and \( 0 \neq 4re \) for some \( re \in R_e \), \( I \) ideal of \( R_e \) and some submodule \( U \) of \( M_g \), then either \( reI \subseteq (N_g : R_e M_g) \) or \( reI \varsubsetneq N_g \) by Lemma 3.1.

**Proof.** By [1, Lemma 3.2], \( N_g \) is a weakly \( 2 \)-absorbing \( R_e \)-submodule of \( M_g \) for every \( g \in G \). Assume that \( reU \subseteq N_g \), \( 0 \neq 4re \), \( reI \varsubsetneq (N_g : R_e M_g) \) and \( reU \varsubsetneq N_g \) for some \( re \in R_e \), \( I \) ideal of \( R_e \) and some submodule \( U \) of \( M_g \). We have to show that \( I \subseteq N_g \). By [3, Lemma 2.15], there exists \( s_e \in I \) such that \( 0 \neq 2rs_eU + 2reseU \) and \( rese \notin \langle N_g : R_e M_g \rangle \). Hence \( 0 \neq 2rs_eU \) and \( reseU \subseteq N_g \). Thus \( s_eU \subseteq N_g \) by Lemma 3.1. Let \( i_e \in I \). Assume that \( 0 \neq 2ri_eU \). Since \( rei_eU \subseteq N_g \) and \( reU \varsubsetneq N_g \), by Lemma 3.1, we have either \( rei_e \in (N_g : R_e M_g) \) or \( i_eU \subseteq N_g \). Thus \( i_e \in ((N_g : R_e M_g) : re) \cup (N_g : R_e U) \). Now, let \( 2rei_eU = 0 \). This yields that \( 0 \neq 2rei_eU = 2re(s_e + i_e)U \) and \( re(s_e + i_e)U \subseteq N_g \). It follows that either \( (se + ie)U \subseteq N_g \) or \( re(s_e + i_e)U \subseteq N_g \). Since \( re \notin (N_g : R_e M_g) \) and \( re(s_e + i_e)U \subseteq N_g \), then \( re(s_e + i_e)U \notin (N_g : R_e M_g) \).

Let \( R_i \) be a graded commutative ring with identity and \( M_i \) be a graded \( R_i \)-module for \( i = 1, 2 \). Let \( R = R_1 \times R_2 \). Then \( M = M_1 \times M_2 \) is a graded \( R \)-module and each graded submodule of \( M \) is of the form \( N = N_1 \times N_2 \) for some graded submodules \( N_1 \) of \( M_1 \) and \( N_2 \) of \( M_2 \).

**Theorem 3.3.** Let \( R = R_1 \times R_2 \) be a \( G \)-graded ring and \( M = M_1 \times M_2 \) be a graded \( R \)-module where \( M_1 \) is a graded \( R_1 \)-module and \( M_2 \) is a graded \( R_2 \)-module. Let \( N_1 \) be a proper graded submodule of \( M_1 \). Then the following statements are equivalent:

(i) \( N_1 \) is a graded 2-absorbing of \( M_1 \);

(ii) \( N_1 \times M_2 \) is a graded 2-absorbing submodule of \( M \);

(iii) \( N_1 \times M_2 \) is a graded weakly 2-absorbing submodule of \( M \).

**Proof.**

(i)⇒(ii) Assume that \( (r_1, r_2)(s_1, s_2)(m_1, m_2) = (r_1s_1m_1, r_2s_2m_2) \in N_1 \times M_2 \), where \( r_1, s_1 \in h(R_1) \), \( r_2, s_2 \in h(R_2) \), \( m_1 \in h(M_1) \), \( m_2 \in h(M_2) \). Then \( r_1s_1m_1 \in N_1 \). Since \( N_1 \) is a graded 2-absorbing of \( M_1 \), we get either \( r_1m_1 \in N_1 \) or \( s_1m_1 \in N_1 \) or \( r_1s_1 \in (N_1 : R_1 M_1) \). If \( r_1m_1 \in N_1 \), then \( (r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2) \in N_1 \times M_2 \). Similarly, if \( s_1m_1 \in N_1 \), then \( (s_1, s_2)(m_1, m_2) = (s_1m_1, s_2m_2) \in N_1 \times M_2 \). Again, if \( r_1s_1 \in (N_1 : R_1 M_1) \), then \( (r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2) \in (N_1 \times M_2 : R M) \). Thus \( N_1 \times M_2 \) is a graded 2-absorbing submodule of \( M \).

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let \( r_1s_1m_1 \in N_1 \) for \( r_1, s_1 \in h(R_1) \) and \( m_1 \in h(M_1) \). Then for each \( 0 \neq m_2 \in h(M_2) \), we have \( (0, 0) \neq (r_1, 1)(s_1, 1)(m_1, m_2) = (r_1s_1m_1, m_2) \in N_1 \times M_2 \). Since \( N_1 \times M_2 \) is a graded weakly 2-absorbing submodule of \( M \), we get either \( (r_1, 1)(m_1, m_2) = (r_1m_1, m_2) \in N_1 \times M_2 \) or \( (s_1, 1)(m_1, m_2) = (s_1m_1, m_2) \in N_1 \times M_2 \) or \( (r_1, 1)(s_1, 1) = (r_1s_1, 1) \in (N_1 \times M_2 : R M) \). It follows that either \( r_1m_1 \in N_1 \) or \( s_1m_1 \in N_1 \) or \( r_1s_1 \in (N_1 : R_1 M_1) \).
**Theorem 3.4.** Let $R = R_1 \times R_2$ be a G-graded ring and $M = M_1 \times M_2$ be a graded $R$-module where $M_1$ is a nonzero graded $R_1$-module and $M_2$ is a nonzero graded $R_2$-module. Let $N_1$ and $N_2$ be proper graded submodules of $M_1$ and $M_2$, respectively.

(i) If $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$, then $N_1$ is a graded weakly prime submodule of $M_1$; moreover, if $0 \neq N_2$, then $N_1$ is a graded classical prime submodule of $M_1$.

(ii) If $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$ and $(N_1 :_{R_1} M_1)M_1 \neq 0$, then $N_2$ is a graded prime submodule of $M_2$.

**Proof.**

(i) Assume that $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$. We show that $N_1$ is a graded weakly prime submodule of $M_1$. Since $N_2 \neq M_2$, there exists $m_2 \in h(M_2) \setminus N_2$. Let $0 \neq rm_1 \in N_1$ for $r \in h(R_1)$ and $m_1 \in h(M_1)$. Then $(0,0) \neq (r,1)(1,0)(m_1,m_2) = (rm_1,0) \in N = N_1 \times N_2$. Since $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$ and $m_2 \notin N_2$, either $(r,1)(1,0) = (r,0) \in (N :_{R} M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ or $(1,0)(m_1,m_2) = (m_1,0) \in N = N_1 \times N_2$. Hence either $m_1 \in N_1$ or $r \in (N_1 :_{R_1} M_1)$ which shows that $N_1$ is a graded weakly prime submodule of $M_1$. Now assume that $0 \neq N_2$ and let $rs_1m_1 \in N_1$ for $r, s \in h(R_1)$ and $m_1 \in h(M_1)$. Let $0 \neq n_2 \in N_2 \cap h(M_2)$. Then $(0,0) \neq (r,1)(s,1)(m_1,n_2) = (rs_1m_1,n_2) \in N = N_1 \times N_2$. Since $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$ and $1 \notin (N_2 :_{R_2} M_2)$, we get either $(r,1)(m_1,n_2) = (rm_1,n_2) \in N = N_1 \times N_2$ or $(s,1)(m_1,n_2) = (sm_1,n_2) \in N = N_1 \times N_2$. Hence, either $rm_1 \in N_1$ or $sm_1 \in N_1$. Therefore $N_1$ is a graded classical prime submodule of $M_1$.

(ii) Assume that $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$ and $(N_1 :_{R_1} M_1)M_1 \neq 0$. Let $rm_2 \in N_2$ for $r \in h(R_2)$ and $m_2 \in h(M_2)$. If $(N_1 :_{R_1} M_1)m_1 = 0$ for each $m_1 \in M_1 \setminus N_1$, then $(M_1 \setminus N_1) \subseteq (0 :_{M_1} (N_1 :_{R_1} M_1))$. Thus $M_1 = N_1 \cup (M_1 \setminus N_1) \subseteq N_1 \cup (0 :_{M_1} (N_1 :_{R_1} M_1))$ and since $M_1 \nsubseteq N_1$, we get $M_1 \subseteq (0 :_{M} (N_1 :_{R_1} M_1))$ by [12, Lemma 2.2]. Hence $(N_1 :_{R_1} M_1)m_1 = 0$, which is a contradiction. Thus there exist $t \in (N_1 :_{R_1} M_1) \cap h(R_1)$ and $m_1 \in h(M_1) \setminus N_1$ with $tm_1 \neq 0$. Then $0 \neq (t,1)(1,r)(m_1,m_2) = (tm_1,rm_2) \in N = N_1 \times N_2$. Since $N = N_1 \times N_2$ is a graded weakly 2-absorbing submodule of $M$ and $m_1 \notin N_1$, we get $(t,1)(1,r) = (t,r) \in (N :_{R} M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$ or $(t,1)(m_1,m_2) = (tm_1,rm_2) \in N$. It follows that either $r \in (N_2 :_{R_2} M_2)$ or $m_2 \in N_2$. Therefore $N_2$ is a graded prime submodule of $M_2$.

**Theorem 3.5.** Let $R = R_1 \times R_2$ be a G-graded ring and $M = M_1 \times M_2$ be a graded $R$-module where $M_1$ is a nonzero graded $R_1$-module and $M_2$ is a nonzero graded $R_2$-module. Let $0 \neq N_1$ be a proper graded submodule of $M_1$ and $(N_1 :_{R_1} M_1)M_1 \neq 0$. Then the graded submodule $N_1 \times 0$ is a graded weakly 2-absorbing submodule of $M_1$ if and only if $N_1$ is a graded weakly prime submodule of $M_1$ and $0$ is a graded prime submodule of $M_2$.

**Proof.**

($\Rightarrow$) By Theorem 3.4.

($\Leftarrow$) Assume that $0,0 \neq (r_1,r_2)(s_1,s_2)(m_1,m_2) = (r_1s_1m_1,r_2s_2m_2) \in N_1 \times 0$, where $r_1, s_1 \in h(R_1), r_2, s_2 \in h(R_2), m_1 \in h(M_1), m_2 \in h(M_2)$. Then $0 \neq r_1s_1m_1 \in N_1$ and $r_2s_2m_2 = 0$. Since $N_1$ is a graded weakly prime submodule of $M_1$, we get either $r_1 \in (N_1 :_{R_1} M_1)$ or $s_1 \in (N_1 :_{R_1} M_1)$ or $m_1 \in N_1$. Since $0$ is a graded prime submodule of $M_2$ and $r_2s_2m_2 = 0$, we get either $r_2 \in (0 :_{R_2} M_2)$ or $s_2 \in (0 :_{R_2} M_2)$ or $m_2 = 0$. It is easy to see that in any of the above cases $(r_1,r_2)(s_1,s_2) \in (N_1 \times 0 :_{R} M)$ or $(r_1,r_2)(m_1,m_2) \in N_1 \times 0$ or $(s_1,s_2)(m_1,m_2) \in N_1 \times 0$. Thus $N_1 \times 0$ is a graded weakly 2-absorbing submodule of $M_1$.

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