# Algorithms for Hammerstein inclusions in certain Banach spaces 

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#### Abstract

Let $E$ be a reflexive smooth and strictly convex real Banach space. Let $F: E \rightarrow 2^{E *}$ and $K: E^{*} \rightarrow E$ be bounded maximal monotone mappings such that $D(F)=E$ and $R(F)=D(K)=E^{*}$. Suppose that the Hammerstein inclusion $0 \in u+K F u$ has a solution in $E$. We present in this paper a new algorithm for approximating solutions of the inclusion $0 \in u+K F u$. Then we prove strong convergence theorems. Our theorems improve and unify most of the results that have been proved in this direction for this important class of nonlinear mappings. Furthermore, our technique of proof is of independent interest.


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## 1. Introduction

Let $X$ and $Y$ be a normed linear spaces. For a multivalued map $A: E \rightarrow 2^{F}$, the domain of $A, D(A)$, the image of a subset $S$ of $E, A(S)$, the range of $A, R(A)$ and the graph of $A, G(A)$ are defined as follows:

$$
\begin{array}{rlrl}
\mathrm{D}(A): & =\{x \in E: A x \neq \emptyset\}, & & A(S):=\cup\{A x: x \in S\} \\
R(A):=A(E), & & G(A):=\{[x, u]: x \in D(A), u \in A x\} .
\end{array}
$$

Let $E$ be a normed linear space with dual E*. A Hammerstein inclusion is any functional inclusion of the form:

$$
\begin{equation*}
0 \in u+K F u, \tag{1.1}
\end{equation*}
$$

where $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ are maps such that $\mathrm{D}(\mathrm{K})=\mathrm{R}(\mathrm{F})=\mathrm{E}^{*}$. This class of inclusions includes nonlinear integral equations of Hammerstein type:

$$
u(x)+\int_{\Omega} k(x, y) f(y, u(y)) d y=0
$$

[^0]where $d y$ is a $\sigma$-finite measure on the measure space $\Omega$; the real kernel k is defined on $\Omega \times \Omega, \mathrm{f}$ is a real-valued function defined on $\Omega \times \mathbb{R}$. The mappings $K$ and $F$ are given by
$$
K v(x)=\int_{\Omega} K(x, y) v(y) d y \text {, a.e. } x \in \Omega, \quad F u(y)=f(y, u(y)) \text { a.e. } y \in \Omega .
$$

There exist various motivations for studying inclusions of type (1.1). For illustration, let us mention two of them.

The study of Hammerstein inclusions is related to nonsmooth calculus of variations (see e.g., the monograph [34]). Suppose that we are interested in minimizing the energy functional

$$
J(u)=\int_{\Omega}(h(u(t))-f(s, u(t))) d s
$$

where $h$ denotes the kinetic energy of the system, and $f$ is a potential energy generating a superposition operator. Assume further that the functional J is not differentiable in the usual sense, but admits a generalized gradient or subgradient in the sense, for instance, of Clarkes generalized gradient, Aubins contingent cone, Ioffes fan, etc. (see e.g. [8, 9]). Consequently, the problem of minimizing the energy functional J leads to the study of boundary value problems for the Euler Lagrange inclusion:

$$
\begin{equation*}
\mathrm{L} u \in \partial \mathrm{~N}_{\mathrm{f}} u, \tag{1.2}
\end{equation*}
$$

where, $L$ is a linear operator on an appropriate function space and $N_{f} u(t)=f(t, u(t))$, and where $\partial N_{f}$ is one of the generalized gradients or subgradients mentioned above. The problem (1.2) in turn is in various function spaces equivalent to the Hammerstein inclusion of type (1.1).

For a second motivation, let $\Omega$ be a smooth bounded open subset of $\mathbb{R}^{N}$ and consider the following boundary value problem :

$$
\begin{equation*}
-\Delta \mathfrak{u}=f(x, u(x)) \text { in } \Omega, u=0 \text { on } \partial \Omega . \tag{1.3}
\end{equation*}
$$

Let K be the operator defined by $\mathrm{Kg}=u$, where $u$ is the unique solution of the corresponding linear boundary value problem

$$
\begin{equation*}
-\Delta \mathfrak{u}=\mathrm{g} \text { in } \Omega, \mathfrak{u}=0 \text { on } \partial \Omega, \tag{1.4}
\end{equation*}
$$

and $F$ the Nemistski operator associated to $f$. Then, there results from (1.3) and (1.4) the operator equation

$$
u=-K F u,
$$

which coincides with the Hammerstein equation (1.1).
Inclusions of the Hammerstein type have been studied by many authors and have been one of the most important domains of application of the ideas and methods of nonlinear functional analysis and in particular of the theory of nonlinear operators of monotone type. We refer to the works of Browder [17], Brezis and Browder[14], Amann [6], Ahmed [1], O'Regan [50] and the references therein. Various applied problems arising in mathematical physics, mechanics and control theory lead to multivalued analogs of the Hammerstein integral equations, the so-called Hammerstein integral inclusions. In this direction we have the works of Lyapin [45], Coffman [35], Glashoff and Sperkels [40], Appell et al. [7], and O'Regan [51].

Let H be a real Hilbert space with inner product $\langle\cdot, \cdot\rangle_{\mathrm{H}}$ and norm $\|\cdot\|_{\mathrm{H}}$. A multivalued operator $A: H \rightarrow 2^{\mathrm{H}}$ with domain $\mathrm{D}(\mathrm{A})$ is called monotone if the following inequality holds:

$$
\langle x-y, u-v\rangle_{H} \geqslant 0, \forall x, y \in D(A), u \in A x, v \in A y,
$$

and it is called strongly monotone if there exists $k \in(0,1)$ such that

$$
\langle x-y, u-v\rangle_{H} \geqslant k\|x-y\|_{H}^{2}, \quad \forall x, y \in D(A), u \in A x, v \in A y
$$

Such operators have been studied extensively (see, e.g., Bruck Jr [20], Chidume [21], Martinet [47], Reich [54], Rockafellar [55]) because of their role in convex analysis, in certain partial differential equations, in nonlinear analysis and in optimization theory.

The extension of the monotonicity definition to operators from a Banach space into its dual has been the starting point for the development of nonlinear functional analysis. The monotone maps constitute the most manageable class because of the very simple structure of the monotonicity condition. They appear in a rather wide variety of contexts since they can be found in many functional equations. Many of them appear also in calculus of variations as subdifferential of convex functions (see, e.g., Pascali and Sburian [52], p. 101, and Rockafellar [55]).

The first extension involves mapping $A$ from $E$ to $2^{\mathrm{E}^{*}}$. Here and in the sequel, $\langle\cdot, \cdot\rangle$ stands for the duality pairing between (a possible normed linear space) E and its dual $\mathrm{E}^{*}$. A mapping $\mathrm{A}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ with domain $D(A)$ is called monotone if the following inequality holds:

$$
\langle x-y, u-v\rangle \geqslant 0, \forall x, y \in D(A), u \in A x, v \in A y,
$$

and $A$ is called strongly monotone if there exists $k \in(0,1)$ such that

$$
\langle x-y, u-v\rangle \geqslant k\|x-y\|^{2}, \forall x, y \in D(A), u \in A x, v \in A y .
$$

The second extension of the notion of monotonicity involves mapping $A$ from $E$ to $2^{E^{*}}$. Let $E$ be a real normed space, $\mathrm{E}^{*}$ its dual space. The map J:E $\rightarrow 2^{\mathrm{E}^{*}}$ defined by:

$$
\mathrm{Jx}:=\left\{x^{*} \in \mathrm{E}^{*}:\left\langle x, x^{*}\right\rangle=\|x\| \cdot\left\|x^{*}\right\|,\left\|x^{*}\right\|=\|x\|\right\}
$$

is called the normalized duality map on E . A mapping $\mathrm{A}: \mathrm{E} \rightarrow 2^{\mathrm{E}}$ with domain $\mathrm{D}(\mathrm{A})$ is called accretive if for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$
\langle u-v, \mathfrak{j}(x-y)\rangle \geqslant 0, \forall u \in A x, v \in A y .
$$

Finally, $A$ is called strongly accretive if there exists $k \in(0,1)$ such that for each $x, y \in D(A)$, there exists $\mathfrak{j}(x-y) \in J(x-y)$ such that

$$
\langle u-v, j(x-y)\rangle \geqslant k\|x-y\|^{2}, \forall u \in A x, v \in A y .
$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, monotonicity and accretivity coincide.

Several existence and uniqueness results have been established for equations of Hammerstein type (see, e.g., Coffman [35], Appell et al. [7] and O'Regan [51]). In general, these equations are nonlinear and there is no known method to find closed form solutions for them. Consequently, methods of approximating solutions of such equations are of interest.

In the special case in which the operator $F$ is angle bounded (defined below) and weakly compact, Brézis and Browder $[13,15$ ] proved the strong convergence of a suitably defined Galerkin approximation to a solution of (1.1). Before we state their results, we need the following definitions.

Let H be a real Hilbert space. A nonlinear operator $\mathrm{A}: \mathrm{H} \rightarrow \mathrm{H}$ is said to be angle-bounded with angle $\beta>0$, if

$$
\begin{equation*}
\langle A x-A z, z-y\rangle \leqslant \beta\langle A x-A y, x-y\rangle \tag{1.5}
\end{equation*}
$$

for any triple of elements $x, y, z \in H$. For $y=z$, inequality (1.5) implies the monotonicity of $A$.
A monotone linear operator $A: H \rightarrow H$ is said to be angle-bounded with angle $\alpha>0$, if

$$
|\langle A x, y\rangle-\langle A y, x\rangle| \leqslant 2 \alpha\langle A x, x\rangle^{\frac{1}{2}}\langle A y, y\rangle^{\frac{1}{2}}
$$

for all $x, y \in H$. It is known (see, e.g., Pascali and Sburlan, [52], Ch. IV, p.189) that for linear operators, the two definitions of angle boundedness are equivalent.

We now state the theorem of Brézis and Browder referred to above.

Theorem 1.1 ([15]). Let H be a separable real Hilbert space and C be a closed subspace of H . Let $\mathrm{K}: \mathrm{H} \rightarrow \mathrm{C}$ be a bounded continuous monotone operator and $\mathrm{F}: \mathrm{C} \rightarrow \mathrm{H}$ be an angle-bounded and weakly compact mapping. For a given $\mathrm{f} \in \mathrm{C}$, consider the Hammerstein equation

$$
\begin{equation*}
(\mathrm{I}+\mathrm{KF}) \mathrm{u}=\mathrm{f} \tag{1.6}
\end{equation*}
$$

and its n-th Galerkin approximation given by

$$
\begin{equation*}
\left(I+K_{n} F_{n}\right) u_{n}=P^{*} f, \tag{1.7}
\end{equation*}
$$

where $\mathrm{K}_{\mathrm{n}}=\mathrm{P}_{\mathrm{n}}^{*} K \mathrm{P}_{\mathrm{n}}: \mathrm{H} \rightarrow \mathrm{C}_{\mathrm{n}}$ and $\mathrm{F}_{\mathrm{n}}=\mathrm{P}_{\mathrm{n}} \mathrm{FP}_{\mathrm{n}}^{*}: \mathrm{C}_{\mathrm{n}} \rightarrow \mathrm{H}$, where the symbols have their usual meanings (see, [52]). Then, for each $n \in \mathbb{N}$, the Galerkin approximation (1.7) admits a unique solution $\mathfrak{u}_{n}$ in $\mathrm{C}_{\mathrm{n}}$ and $\left\{\mathrm{u}_{\mathrm{n}}\right\}$ converges strongly in H to the unique solution $\mathrm{u} \in \mathrm{C}$ of the equation (1.6).

It is obvious that if an iterative algorithm can be developed for the approximation of solutions of equations of Hammerstein type (1.1), this will certainly be preferred.

We first note that for the iterative approximation of zeros of accretive type operators, the monotonicity/accretivity of operators is crucial. The Mann type iteration scheme (see, e.g., Mann [46]) has successfully been employed (see, e.g., the recent monographs of Berinde [12] and Chidume [21] for results obtained within the past 40 years, or so). One drawback of the Mann iterative scheme, however, is that in general, it only yields weak convergence (see, e.g., Matouskova and Reich [11]). All attempts to use the Mann type iteration scheme directly to approximate solutions of equations of Hammertsein type (1.1) did not yield satisfactory results (see Chidume and Osilike [29]). The recurrence formulas used in early attempts involved $\mathrm{K}^{-1}$ which is also required to be strongly monotone, and this, apart from limiting the class of mappings to which such iterative schemes are applicable, is also not convenient in application. Part of the difficulty is the fact that the composition of two monotone operators need not be monotone.

The first satisfactory results on iterative methods for approximating solutions of Hammerstein equations, as far as we know, were obtained by Chidume and Zegeye [31-33] under the setting of a real Hilbert space H . The method of proof used by Chidume and Zegeye provided the clue to the establishment of the following coupled explicit algorithm for computing a solution of the equation $u+K F u=0$ in the original space $X$. With initial vectors $u_{0}, v_{0} \in X$, sequences $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ in $X$ are defined iteratively as follows:

$$
\begin{align*}
& u_{n+1}=u_{n}-\alpha_{n}\left(F u_{n}-v_{n}\right), n \geqslant 0,  \tag{*}\\
& v_{n+1}=v_{n}-\alpha_{n}\left(K v_{n}+u_{n}\right), n \geqslant 0, \tag{**}
\end{align*}
$$

where $\left\{\alpha_{n}\right\}$ is a sequence in $(0,1)$ satisfying appropriate conditions. The recursion formulas $\left(^{*}\right)$ and $\left({ }^{* *}\right)$ have been used successfully to approximate solutions of Hammerstein equations involving nonlinear accretive-type operators. Following this, Chidume and Djitte studied this explicit coupled iterative algorithms and proved several strong convergence theorems (see, Chidume and Djitte [23, 24] ). For recent results using these recursion formulas or their modifications, the reader may consult any of the following references Chidume and Djitte [25-27], Djitte and Sene [37, 38], Chidume and Ofeodu [28], Chidume and Shehu [30] and also Chapter 13 of [21].

For Hammerstein equations involving monotone mappings from $E$ to $E^{*}$, very little has been achieved. Interestingly enough, almost all the existence theorems proved for Hammerstein equations involve monotone mappings (see, e.g., Brézis and Browder [13-15], Browder [16], Browder et al. [18], and Browder and Gupta [19]). We note that it has been remarked that in dealing with the Nemistkyi operator, which is intimately connected with the Hammertsein integral equation, its properties are distinguished, in applications, according to two important cases: $\mathrm{L}_{\mathrm{p}}(\Omega)$ spaces, $1<p<\infty$, and $\mathrm{L}_{1}(\Omega)$, (see Pascali and Sburlan [52], Chapter IV, pp. 165, 172). Thus, developing iterative methods for approximating solutions of nonlinear Hammerstein integral equations in these cases is of paramount importance.

Motivated by approximating solutions of integral equations of Hammerstein type, in [48], Ofoedu and Onyi proposed an iterative scheme and they obtained strong convergence results in the setting of Hilbert spaces. In fact, they proved the following theorem.

Theorem 1.2 ([48]). Let H be a real Hilbert space. Let $\mathrm{F}, \mathrm{K}: \mathrm{H} \rightarrow \mathrm{H}$ be Lipschitz monotone mappings. Let the sequence $\left\{\left(\mathfrak{u}_{n}, v_{n}\right)\right\}_{n} \geqslant 1$ in $\mathrm{H} \times \mathrm{H}$ be generated iteratively by $\left(\mathfrak{u}_{1}, v_{1}\right) \in \mathrm{H} \times \mathrm{H}$,

$$
\left\{\begin{array}{l}
u_{n+1}=\left(1-\sigma_{n}\right) u_{n}+\sigma_{n}\left(u_{n}-F u_{n}+v_{n}\right)-\sigma_{n} \xi_{n} \alpha_{n} u_{n}, \\
v_{n+1}=\left(1-\sigma_{n}\right) v_{n}+\sigma_{n}\left(v_{n}-K v_{n}-u_{n}\right)-\sigma_{n} \xi_{n} \alpha_{n} v_{n},
\end{array}\right.
$$

where $\left\{\sigma_{n}\right\}_{n \geqslant 1},\left\{\xi_{n}\right\}_{n \geqslant 1}$ and $\left\{\alpha_{n}\right\}_{n \geqslant 1}$ are decreasing sequences in $(0,1)$ such that
(i) $\lim _{n \rightarrow \infty} \xi_{n}=0$;
(ii) $\lim _{n \rightarrow \infty} \alpha_{n}=0$;
(iii) $\lim _{n \rightarrow \infty} \frac{\sigma_{n}}{\alpha_{n} \xi_{n}}=0$;
(iv) $\lim _{n \rightarrow \infty} \frac{\alpha_{n-1}-\alpha_{n}}{\sigma_{n} \varepsilon_{n} \alpha_{n}^{2}}=0$;
(v) $\lim _{n \rightarrow \infty} \frac{\xi_{n-1}-\xi_{n}}{\sigma_{n} \varepsilon_{n}^{2} \alpha_{n}^{2}}=0$.

Then the sequence $\left\{\left(u_{n}, v_{n}\right)\right\}_{n \geqslant 1}$ is bounded. Moreover, if the Hammerstein equation $u+K F u=0$ has some solutions in H , then $\left\{\mathrm{u}_{n}\right\}_{n} \geqslant 1$ converges strongly to a solution $\mathfrak{u}^{*}$ of $\mathfrak{u}+\mathrm{KFu}=0$.

Recently, Chidume and Bello [22] constructed a new iterative algorithm for approximating solutions of Hammerstein equations in $L_{p}$-spaces, and where the operators $K$ and $F$ are assumed to be bounded and strongly monotone. They obtained the following theorem.

Theorem 1.3. Let $\mathrm{E}=\mathrm{L}_{\mathrm{p}}, 1<\mathrm{p} \leqslant 2$ with dual $\mathrm{E}^{*}$ and $\mathrm{F}: \mathrm{E} \rightarrow \mathrm{E}^{*}, \mathrm{~K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be strongly monotone and bounded mappings with $\mathrm{D}(\mathrm{K})=\mathrm{R}(\mathrm{F})=\mathrm{E}^{*}$. For given $\mathrm{u}_{1} \in \mathrm{E}$ and $v_{1} \in \mathrm{E}^{*}$, let $\left\{\mathrm{u}_{\mathrm{n}}\right\}$ and $\left\{v_{n}\right\}$ be generated iteratively by:

$$
u_{n+1}=J^{-1}\left(J u_{n}-\lambda\left(F u_{n}-v_{n}\right)\right), \quad n \geqslant 1, \quad v_{n+1}=J\left(J^{-1} v_{n}-\lambda\left(k v_{n}+u_{n}\right)\right), \quad n \geqslant 1,
$$

where J is the normalized duality mapping from E into $\mathrm{E}^{*}$ and $\left\{\alpha_{\mathrm{n}}\right\} \subset(0,1)$ satisfies the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_{n}=\infty$;
(ii) $\sum_{n=0}^{\infty} \alpha_{n}^{2}<\infty$;
(iii) $\sum_{n=0}^{\infty} \alpha_{n}^{\frac{q}{q-1}}<\infty$, where $q$ is the conjugate of p .

Suppose that the equation $u+\mathrm{KFu}=0$ has a unique solution $\mathfrak{u}^{*}$. Then, there exists $\gamma_{0}>0$ such that if $\alpha_{n}<\gamma_{0}$ for all $n \geqslant 1$, the sequence $\left\{u_{n}\right\}$ converges strongly to $u^{*}$, the sequence $\left\{v_{n}\right\}$ converges strongly to $v^{*}$, with $v^{*}=F u^{*}$.

Using the same scheme, still in [22], they also proved a similar result in $L_{p}$, for $2 \leqslant p<\infty$.
Let $E$ be a normed linear space. A monotone multivalued mapping $A: E \rightarrow 2^{E^{*}}$, with domain $D(A)$ is said to be maximal if its graph $G(A)=\left\{(x, y) \in E \times E^{*}: x \in D(A), y \in A x\right\}$ is not properly contained in the graph of any other monotone mapping. It is known that if $A$ is maximal monotone, then the zero of $A, A^{-1}(0):=\{x \in E: 0 \in A x\}$, is closed and convex.

Motivated by the discussion above, it is our purpose in this paper to construct a new algorithm for approximation solutions of inclusions of Hammerstein type, $0 \in \mathfrak{u}+\mathrm{KFu}$. The operators $F$ and $K$, defined in 2-uniformly convex and $q$-uniformly smooth ( $q>1$ ) or s-uniformly convex ( $s>1$ ) and 2-uniformly smooth real Banach spaces are assumed to be bounded and maximal monotone. Our results extend and unify most of the results that have been proved in this direction for this important class of nonlinear mappings.

## 2. Preliminaries

Let E be a normed linear space. Then, E is said to be strictly convex if the following holds. For all $x, y \in E$ such that $\|x\|=\|y\|=1$ and $x \neq y,\left\|\frac{x+y}{2}\right\|<1$ holds. The modulus of convexity of $E$ is the function $\delta_{E}:(0,2] \rightarrow[0,1]$ defined by

$$
\delta_{E}(\epsilon):=\inf \left\{1-\frac{1}{2}\|x+y\|:\|x\|=\|y\|=1,\|x-y\| \geqslant \epsilon\right\} .
$$

$E$ is uniformly convex if and only if $\delta_{E}(\epsilon)>0$ for every $\epsilon \in(0,2]$. For a real number $p>1, E$ is said to be $p$-uniformly convex if there exists a constant $c>0$ such that $\delta_{E}(\epsilon) \geqslant c \epsilon^{p}$ for all $\epsilon \in(0,2]$.

Let $E$ be a real normed space and let $S:=\{x \in E:\|x\|=1\}$. E is said to be smooth if the limit

$$
\begin{equation*}
\lim _{t \rightarrow 0^{+}} \frac{\|x+t y\|-\|x\|}{t} \tag{2.1}
\end{equation*}
$$

exists for each $x, y \in S$. E is said to be Fréchet differentiable if it is smooth and the limit in (2.1) is attained uniformly for $y \in S_{E}$. Finally $E$ is uniformly smooth if it is smooth and the limit in (2.1) is attained uniformly for each $x, y \in S_{E}$. If $E$ is a normed linear space of dimension $\geqslant 2$, then, the modulus of smoothness of $\mathrm{E}, \rho_{\mathrm{E}}$, is defined by

$$
\rho_{\mathrm{E}}(\tau):=\sup \left\{\frac{\|x+y\|+\|x-y\|}{2}-1:\|x\|=1,\|y\|=\tau\right\} ; \quad \tau>0 .
$$

A normed linear space $E$ is called uniformly smooth if

$$
\lim _{\tau \rightarrow 0} \frac{\rho_{\mathrm{E}}(\tau)}{\tau}=0 .
$$

If there exists a constant $c>0$ and a real number $q>1$ such that $\rho_{E}(\tau) \leqslant c \tau^{q}$, then $E$ is said to be q-uniformly smooth.

Typical examples of such spaces are the $L_{p}, \ell_{p}$ and $W_{p}^{m}$ spaces for $1<p<\infty$ where

$$
L_{p}\left(\text { or } l _ { p } \text { ) or } W _ { p } ^ { m } \text { is } \left\{\begin{array}{l}
\text { 2-uniformly smooth and } p \text {-uniformly convex if } 2 \leqslant p<\infty, \\
\text { 2-uniformly convex and } p \text {-uniformly smooth if } 1<p<2 .
\end{array}\right.\right.
$$

Let $\mathrm{J}_{\mathrm{q}}$ denote the generalized duality mapping from E to $2^{\mathrm{E}^{*}}$ defined by

$$
\mathrm{J}_{\mathrm{q}}(x):=\left\{\mathrm{f} \in \mathrm{E}^{*}:\langle x, f\rangle=\|x\|^{\mathbf{q}} \text { and }\|f\|=\|x\|^{\boldsymbol{q}-1}\right\} .
$$

$\mathrm{J}_{2}$ is called the normalized duality mapping and is denoted simply by J .
It is well-known that $E$ is smooth if and only if $J$ is single-valued. Moreover, if $E$ is a reflexive, smooth and strictly convex real Banach space, then $\mathrm{J}^{-1}$ is single-valued, one-to-one, surjective and it is the duality mapping from $E^{*}$ into $E$.

Let $E$ be a normed linear space. A monotone mapping $A: E \rightarrow 2^{E^{*}}$, with domain $D(A)$ is said to be maximal if its graph $G(A)=\left\{(x, y) \in E \times E^{*}: x \in D(A), y \in A x\right\}$ is not properly contained in the graph of any other monotone mapping. It is known that if $A$ is maximal monotone, then the zero of $A$, $A^{-1}(0):=\{x \in E: 0 \in A x\}$, is closed and convex.
Remark 2.1. The maximality of $A$ is equivalent to: if $(x, u) \in E \times E^{*}$ is such that $\langle u-v, x-y\rangle \geqslant 0$ for every $y \in D(A), v \in A y$, then $x \in D(A)$ and $u=A x$.

Let E be a reflexive, smooth and strictly convex real Banach space, and let $A: E \rightarrow E^{*}$ be a monotone operator. Then $A$ is maximal if and only if $R(J+r A)=E^{*}$ for all $r>0$ (see, e.g., Barbu [10]). If $A: E \rightarrow E^{*}$ is a maximal monotone operator, then for each $r>0$ and $x \in E$, there exists a unique element $x_{r} \in D(A)$ satisfying $J(x) \in J\left(x_{r}\right)+r A x_{r}$. We define the resolvent of $A$ by $J_{r}^{A} x=x_{r}$. In other words,

$$
\mathrm{J}_{\mathrm{r}}^{\mathrm{A}}=(\mathrm{J}+\mathrm{r} \mathrm{~A})^{-1} \mathrm{~J}, \forall \mathrm{r}>0 .
$$

Definition 2.2. An operator $A: E \longrightarrow 2^{E^{*}}$ is called demiclosed if the conditions $x_{n} \rightarrow x, y_{n} \rightarrow y$ or $x_{n} \rightharpoonup x, y_{n} \rightarrow y$, where $y_{n} \in A x_{n}$, imply that $y \in A x$.
Lemma 2.3 ([5]). Any maximal monotone operator $\mathrm{A}: \mathrm{E} \longrightarrow 2^{\mathrm{E}^{*}}$ is demiclosed.
Lemma 2.4 ([5]). Let $A: E \longrightarrow 2^{\mathrm{E}^{*}}$ be a monotone demiclosed operator such that $\mathrm{D}(\mathrm{A})=\mathrm{E}$ and for each $\mathrm{x} \in \mathrm{E}$ $A x$ is a nonempty convex subset of $\mathrm{E}^{*}$. Then A is maximal monotone.

In the sequel, we shall need the following results and definitions.
Lemma 2.5 ([5]). Let E be a uniformly smooth and strictly convex Banach space. Then there exists $\mathrm{L}>0$ such that for any $x, y \in E$ such that $\|x\| \leqslant R$ and $\|y\| \leqslant R$ the following inequality holds

$$
\langle J x-J y, x-y\rangle \geqslant L \delta_{\mathrm{E}^{*}}\left(\mathrm{c}_{2}^{-1}\|J x-J y\|\right)
$$

where $\mathrm{c}_{2}=2 \max \{1, \mathrm{R}\}$. Moreover the constant L is called Figiel constant and is such that $1<\mathrm{L}<1.7$.
Lemma 2.6 ([5]). Let $\mathrm{p} \geqslant 2, \mathrm{q}>1$, and let E be a p -uniformly convex and q -uniformly smooth real Banach space. Then, the duality mapping $\mathrm{J}: \mathrm{E} \rightarrow \mathrm{E}^{*}$ is Lipschitz on the bounded sets; that is, for all $\mathrm{R}>0$, there exists a positive constant $\mathrm{m}_{1}$ such that

$$
\|J x-J y\| \leqslant m_{1}\|x-y\|
$$

for all $x, y \in E$, with $\|x\| \leqslant R,\|y\| \leqslant R$.
Lemma 2.7. Let $\mathrm{p} \geqslant 2$ and E be a 2 -uniformly smooth and p -uniformly convex real Banach space. Then $\mathrm{J}^{-1}$ is Lipschitz on the bounded sets; that is, for all $\mathrm{R}>0$, there exists a positive constant $\mathrm{m}_{2}$ such that

$$
\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\| \leqslant m_{2}\left\|x^{*}-y^{*}\right\|
$$

for all $x^{*}, y^{*} \in E^{*}$, with $\left\|x^{*}\right\| \leqslant R,\left\|y^{*}\right\| \leqslant R$.
Proof. Since E is 2-uniformly smooth and p-uniformly convex, then E* is 2-uniformly convex and quniformly smooth where $\frac{1}{p}+\frac{1}{q}=1$. Therefore, the proof follows from Lemma 2.6 and the fact that $\mathrm{J}^{-1}=\mathrm{J}_{*}$ where $\mathrm{J}_{*}$ is the normalized duality mapping of $\mathrm{E}^{*}$.

We deduce the following useful result.
Lemma 2.8. For $\mathrm{q} \geqslant 1$, let E be a 2 -uniformly convex and q -uniformly smooth real Banach space. Then for every $R>0$ there exists a constant $\mathrm{d}_{1}>0$ such that for any $x^{*}, \mathrm{y}^{*} \in \mathrm{E}^{*}$ such that $\left\|x^{*}\right\| \leqslant R$ and $\left\|y^{*}\right\| \leqslant R$ the following inequality holds

$$
\left\langle J^{-1} x^{*}-J^{-1} y^{*}, x^{*}-y^{*}\right\rangle \geqslant d_{1}\left\|x^{*}-y^{*}\right\|^{2}
$$

Proof. Since E is 2-uniformly convex and q-uniformly smooth, then $\mathrm{E}^{*}$ is 2-uniformly smooth and puniformly convex with $\frac{1}{p}+\frac{1}{q}=1$. Moreover $E$ is reflexive. Hence from the fact that $J^{-1}=J_{*}$, using successively Lemma 2.5 , Lemma 2.6 and the 2 -uniform convexity of $E$ we have, for any $x^{*}, y^{*} \in E^{*}$ such that $\left\|x^{*}\right\| \leqslant R$ and $\left\|y^{*}\right\| \leqslant R$, the following holds.

$$
\left\langle J^{-1} x^{*}-J^{-1} y^{*}, x^{*}-y^{*}\right\rangle \geqslant \operatorname{L} \delta_{E}\left(c_{2}^{-1}\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\|\right) \geqslant L c_{2}^{-2}\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\|^{2} \geqslant L\left(m_{1} c_{2}\right)^{-2}\left\|x^{*}-y^{*}\right\|^{2}
$$

Let $d_{1}=L\left(m_{1} c_{2}\right)^{-2}$, we obtain

$$
\left\langle J^{-1} x^{*}-J^{-1} y^{*}, x^{*}-y^{*}\right\rangle \geqslant d_{1}\left\|x^{*}-y^{*}\right\|^{2}
$$

As a corollary of Lemma 2.8 we have the next lemma.

Lemma 2.9. Let E be a 2-uniformly convex and q -uniformly smooth real Banach space. Then $\mathrm{J}^{-1}$ is Lipschitz on the bounded sets; that is, for all $\mathrm{R}>0$, there exists a positive constant $\mathrm{m}_{3}$ such that

$$
\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\| \leqslant m_{3}\left\|x^{*}-y^{*}\right\|
$$

for all $x^{*}, y^{*} \in \mathrm{E}^{*}$, with $\left\|x^{*}\right\| \leqslant R,\left\|y^{*}\right\| \leqslant R$.
Proof. As in the proof of Lemma 2.8, for any $x^{*}, y^{*} \in E^{*}$ such that $\left\|x^{*}\right\| \leqslant R$ and $\left\|y^{*}\right\| \leqslant R$ we have

$$
\left\langle J^{-1} x^{*}-J^{-1} y^{*}, x^{*}-y^{*}\right\rangle \geqslant \operatorname{L\delta } \delta_{\mathrm{E}}\left(c_{2}^{-1}\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\|\right) \geqslant L c_{2}^{-2}\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\|^{2}
$$

Using Schwartz inequality and simplifying we obtain:

$$
\left\|J^{-1} x^{*}-J^{-1} y^{*}\right\| \leqslant m_{3}\left\|x^{*}-y^{*}\right\|,
$$

where $\mathrm{m}_{3}=\mathrm{L}^{-1} \mathrm{c}_{2}^{2}$
Lemma 2.10 ([56]). Let $\mathrm{p}>1$ be a real number and E be a Banach space. Then the following assertions are equivalent.
(i) E is p-uniformly convex.
(ii) There exists a constant $d_{2}>0$ such that for all $x, y \in E$ and $f_{x} \in J_{p}(x), f_{y} \in J_{p}(y)$, one has:

$$
\left\langle x-y, f_{x}-f_{y}\right\rangle \geqslant d_{2}\|x-y\|^{p} .
$$

Let $E$ be a smooth real Banach space with dual space $E^{*}$. The function $\phi: E \times E \rightarrow \mathbb{R}$, defined by

$$
\begin{equation*}
\phi(x, y)=\|x\|^{2}-2\langle x, J y\rangle+\|y\|^{2}, x, y \in E \tag{2.2}
\end{equation*}
$$

where $J$ is the normalized duality mapping from $E$ into $E^{*}$, introduced by Alber has been studied by Alber [3], Alber and Guerre-Delabriere [4], Kamimura and Takahashi[42], Reich[53] and a host of other authors. This functional $\phi$ will play a central role in what follows. If $E=H$, a real Hilbert space, then equation (2.2) reduces to $\phi(x, y)=\|x-y\|^{2}$ for $x, y \in H$. It is obvious from the definition of the function $\phi$ that

$$
\begin{equation*}
(\|x\|-\|y\|)^{2} \leqslant \phi(x, y) \leqslant(\|x\|+\|y\|)^{2}, \forall x, y \in E \tag{2.3}
\end{equation*}
$$

Let $V: E \times E^{*} \rightarrow \mathbb{R}$ be the functional defined by:

$$
\mathrm{V}\left(x, x^{*}\right)=\|x\|^{2}-2\left\langle x, x^{*}\right\rangle+\left\|x^{*}\right\|^{2}, \forall x \in \mathrm{E}, x^{*} \in \mathrm{E}^{*} .
$$

Then, it is easy to see that

$$
\mathrm{V}\left(x, x^{*}\right)=\phi\left(x, J^{-1} x^{*}\right), \forall x \in E, x^{*} \in \mathrm{E}^{*} .
$$

Lemma 2.11 ([3]). Let X be a reflexive strictly convex and smooth real Banach space with $\mathrm{X}^{*}$ as its dual. Then,

$$
V\left(x, x^{*}\right)+2\left\langle J^{-1} x^{*}-x, y^{*}\right\rangle \leqslant V\left(x, x^{*}+y^{*}\right)
$$

for all $x \in X$ and $x^{*}, y^{*} \in X^{*}$.
Lemma 2.12 ([2]). Let E be a smooth real Banach space. Then

$$
V(x, y)=V(x, z)+V(z, y)+2\langle x-z, J z-J y\rangle, \forall x, y, z \in E
$$

From the definition of $\phi$ and inequality (2.3), we can observe that for all $x, y \in E, \phi(y, x) \geqslant 0$ and

$$
2\langle x-y, J x-J y\rangle-\phi(x, y)=\phi(y, x) .
$$

This leads to the following.

Lemma 2.13. Let E be a smooth real Banach space. Then, for all $\mathrm{x}, \mathrm{y} \in \mathrm{E}$, the following holds

$$
\phi(x, y) \leqslant 2\langle J y-J x, y-x\rangle
$$

Lemma 2.14 ([42]). Let $X$ be a smooth and uniformly convex real Banach space, and let $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ be two sequences of $X$. If either $\left\{x_{n}\right\}$ or $\left\{y_{n}\right\}$ is bounded and $\phi\left(x_{n}, y_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$, then $\left\|x_{n}-y_{n}\right\| \rightarrow 0$ as $n \rightarrow \infty$.

Similarly, if E is a reflexive smooth and strictly convex real Banach space, we introduce the functional $\phi_{*}: \mathrm{E}^{*} \times \mathrm{E}^{*} \rightarrow \mathbb{R}$, defined by:

$$
\phi_{*}\left(x^{*}, y^{*}\right)=\left\|x^{*}\right\|^{2}-2\left\langle J^{-1} y^{*}, x^{*}\right\rangle+\left\|y^{*}\right\|^{2}, x^{*}, y^{*} \in \mathrm{E}^{*},
$$

and the functional $V_{*}: E^{*} \times E \rightarrow \mathbb{R}$ defined from $E^{*} \times E$ to $\mathbb{R}$ by:

$$
V_{*}\left(x^{*}, x\right)=\left\|x^{*}\right\|^{2}-2\left\langle x, x^{*}\right\rangle+\|x\|^{2}, x \in E, x^{*} \in E^{*} .
$$

It is easy to see that

$$
V_{*}\left(x^{*}, x\right)=\phi_{*}\left(x^{*}, J x\right), \quad \forall x \in E, x^{*} \in E^{*} .
$$

In what follows, the product space $E \times E^{*}$ is equipped with the following norm:

$$
\left\|w_{1}-w_{2}\right\|=\left(\|x-y\|^{2}+\left\|x^{*}-y^{*}\right\|^{2}\right)^{\frac{1}{2}}, \forall w_{1}=\left(x, x^{*}\right) \in E \times E^{*}, w_{2}=\left(y, y^{*}\right) \in E \times E^{*}
$$

Finally, we introduce the functional $\psi:\left(E \times E^{*}\right) \times\left(E \times E^{*}\right) \rightarrow \mathbb{R}$ defined by:

$$
\psi\left(w_{1}, w_{2}\right):=\phi(x, y)+\phi_{*}\left(x^{*}, y^{*}\right), \quad \forall w_{1}=\left(x, x^{*}\right) \in \mathrm{E} \times \mathrm{E}^{*}, w_{2}=\left(y, y^{*}\right) \in \mathrm{E} \times \mathrm{E}^{*} .
$$

Lemma 2.15 ([57]). Let $\left\{\rho_{n}\right\}$ be a sequence of non-negative real numbers satisfying the following inequality

$$
\rho_{n+1} \leqslant\left(1-\alpha_{n}\right) \rho_{n}+\alpha_{n} \sigma_{n}+\gamma_{n},
$$

where $\left\{\alpha_{n}\right\},\left\{\sigma_{n}\right\}$, and $\left\{\gamma_{n}\right\}$ are real sequences satisfying:
(i) $\left.\left\{\alpha_{n}\right\} \subset\right] 0,1\left[, \sum \alpha_{n}=\infty\right.$;
(ii) $\limsup \sigma_{n} \leqslant 0$;
(iii) $\gamma_{n} \geqslant 0, \sum \gamma_{n}<\infty$.

Then, the sequence ( $\rho_{\mathrm{n}}$ ) converges to zero as $\mathrm{n} \rightarrow \infty$.

## 3. Main results

We start by a presentation of our iterative algorithm. Let E be a smooth, strictly convex and reflexive Banach space with norm $\|\cdot\|$ and dual space $E^{*}$. For $F: E \rightarrow 2^{E^{*}}$ and $K: E^{*} \rightarrow E$ mappings, let the sequences $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ be generated iteratively from $\left(u_{1}, v_{1}\right) \in E \times E^{*}$ by:

$$
\begin{gather*}
u_{n+1}=J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right), p_{n} \in F u_{n},  \tag{3.1}\\
v_{n+1}=J\left(J^{-1} v_{n}-\alpha_{n}\left(K v_{n}+u_{n}\right)-\lambda_{n} \theta_{n}\left(J^{-1} v_{n}-J^{-1} v_{1}\right)\right), n \geqslant 1,
\end{gather*}
$$

where $J$ is the normalized duality mapping from $E$ onto $E^{*}$ and $\left\{\lambda_{n}\right\},\left\{\theta_{n}\right\}$ are real sequences in $(0,1)$ satisfying, here and elsewhere, the following conditions:
(i) $\lim _{n \rightarrow \infty} \theta_{n}=0$;
(ii) $\sum_{n=1}^{\infty} \lambda_{n} \theta_{n}=\infty, \quad \lambda_{n}=o\left(\theta_{n}\right)$;
(iii) $\limsup _{n \rightarrow \infty} \frac{\left(\frac{\theta_{n-1}}{\theta_{n}}-1\right)}{\lambda_{n} \theta_{n}} \leqslant 0, \sum_{n=1}^{\infty} \lambda_{n}^{2}<\infty$.

Remark 3.1. Real sequences that satisfy conditions (i)-(iii) are $\lambda_{n}=(n+1)^{-a}$ and $\theta_{n}=(n+1)^{-b}, n \geqslant 1$ with $0<b<a, \frac{1}{2}<a<1$ and $a+b<1$.

In fact, (i), (ii), and the second part of (iii) are easy to check. For the first part of condition (iii), using the fact that $(1+x)^{s} \leqslant 1+s x$, for $x>-1$ and $0<s<1$, we have

$$
\begin{aligned}
0 \leqslant \frac{\left(\frac{\theta_{n-1}}{\theta_{n}}-1\right)}{\lambda_{n} \theta_{n}} & =\left[\left(1+\frac{1}{n}\right)^{b}-1\right] \cdot(n+1)^{a+b} \\
& \leqslant b \cdot \frac{(n+1)^{a+b}}{n}=b \cdot \frac{n+1}{n} \cdot \frac{1}{(n+1)^{1-(a+b)}} \rightarrow 0 \text { as } n \rightarrow \infty
\end{aligned}
$$

Remark 3.2. Note also that a duality mapping exists in each Banach space. From [2], this mapping is known precisely in $\ell_{p}, L_{p}, W^{m, p}$-spaces, $1<p<\infty$ and is given by:
(i) $\ell_{p}: J x=\|x\|_{\ell_{p}}^{2-p} y \in l_{q}, x=\left(x_{1}, x_{2}, \ldots, x_{n}, \ldots\right), y=\left(x_{1}\left|x_{1}\right|^{p-2}, x_{2}\left|x_{2}\right|^{p-2}, \ldots, x_{n}\left|x_{n}\right|^{p-2}, \ldots\right)$;
(ii) $\mathrm{L}_{\mathrm{p}}: J u=\|u\|_{\mathrm{L}_{p}}^{2-\mathrm{p}}|\mathrm{u}|^{p-2} u \in \mathrm{~L}_{\mathrm{q}}$, with $1 / p+1 / \mathrm{q}=1$;
(iii) $W^{m, p}: J u=\|u\|_{W^{m, p}}^{2-p} \sum_{|\alpha \leqslant m|}(-1)^{|\alpha|} D^{\alpha}\left(\left|D^{\alpha} u\right|^{p-2} D^{\alpha} u\right) \in W^{-m, q}$, with $1 / p+1 / q=1$.

Next, we introduce the auxiliary map we referred and the normalized duality mapping in the Cartesian product space $X:=E \times E^{*}$ with the norm $\|w\|_{X}=\left(\|u\|^{2}+\|v\|_{*}^{2}\right)^{\frac{1}{2}}$ for $w=(u, v) \in X$, where $\|\cdot\|_{*}$ denotes the norm in $E^{*}$. For mappings $F: E \rightarrow 2^{E^{*}}$ and $K: E^{*} \rightarrow E$, observing that $E$ is reflexive and so $X^{*}=E^{*} \times E$, we define

$$
\mathrm{JX}_{\mathrm{X}}: \mathrm{X} \rightarrow \mathrm{X}^{*} \text { by: } \mathrm{J}_{\mathrm{X}}(w)=\left(\mathrm{J}(\mathrm{u}), \mathrm{J}^{-1}(v)\right) \forall w=(\mathrm{u}, v) \in \mathrm{X}
$$

and

$$
\Lambda: X \rightarrow X^{*} \text { by: } \Lambda w=\{(p-v, K v+u): p \in F u\} \forall w=(u, v) \in X
$$

Remark 3.3. Note that the zeros of $\Lambda$ give the solutions of the Hammerstein inclusion $0 \in u+K F u$. More precisely, for $w=(u, v) \in E \times E^{*}, 0 \in \Lambda(u, v)$ if and only if $0 \in u+K F u$ and $v \in F u$.

The following results will be crucial in the sequel.
Lemma 3.4. Let E be a reflexive, smooth and strictly convex real Banach space. Then $\mathrm{J}_{\mathrm{X}}$ is the normalized duality mapping of $X$.

Proof. For arbitrary $w=\left(u_{1}, v_{1}\right) \in X$ and $h=\left(v_{2}, u_{2}\right) \in X^{*}$, the duality pairing $\langle\cdot, \cdot\rangle_{X}$ is given by

$$
\langle w, h\rangle_{x}=\left\langle u_{1}, v_{2}\right\rangle+\left\langle u_{2}, v_{1}\right\rangle
$$

Now let $w=\left(u_{1}, v_{1}\right) \in X$. Set $h=J X(w)$. Then, we have $h=\left(J\left(u_{1}\right), J^{-1}\left(v_{1}\right)\right)$. So, it follows that

$$
\begin{aligned}
\left\langle w, \mathrm{~J}_{\mathrm{X}}(w)\right\rangle_{\mathrm{x}} & =\langle w, \mathrm{~h}\rangle \\
& =\left\langle\mathrm{u}_{1}, \mathrm{~J}\left(\mathrm{u}_{1}\right)\right\rangle+\left\langle\mathrm{J}^{-1}\left(v_{1}\right), v_{1}\right\rangle \\
& =\left\|\mathrm{u}_{1}\right\|^{2}+\left\|v_{1}\right\|_{*}^{2} \\
& =\left(\left\|u_{1}\right\|^{2}+\left\|v_{1}\right\|_{*}^{2}\right)^{\frac{1}{2}}\left(\left\|u_{1}\right\|^{2}+\left\|v_{1}\right\|_{*}^{2}\right)^{\frac{1}{2}} \\
& =\left(\left\|u_{1}\right\|^{2}+\left\|v_{1}\right\|_{*}^{2}\right)^{\frac{1}{2}}\left(\left\|\mathrm{~J}\left(u_{1}\right)\right\|^{2}+\left\|\mathrm{J}\left(v_{1}\right)\right\|_{*}^{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
=\|w\|_{X}\|\cdot\| h\left\|_{X^{*}}=\right\| w \|_{X}^{2} .
$$

This proves that $\mathrm{J}_{\mathrm{X}}$ is the normalized duality mapping of X .
Lemma 3.5. Let E be a reflexive real Banach space. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be two demiclosed operators such that for each $x \in \mathrm{D}(\mathrm{F})=\mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Then the map $\wedge$ is demiclosed and $\wedge w$ is convex for all $w \in \mathrm{D}(\Lambda)$.
Proof. Let $w_{n} \rightarrow w$ and $\Gamma_{n} \rightharpoonup \Gamma$ with $\Gamma_{n}=\left(\Gamma_{n}^{1}, \Gamma_{n}^{2}\right) \in \Lambda w_{n}$. Let us show that $\Gamma \in \Lambda w$. Let $w_{n}=\left(u_{n}, \mathfrak{u}_{n}^{*}\right)$, $w=\left(u, u^{*}\right)$ and $\Gamma=\left(\Gamma^{1}, \Gamma^{2}\right)$. Hence

$$
\Gamma_{\mathrm{n}} \in \Lambda_{\mathrm{n}} \Longleftrightarrow \Gamma_{\mathrm{n}}=\left(\gamma_{\mathrm{n}}^{*}-u_{n}^{*}, K u_{n}^{*}+u_{n}\right) \quad \text { where } \quad \gamma_{n}^{*} \in F u_{n} .
$$

On one hand we have $u_{n} \rightarrow u$ and $\gamma_{n}^{*}=u_{n}^{*}+\Gamma_{n}^{1} \rightharpoonup u^{*}+\Gamma^{1}$. Since $F$ is demiclosed, this implies that $u^{*}+\Gamma^{1} \in F u$ that is there exists $\gamma^{*} \in F u$ such that $\Gamma^{1}=\gamma^{*}-u^{*}$. Similarly $\Gamma^{2}=K u^{*}+u$. Therefore $\left(\Gamma^{1}, \Gamma^{2}\right) \in \Lambda w$. Similar arguments show that if $w_{n} \rightharpoonup w$ and $\Gamma_{n} \rightarrow \Gamma$ with $\Gamma_{n} \in \Lambda w_{n}$ then $\Gamma \in \Lambda w$. Hence $\Lambda$ is demiclosed. Moreover since $F$ is of nonempty convex-valued it ensues that $\Lambda$ is of nonempty convex-valued.

Lemma 3.6. Let E be a reflexive real Banach space. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be two maximal monotone operators such that for each $x \in \mathrm{D}(\mathrm{F})=\mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Then the map $\Lambda$ is maximal monotone.

Proof. It suffices to show that $\Lambda$ is monotone and demiclosed. For the monotonicity, let $w_{1}=\left(u_{1}, v_{1}\right), w_{2}=$ $\left(\mathfrak{u}_{2}, v_{2}\right) \in X$. Using the fact that $K$ and $F$ are monotone, we have, for $\tau_{w_{1}}=\left(p_{1}-v_{1}, K v_{1}+u_{1}\right) \in \Lambda w_{1}$ and $\tau_{w_{2}}=\left(p_{2}-v_{2}, K v_{2}+u_{2}\right) \in \Lambda w_{2}$

$$
\begin{aligned}
\left\langle w_{1}-w_{2}, \tau_{w_{1}}-\tau_{w_{2}}\right\rangle_{\mathrm{X}} & =\left\langle\left(\mathfrak{u}_{1}-\mathfrak{u}_{2}, v_{1}-v_{2}\right),\left(\mathfrak{p}_{1}-p_{2}+v_{2}-v_{1}, K v_{1}-K v_{2}+\mathfrak{u}_{1}-\mathfrak{u}_{2}\right)\right\rangle_{\mathrm{X}} \\
& =\left\langle\mathfrak{u}_{1}-\mathfrak{u}_{2}, \boldsymbol{p}_{1}-\mathfrak{p}_{2}+v_{2}-v_{1}\right\rangle+\left\langle K v_{1}-K v_{2}+\mathfrak{u}_{1}-\mathfrak{u}_{2}, v_{1}-v_{2}\right\rangle \\
& =\left\langle\mathfrak{u}_{1}-\mathfrak{u}_{2}, \boldsymbol{p}_{1}-\mathfrak{p}_{2}\right\rangle+\left\langle K v_{1}-K v_{2}, v_{1}-v_{2}\right\rangle \geqslant 0 .
\end{aligned}
$$

This implies that $\Lambda$ is monotone. F and K being maximal monotone, Lemma 2.3 implies that they are demiclosed. Therefore it follows from Lemma 3.5 that $\Lambda$ is demiclosed and is of nonempty and convexvalued in $X^{*}$. Hence by Lemma $2.4 \Lambda$ is maximal monotone.

### 3.1. Implicit scheme for integral inclusions

We recall the following result.
Lemma 3.7 ([44]). Let E be a uniformly convex real Banach space with Fréchet differentiable norm. Let A : $\mathrm{E}^{*} \rightarrow 2^{\mathrm{E}}$ be a maximal monotone mapping with $\mathrm{A}^{-1}(0) \neq \emptyset$. Then for $\mathrm{u} \in \mathrm{E}$ and $\lambda>0, \lim _{\lambda \rightarrow \infty}(\mathrm{I}+$ $\lambda A J)^{-1} \mathfrak{u}$ exists and belongs to $(\mathrm{AJ})^{-1}(0)$, where J is the normalized duality mapping from E into $\mathrm{E}^{*}$. Moreover, if $R u:=y^{*}=\lim _{\lambda \rightarrow \infty}(I+\lambda A J)^{-1} u$, then $R$ is a sunny generalized nonexpansive retraction of $E$ into $(A J)^{-1}(0)$.

The next theorem is a consequence of the above lemma.
Theorem 3.8. Let E be a uniformly convex and uniformly smooth real Banach space with dual $\mathrm{E}^{*}$. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be two maximal monotone operators such that for each $\mathrm{x} \in \mathrm{D}(\mathrm{F})=\mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Assume that $0 \in \mathfrak{u}+\mathrm{KFu}$ has a solution in E . For given $\mathrm{u}_{1} \in \mathrm{E}$ and $v_{1} \in \mathrm{E}^{*}$, there exist a sequence $\left\{z_{n}\right\}$ in $E \times E^{*}$ with $z_{n}=\left(x_{n}, y_{n}\right)$ and a sequence $\left\{q_{n}\right\}$ in $E^{*}$ satisfying the following

$$
\begin{align*}
\theta_{n}\left(J x_{n}-J u_{1}\right)+q_{n}-y_{n} & =0, \quad q_{n} \in F x_{n}, \forall n \geqslant 1,  \tag{3.2}\\
\theta_{n}\left(J^{-1} y_{n}-J^{-1} v_{1}\right)+K y_{n}+x_{n} & =0, \quad \forall n \geqslant 1 . \tag{3.3}
\end{align*}
$$

And furthermore

$$
x_{n} \rightarrow x^{*}, y_{n} \rightarrow y^{*} \text { with } 0 \in x^{*}+K F x^{*} \text { and } y^{*} \in F x^{*} .
$$

Proof. Let $\Lambda$ as defined above. It follows from Lemma 3.6 that $\Lambda$ is maximal monotone from $X$ to $X^{*}$. Let $x^{*} \in E$ such that $0 \in x^{*}+K F x^{*}$ with $y^{*} \in F x^{*}$ such that $x^{*}+K y^{*}=0$, we have $0 \in \Lambda\left(x^{*}, y^{*}\right)$. This implies that $\Lambda^{-1}(0) \neq \emptyset$. Given $w_{1}=\left(u_{1}, v_{1}\right) \in X$, if follows from Lemma 3.7 that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(I^{*}+\theta_{n}^{-1} \wedge J_{x}^{-1}\right)^{-1} J \chi w_{1} \text { exists and belongs to }\left(\Lambda J_{X}^{-1}\right)^{-1}(0) \tag{3.4}
\end{equation*}
$$

where $I^{*}: X^{*} \rightarrow X^{*}$ is the identity. Let $\left(a_{n}, b_{n}\right)$ be a sequence in $X^{*}$ defined as follows

$$
\left(a_{n}, b_{n}\right)=\left(I^{*}+\theta_{n}^{-1} \Lambda J_{X}^{-1}\right)^{-1} J X w_{1} .
$$

Equation (3.4) implies that

$$
\Lambda J_{x}^{-1}\left(a_{n}, b_{n}\right) \rightarrow 0
$$

Furthermore we have

$$
\left(J u_{1}, J^{-1} v_{1}\right) \in\left(a_{n}, b_{n}\right)+\theta_{n}^{-1} \wedge\left(J^{-1} a_{n}, J b_{n}\right) \in\left(a_{n}, b_{n}\right)+\theta_{n}^{-1}\left(F J^{-1} a_{n}-J b_{n}\right) \times\left\{K J b_{n}+J^{-1} a_{n}\right\} .
$$

This implies

$$
\mathrm{FJ}^{-1} \mathrm{a}_{\mathrm{n}}-\mathrm{J} \mathrm{~b}_{\mathrm{n}}+\theta_{\mathrm{n}}\left(\mathrm{a}_{\mathrm{n}}-\mathrm{J} \mathrm{u}_{1}\right) \ni 0 \quad n \geqslant 1
$$

and

$$
K J b_{n}+J^{-1} a_{n}+\theta_{n}\left(b_{n}-J^{-1} v_{1}\right)=0 \quad n \geqslant 1 .
$$

Let $\left(x_{n}, y_{n}\right)=J_{X}^{-1}\left(a_{n}, b_{n}\right)=\left(J^{-1} a_{n}, J b_{n}\right)$, we have the following

$$
0 \in F x_{n}-y_{n}+\theta_{n}\left(J x_{n}-J u_{1}\right) \quad n \geqslant 1,
$$

and

$$
K y_{n}+x_{n}+\theta_{n}\left(J^{-1} y_{n}-J^{-1} v_{1}\right)=0 \quad n \geqslant 1 .
$$

Therefore there exists $\left\{q_{n}\right\}_{n} \subset E^{*}$ such that

$$
\begin{aligned}
\theta_{\mathfrak{n}}\left(J x_{n}-J u_{1}\right)+q_{n}-y_{n} & =0, \quad q_{n} \in F x_{n}, \forall n \geqslant 1, \\
\theta_{n}\left(J^{-1} y_{n}-J^{-1} v_{1}\right)+K y_{n}+x_{n} & =0, \quad \forall n \geqslant 1 .
\end{aligned}
$$

Moreover

$$
\Lambda\left(x_{n}, y_{n}\right)=\Lambda J_{X}^{-1}\left(a_{n}, b_{n}\right) \rightarrow 0
$$

This completes the proof.

### 3.2. Convergence in $\ell_{\mathrm{p}}, \mathrm{L}_{\mathrm{p}}$ and $\mathrm{W}^{\mathrm{m}, \mathrm{p}}$-spaces, $1<\mathrm{p} \leqslant 2$

Theorem 3.9. For $\mathrm{q}>1$, let E be a 2 -uniformly convex and q -uniformly smooth real Banach space with dual $\mathrm{E}^{*}$. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be bounded, maximal monotone mappings such that $\mathrm{D}(\mathrm{F})=\mathrm{E}$ and $\mathrm{D}(\mathrm{K})=\mathrm{R}(\mathrm{F})=\mathrm{E}^{*}$ and for each $\mathrm{x} \in \mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Suppose that the Hammerstein inclusion $0 \in \mathfrak{u}+K F u$ has a solution. Then, there exists $\gamma_{0}>0$ such that if $\lambda_{n}<\gamma_{0} \theta_{n}$ for all $n \geqslant 1$, the sequence $\left\{\left(u_{n}, v_{n}\right)\right\}$ given by (3.1) converges strongly to $\left(u^{*}, v^{*}\right)$, where $u^{*}$ is a solution of the Hammerstein equation $0 \in u+\mathrm{KFu}$ and $v^{*} \in \mathrm{Fu}^{*}$.

Proof.
Step 1: We prove that $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are bounded. Before starting the proof, let us mention that $E$ is 2 -uniformly convex and p-uniformly smooth. So it satisfies the conditions of the lemmas 2.6 and 2.9. Let $w_{n}=\left(u_{n}, v_{n}\right) \in X$ and $w_{1}=\left(u_{1}, v_{1}\right) \in X, w^{*}=\left(u^{*}, v^{*}\right) \in X$ with $v^{*} \in F u^{*}$ and $u^{*}+K v^{*}=0$. There exists
$r>0$ large enough such that

$$
\begin{equation*}
\max \left\{\sqrt{\psi\left(w^{*}, w_{1}\right)}, 24 m\left\|w_{1}-w^{*}\right\|,\left\|w^{*}\right\|\right\}<\sqrt{r} \quad \text { with } \quad m=m_{1}+\max \left\{m_{2}, m_{3}\right\} . \tag{3.5}
\end{equation*}
$$

Since $F$ and $K$ are bounded we have:

$$
M_{1}:=\sup \left\{\left\|p-v+\theta\left(J u-J u_{1}\right)\right\|_{*}: p \in F u, \psi\left(w^{*},(u, v)\right)<r, 0<\theta<1\right\}<\infty
$$

and

$$
M_{2}:=\sup \left\{\left\|K u+v+\theta\left(\mathrm{J}^{-1} u-\mathrm{J}^{-1} v_{1}\right)\right\|: \psi\left(w^{*},(\mathrm{u}, v)\right)<\mathrm{r}, 0<\theta<1\right\}<\infty .
$$

From the local Lipschitz property of J (Lemma 2.6) and $\mathrm{J}^{-1}$ (Lemma 2.9) on bounded sets there exist $\mathrm{m}_{1}>0$ and $\mathrm{m}_{2}>0$ such that

$$
\left\|J^{-1}\left(J u-\lambda(p-v)-\lambda \theta\left(J u-J u_{1}\right)\right)-J^{-1}(J u)\right\| \leqslant \lambda m_{1} M_{1},
$$

for all $\lambda, \theta \in(0,1), p \in F u(u, v) \in X: \psi\left(w^{*},(u, v)\right) \leqslant r$, and

$$
\left\|J\left(J^{-1} v-\lambda(\mathrm{K} v+u)-\lambda \theta\left(\mathrm{J}^{-1} v-\mathrm{J}^{-1} v_{1}\right)\right)-\mathrm{J}\left(\mathrm{~J}^{-1} v\right)\right\| \leqslant \lambda m_{2} \mathrm{M}_{2}
$$

for all $\lambda, \theta \in(0,1),(u, v) \in X: \psi\left(w^{*},\left(u_{1}, v_{1}\right)\right) \leqslant r$.
Set

$$
M:=M_{1}^{2}+M_{2}^{2}, \quad \text { and } \quad \bar{m}^{2}=m_{1}^{2}+\max \left\{m_{2}^{2}, m_{3}^{2}\right\} .
$$

Define the constant $\gamma_{0}$ as follows

$$
\gamma_{0}:=\min \left\{1, \frac{r}{8 \mathrm{~m}^{2}}\right\} .
$$

Let $\mathfrak{n} \in \mathbb{N}$ such that $\psi\left(w^{*}, w_{n}\right)<r$. From the definition of $u_{n}$ we have the following

$$
\begin{aligned}
\phi\left(u^{*}, u_{n+1}\right) & =\phi\left(u^{*}, J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right)\right) \\
& =V\left(u^{*}, J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right) .
\end{aligned}
$$

Using Lemma 2.11 with $y^{*}=\lambda_{n}\left(p_{n}-v_{n}\right)+\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)$ we have

$$
\begin{aligned}
\phi\left(u^{*}, u_{n+1}\right) \leqslant & V\left(u^{*}, J u_{n}\right) \\
& -2\left\langle J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)-u^{*}, \lambda_{n}\left(p_{n}-v_{n}\right)+\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle\right. \\
\leqslant & \phi\left(u^{*}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-u^{*}, p_{n}-v_{n}+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle \\
& -2 \lambda_{n}\left\langle J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right)-u_{n}\left(p_{n}-v_{n}\right)+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle \\
\leqslant & \phi\left(u^{*}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-u^{*}, p_{n}-v_{n}+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle+2 m_{1} M_{1}^{2} \lambda_{n}^{2} .
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\left\langle\mathbf{u}_{n}-u^{*}, p_{n}-v_{n}+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle= & \left\langle u_{n}-u^{*}, p_{n}-v^{*}\right\rangle+\left\langle u_{n}-u^{*}, v^{*}-v_{n}+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle \\
= & \left\langle\mathfrak{u}_{n}-u^{*}, \boldsymbol{p}_{n}-v^{*}\right\rangle+\left\langle u_{n}-\mathfrak{u}^{*}, v^{*}-v_{n}\right\rangle \\
& +\theta_{n}\left\langle u_{n}-u^{*}, J u_{n}-J u^{*}\right\rangle+\theta_{n}\left\langle u_{n}-u^{*}, J u^{*}-J u_{1}\right\rangle .
\end{aligned}
$$

Therefore using Lemma 2.13 and the fact that $F$ is monotone we get

$$
\begin{align*}
\phi\left(u^{*}, u_{n+1}\right) \leqslant & \phi\left(u^{*}, u_{n}\right)+2 \lambda_{n}\left\langle u_{n}-u^{*}, v_{n}-v^{*}\right\rangle \\
& -\lambda_{n} \theta_{n} \phi\left(u^{*}, u_{n}\right)-2 \lambda_{n} \theta_{n}\left\langle u_{n}-u^{*}, J u_{1}-J u^{*}\right\rangle+2 m_{1} M_{1}^{2} \lambda_{n}^{2} . \tag{3.6}
\end{align*}
$$

The same arguments lead to the following

$$
\begin{align*}
\phi_{*}\left(v^{*}, v_{n+1}\right) \leqslant & \phi_{*}\left(v^{*}, v_{n}\right)-2 \lambda_{n}\left\langle v_{n}-v^{*}, \mathbf{u}_{n}-u^{*}\right\rangle \\
& -\lambda_{n} \theta_{n} \phi_{*}\left(v^{*}, v_{n}\right)-2 \lambda_{n} \theta_{n}\left\langle v_{n}-v^{*}, \mathrm{~J}^{-1} v_{1}-\mathrm{J}^{-1} v^{*}\right\rangle+2 m_{2} M_{2}^{2} \lambda_{n}^{2} . \tag{3.7}
\end{align*}
$$

Since $\left\|\mathfrak{u}_{n}-\mathfrak{u}^{*}\right\| \leqslant\left\|w_{n}-w^{*}\right\|$ for all $n \geqslant 1$, Schwartz inequality together with Lemma 2.6 yields

$$
\left|\left\langle\mathfrak{u}_{n}-u^{*}, \boldsymbol{J} u_{1}-J u^{*}\right\rangle\right| \leqslant \mathfrak{m}_{1}\left\|w_{n}-w^{*}\right\| \cdot\left\|w_{1}-w^{*}\right\| .
$$

Likewise Schwartz inequality together with Lemma 2.9 yields

$$
\left|\left\langle v_{n}-v^{*}, J^{-1} v_{1}-\mathrm{J}^{-1} v^{*}\right\rangle\right| \leqslant \mathfrak{m}_{3}\left\|w_{n}-w^{*}\right\| \cdot\left\|w_{1}-w^{*}\right\| .
$$

Adding (3.6) and (3.7) we obtain

$$
\psi\left(w^{*}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(w^{*}, w_{n}\right)+2 m \lambda_{n} \theta_{n}\left\|w_{n}-w^{*}\right\| \cdot\left\|w_{1}-w^{*}\right\|+2 m M^{2} \lambda_{n}^{2} .
$$

From (2.3), the induction assumption and inequality (3.5) we have

$$
\left\|w_{n}-w^{*}\right\| \leqslant 3 \sqrt{r} \quad \text { and } \quad 2 m\left\|w_{n}-w^{*}\right\| \cdot\left\|w_{1}-w^{*}\right\| \leqslant \frac{r}{4} .
$$

This implies the following

$$
\psi\left(w^{*}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(w^{*}, w_{n}\right)+\lambda_{n} \theta_{n} \frac{r}{4}+\lambda_{n} \theta_{n} \frac{r}{4} \leqslant\left(1-\frac{\lambda_{n} \theta_{n}}{2}\right) r .
$$

Therefore $\psi\left(w^{*}, w_{n+1}\right) \leqslant r$. So by induction, $\psi\left(w^{*}, w_{n}\right) \leqslant r$ for all $n \geqslant 1$. Hence $w_{n}$ is bounded.
Step 2: Let $z_{n}=\left(x_{n}, y_{n}\right)$ given in Theorem 3.8, let us show that $\psi\left(w_{n}, z_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$. We have

$$
\begin{aligned}
\phi\left(x_{n}, u_{n+1}\right) & =\phi\left(x_{n}, J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right)\right) \\
& =V\left(x_{n}, J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right) .
\end{aligned}
$$

Using Lemma 2.11 with $y^{*}=\lambda_{n}\left(p_{n}-v_{n}\right)+\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)$ we obtain

$$
\begin{aligned}
\phi\left(x_{n}, u_{n+1}\right) \leqslant & \phi\left(x_{n}, u_{n}\right) \\
& -2 \lambda_{n}\left\langle J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right)-x_{n},\left(p_{n}-v_{n}\right)+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle \\
= & \phi\left(x_{n}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-x_{n},\left(p_{n}-v_{n}\right)+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle \\
& -2 \lambda_{n}\left\langle J^{-1}\left(J u_{n}-\lambda_{n}\left(p_{n}-v_{n}\right)-\lambda_{n} \theta_{n}\left(J u_{n}-J u_{1}\right)\right)-u_{n},\left(p_{n}-v_{n}\right)+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle \\
\leqslant & \phi\left(x_{n}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-x_{n}, p_{n}-q_{n}\right\rangle \\
& -2 \lambda_{n}\left\langle u_{n}-x_{n}, q_{n}-v_{n}+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle+2 \lambda_{n}^{2} m_{1} M^{2} \\
\leqslant & \phi\left(x_{n}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-x_{n}, y_{n}-\theta_{n}\left(J x_{n}-J u_{1}\right)-v_{n}+\theta_{n}\left(J u_{n}-J u_{1}\right)\right\rangle+2 \lambda_{n}^{2} m_{1} M^{2} \\
\leqslant & \phi\left(x_{n}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-x_{n}, y_{n}-v_{n}+\theta_{n}\left(J u_{n}-J x_{n}\right)\right\rangle+2 \lambda_{n}^{2} m_{1} M^{2} \\
\leqslant & \phi\left(x_{n}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-x_{n}, y_{n}-v_{n}\right\rangle-2 \lambda_{n} \theta_{n}\left\langle u_{n}-x_{n}, J u_{n}-J x_{n}\right\rangle+2 \lambda_{n}^{2} m_{1} M_{1}^{2} .
\end{aligned}
$$

This implies the following

$$
\phi\left(x_{n}, u_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \phi\left(x_{n}, u_{n}\right)-2 \lambda_{n}\left\langle u_{n}-x_{n}, y_{n}-v_{n}\right\rangle+2 \lambda_{n}^{2} m_{1} M^{2}
$$

Similar arguments give

$$
\phi_{*}\left(y_{n}, v_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \phi_{*}\left(y_{n}, v_{n}\right)-2 \lambda_{n}\left\langle y_{n}-v_{n}, x_{n}-u_{n}\right\rangle+2 \lambda_{n}^{2} m_{2} M_{2}^{2}
$$

Therefore by adding the two last inequalities we obtain

$$
\psi\left(z_{n}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(z_{n}, w_{n}\right)+2 \lambda_{n}^{2} m M^{2} .
$$

Observe that

$$
\psi\left(z_{n}, w_{n+1}\right)=\psi\left(z_{n}, z_{n+1}\right)+\psi\left(z_{n+1}, w_{n+1}\right)+2\left\langle z_{n+1}-z_{n}, J_{X} w_{n+1}-J_{X} z_{n+1}\right\rangle
$$

Since $\psi\left(z_{n}, z_{n+1}\right) \geqslant 0$, using this in the above inequality we get

$$
\psi\left(z_{n+1}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(z_{n}, w_{n}\right)+2\left\langle z_{n}-z_{n+1}, J_{X} w_{n+1}-J J_{X} z_{n+1}\right\rangle+2 \lambda_{n}^{2} m M^{2}
$$

Using the fact that $\left\{w_{n}\right\}$ and $\left\{z_{n}\right\}$ are bounded we have

$$
\begin{equation*}
\psi\left(z_{n+1}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(z_{n}, w_{n}\right)+C\left\|z_{n}-z_{n+1}\right\|_{x}+2 \lambda_{n}^{2} m M^{2} \tag{3.8}
\end{equation*}
$$

for some constant $C>0$.
Now from (3.2) and (3.3) in Theorem 3.8 we have respectively

$$
\begin{equation*}
J x_{n}-J x_{n+1}+\frac{1}{\theta_{n+1}}\left(q_{n}-y_{n}-q_{n+1}+y_{n+1}\right)=\left(\frac{\theta_{n+1}-\theta_{n}}{\theta_{n+1}}\right)\left(J x_{n}-J u_{1}\right) \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
J^{-1} y_{n}-J^{-1} y_{n+1}+\frac{1}{\theta_{n+1}}\left(K y_{n}+x_{n}-K y_{n+1}-x_{n+1}\right)=\left(\frac{\theta_{n+1}-\theta_{n}}{\theta_{n+1}}\right)\left(J^{-1} y_{n}-J^{-1} v_{1}\right) \tag{3.10}
\end{equation*}
$$

Taking the duality pairing with $x_{n}-x_{n+1}$ and with $y_{n}-y_{n+1}$ respectively in (3.9) and (3.10) and using the monotonicity of F and K we obtain the following estimates:

$$
\begin{equation*}
\left\langle J x_{n}-J x_{n+1}, x_{n}-x_{n+1}\right\rangle+\frac{1}{\theta_{n+1}}\left\langle y_{n+1}-y_{n}, x_{n}-x_{n+1}\right\rangle \leqslant\left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)\left\langle J x_{n}-J x_{1}, x_{n}-x_{n+1}\right\rangle \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle J^{-1} y_{n}-J^{-1} y_{n+1}, y_{n}-y_{n+1}\right\rangle+\frac{1}{\theta_{n+1}}\left\langle x_{n}-x_{n+1}, y_{n}-y_{n+1}\right\rangle \leqslant\left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)\left\langle J^{-1} y_{n}-J^{-1} y_{1}, y_{n}-y_{n+1}\right\rangle . \tag{3.12}
\end{equation*}
$$

Adding up (3.11) and (3.12), using Schwartz inequality, Lemmas 2.6, 2.8-2.10, and the boundedness of $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ we have:

$$
\begin{aligned}
d_{2}\left\|x_{n}-x_{n+1}\right\|^{2}+d_{1}\left\|y_{n}-y_{n+1}\right\|^{2} \leqslant & \left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)\left(\left\langle J x_{n}-J x_{1}, x_{n}-x_{n+1}\right\rangle+\left\langle J^{-1} y_{n}-J^{-1} y_{1}, y_{n}-y_{n+1}\right\rangle\right) \\
\leqslant & \left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)\left(m_{1}\left\|x_{n}-x_{1}\right\|\left\|x_{n}-x_{n+1}\right\|\right. \\
& \left.+m_{2}\left\|y_{n}-y_{1}\right\|_{*}\left\|y_{n}-y_{n+1}\right\|_{*}\right) \\
\leqslant & C_{1}\left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)\left(\left\|x_{n}-x_{n+1}\right\|+\left\|y_{n}-y_{n+1}\right\|_{*}\right) \\
\leqslant & 2 C_{1}\left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)\left\|z_{n}-z_{n+1}\right\| x
\end{aligned}
$$

for some constant $C_{1}$. So,

$$
\begin{equation*}
\left\|z_{n}-z_{n+1}\right\|_{x} \leqslant K\left(\frac{\theta_{n+1}-\theta_{n}}{\theta_{n+1}}\right) \tag{3.13}
\end{equation*}
$$

where, $\mathrm{K}:=\frac{2 \mathrm{CC}_{1}}{\min \left\{\mathrm{~d}_{1}, \mathrm{~d}_{2}\right\}}$.

Therefore, combining inequalities (3.8) and (3.13), we obtain

$$
\psi\left(z_{n+1}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(z_{n}, w_{n}\right)+K\left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)+2 \lambda_{n}^{2} m M^{2} .
$$

Finally, we have

$$
\psi\left(z_{n+1}, w_{n+1}\right) \leqslant\left(1-\lambda_{n} \theta_{n}\right) \psi\left(z_{n}, w_{n}\right)+\lambda_{n} \theta_{n} \sigma_{n}+\gamma_{n},
$$

with $\sigma_{n}:=K\left(\frac{\theta_{n}-\theta_{n+1}}{\theta_{n+1}}\right)$ and $\gamma_{n}=2 \lambda_{n}^{2} m M^{2}$. So, using Lemma 2.15, it follows that $\psi\left(z_{n}, w_{n}\right) \rightarrow 0$, as $n \rightarrow \infty$. Therefore, from Lemma 2.14, we have that $\left\|w_{n}-z_{n}\right\| \rightarrow 0$ as $n \rightarrow \infty$. Hence, the conclusion follows from Theorem 3.8.

Corollary 3.10. Let E be a Banach space either $\ell_{\mathrm{p}}$ or $\mathrm{L}_{\mathrm{p}}$ or $\mathrm{W}^{\mathrm{m}, \mathrm{p}}, 1<\mathrm{p} \leqslant 2$ with dual $\mathrm{E}^{*}$. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be bounded, maximal monotone mappings such that $\mathrm{D}(\mathrm{F})=\mathrm{E}$ and $\mathrm{D}(\mathrm{K})=\mathrm{R}(\mathrm{F})=\mathrm{E}^{*}$ and for each $x \in \mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Suppose that the Hammerstein inclusion $0 \in u+\mathrm{KFu}$ has a solution. Then, there exists $\gamma_{0}>0$ such that if $\lambda_{n}<\gamma_{0} \theta_{n}$ for all $n \geqslant 1$, the sequence $\left\{\left(u_{n}, v_{n}\right)\right\}$ given by (3.1) converges strongly to $\left(u^{*}, v^{*}\right)$, where $u^{*}$ is a solution of the Hammerstein equation $0 \in u+K F u$ and $v^{*} \in \mathrm{Fu}^{*}$.
3.3. Convergence in $\ell_{\mathrm{p}}, \mathrm{L}_{\mathrm{p}}$ and $\mathrm{W}^{\mathrm{m}, \mathrm{p}}$-spaces, $2 \leqslant \mathrm{p}<\infty$

Theorem 3.11. For $s>1$, let E be a s-uniformly convex and 2 -uniformly smooth real Banach space with dual $\mathrm{E}^{*}$. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be bounded, maximal monotone mappings such that $\mathrm{D}(\mathrm{F})=\mathrm{E}$ and $\mathrm{D}(\mathrm{K})=\mathrm{R}(\mathrm{F})=\mathrm{E}^{*}$ and for each $\mathrm{x} \in \mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Suppose that the Hammerstein inclusion $0 \in \mathfrak{u}+\mathrm{KFu}$ has a solution. Then, there exists $\gamma_{0}>0$ such that if $\lambda_{n}<\gamma_{0} \theta_{\mathrm{n}}$ for all $\mathrm{n} \geqslant 1$, the sequence $\left\{\left(u_{n}, v_{n}\right)\right\}$ given by (3.1) converges strongly to $\left(u^{*}, v^{*}\right)$, where $u^{*}$ is a solution of the Hammerstein equation $0 \in u+K F u$ and $v^{*} \in \mathrm{Fu}^{*}$.

Proof. For $\mathrm{p} \geqslant 2, \mathrm{E}$ is p -uniformly convex and 2-uniformly smooth. So it satisfies the conditions of the Lemmas 2.6 and 2.7. Therefore we deduce the result by the arguments in Theorem 3.9.

Corollary 3.12. Let E be a Banach space either $\ell_{\mathrm{p}}$ or $\mathrm{L}_{\mathrm{p}}$ or $\mathrm{W}^{\mathrm{m}, \mathrm{p}}, 2 \leqslant \mathrm{p}<\infty$ with dual $\mathrm{E}^{*}$. Let $\mathrm{F}: \mathrm{E} \rightarrow 2^{\mathrm{E}^{*}}$ and $\mathrm{K}: \mathrm{E}^{*} \rightarrow \mathrm{E}$ be bounded, maximal monotone mappings such that $\mathrm{D}(\mathrm{F})=\mathrm{E}$ and $\mathrm{D}(\mathrm{K})=\mathrm{R}(\mathrm{F})=\mathrm{E}^{*}$ and for each $x \in \mathrm{E}, \mathrm{Fx}$ is a nonempty convex subset of $\mathrm{E}^{*}$. Suppose that the Hammerstein inclusion $0 \in u+\mathrm{KFu}$ has a solution. Then, there exists $\gamma_{0}>0$ such that if $\lambda_{n}<\gamma_{0} \theta_{n}$ for all $n \geqslant 1$, the sequence $\left\{\left(u_{n}, v_{n}\right)\right\}$ given by (3.1) converges strongly to $\left(\mathfrak{u}^{*}, v^{*}\right)$, where $\mathfrak{u}^{*}$ is a solution of the Hammerstein equation $0 \in \mathfrak{u}+\mathrm{KFu}$ and $v^{*} \in \mathrm{Fu}^{*}$.

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