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Symmetry Lie algebra and exact solutions of some fourth-order difference equations

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Abstract

In this paper, all the Lie point symmetries of difference equations of the form

$$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}},$$

where, $(A_n)_{n\geqslant 0}$ and $(B_n)_{n\geqslant 0}$ are sequences of real numbers, are obtained. We perform reduction of order using the invariant of the group of transformations. Furthermore, we obtain their solutions. In particular, our work generalizes some results in the literature.

Keywords: Difference equation, symmetry, group invariant solutions.

2010 MSC: 39A10, 39A99, 39A13.

1. Introduction

Lie symmetry analysis is a powerful method not only for differential equations but also difference equations. Its demonstration on difference equations has been successful and there has been progress in this area (see [16, 19, 20]). The symmetry method has been used to find traveling wave solutions. For more on traveling waves, refer to [12, 21–23].

In this method, one finds a group of mappings that map the set of solutions to the difference equation under study onto itself. However, it should be stated that the computational difficulty in employing this method can increase with increasing order of the equation being studied.

Elsayed [9] obtained the exact solutions of

$$x_{n+1} = \frac{x_{n-3}}{\pm 1 \pm x_{n-1}x_{n-3}}. \tag{1.1}$$

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Related work has been done (see [1–11, 13–15, 17, 18]). In this paper we obtain solutions of the following difference equations via the invariant of their group of transformations:

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1} x_{n-3}}$$

for some arbitrary sequences of real numbers (a_n) and (b_n). Thus the solution to the difference equation above extends the solution of Elsayed [9] to equation (1.1) to a more general setting.

Without loss of generality, we instead study the difference equation

$$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}}.$$

1.1. *Preliminaries*

Consider the difference equation

$$u_{n+4} = f(n, u_n, u_{n+1}, \dots, u_{n+3}), \tag{1.2}$$

where f is an arbitrary function. Suppose that the point transformations are of the form

$$(n, u_n) \mapsto (n, u_n + \epsilon Q(n, u_n)), \tag{1.3}$$

where Q is the characteristic and

$$X = Q(n, u_n)\frac{\partial}{\partial u_n} + Q(n+1, u_{n+1})\frac{\partial}{\partial u_{n+1}} + \cdots + Q(n+3, u_{n+3})\frac{\partial}{\partial u_{n+3}}$$

is the corresponding symmetry generator. We have the following linearized symmetry condition, obtained using (1.3):

$$Q(n+4, u_{n+4}) - \frac{\partial f}{\partial u_n} Q(n, u_n) - \frac{\partial f}{\partial u_{n+2}} Q(n+2, u_{n+2}) - \frac{\partial f}{\partial u_{n+3}} Q(n+3, u_{n+3}) = 0, \tag{1.4}$$

whenever (1.2) holds. In solving our difference equation under study, we will use a canonical coordinate, that is the variable S such that

$$XS = 1.$$

We make use of the known choice of S given by (see [16])

$$S = \int \frac{du_n}{Q(n, u_n)}.$$

Equation (1.4) appears simpler, although it is generally hard to solve.

2. Main results

We apply the symmetry condition (1.4) to

$$u_{n+4} = \frac{u_n}{A_n + B_n u_n u_{n+2}} \tag{2.1}$$

to solve for the characteristic Q. This yields

$$Q(n+4, u_{n+4}) + \frac{B_n {u_n}^2}{(B_n u_n u_{n+2} + A_n)^2} Q\left(n+2, u_{n+2}\right) - \frac{A_n}{(B_n u_n u_{n+2} + A_n)^2} Q\left(n, u_n\right). \tag{2.2}$$

In oder to get an equation that involves u_n only, we first differentiate implicitly (2.2) with respect to u_n (keeping u_{n+4} fixed). This results in

$$\frac{\partial}{\partial u_{n+2}} Q\left(n+2, u_{n+2}\right) - \frac{\partial}{\partial u_n} Q\left(n, u_n\right) + \frac{2}{u_n} Q\left(n, u_n\right) = 0. \tag{2.3}$$

Secondly, we differentiate (2.3) with respect to u_n. This yields

$$\frac{d}{du_n}\left(-\frac{\partial}{\partial u_n} Q\left(n, u_n\right) + \frac{2}{u_n} Q\left(n, u_n\right)\right) = 0.$$

So,

$$-\frac{\partial}{\partial u_n} Q\left(n, u_n\right) + \frac{2}{u_n} Q\left(n, u_n\right) = f_2(n).$$

Thus Q takes the form

$$Q\left(n, u_n\right) = f_1(n) {u_n}^2 + f_2(n) u_n, \tag{2.4}$$

where f_1 and f_2 are functions of n. Finally, we substitute (2.4) into (2.2) and do the separation by powers of shifts of u_n. This yields a system of equations that simplifies to

$$\begin{cases} f_1(n) = 0, \\ f_2(n) + f_2(n+2) = 0. \end{cases} \tag{2.5}$$

Hence,

$$f_1(n) = 0,\ f_2(n) = \alpha^n,\ f_2(n) = \bar{\alpha}^n,$$

where $\alpha = \exp\{i\pi/2\}$ and $\bar{\alpha}$ is the complex conjugate. Hence, we have two generators given as follows:

$$\begin{aligned} X_0 =& \alpha^n u_n \partial_{u_n} + \alpha^{n+1} u_{n+1} \partial_{u_{n+1}} + \alpha^{n+2} u_{n+2} \partial_{u_{n+2}} + \alpha^{n+3} u_{n+3} \partial_{u_{n+3}}, \\ X_1 =& \bar{\alpha}^n u_n \partial_{u_n} + \bar{\alpha}^{n+1} u_{n+1} \partial_{u_{n+1}} + \bar{\alpha}^{n+2} u_{n+2} \partial_{u_{n+2}} + \bar{\alpha}^{n+3} u_{n+3} \partial_{u_{n+3}}. \end{aligned}$$

Let

$$S_n = \int \frac{du_n}{\alpha^n u_n} = \frac{1}{\alpha^n} \ln |u_n|.$$

We perform a change of variable, thanks to (2.5):

$$\tilde{V}_n = S_n \alpha^n + S_{n+2} \alpha^{n+2}.$$

We have

$$X_0 \tilde{V}_n = X_1 \tilde{V}_n = 0$$

so that $\tilde{V}_n$ is an invariant of the group of transformations (1.3). Considering the fact that the equation being studied is rational, it is convenient to utilise

$$|V_n| = \exp\{-\tilde{V}_n\}.$$

Using V_n with (2.1), we obtain

$$V_{n+2} = A_n V_n \pm B_n. \tag{2.6}$$

Now we make the choice of using the plus sign and thus write the solution of (2.6) as

$$V_{2n+i} = V_i\left(\prod_{k_1=0}^{n-1} A_{2k_1+i}\right) + \sum_{l=0}^{n-1}\left(B_{2l+i}\prod_{k_2=l+1}^{n-1} A_{2k_2+i}\right), \tag{2.7}$$

where $i = 0, 1$. Thus the solution of (2.1) can be obtained by reversing the changes of variables. We have

$$\begin{aligned} |u_n| = \exp(\alpha_n S_n) &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \frac{1}{2}\sum_{k_1=0}^{n-1}\alpha^n\bar{\alpha}^{k_1}\ln|V_{k_1}| + \frac{1}{2}\sum_{k_2=0}^{n-1}\bar{\alpha}^n\alpha^{k_2}\ln|V_{k_2}|\right)\\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \frac{1}{2}\sum_{k=0}^{n-1}(\alpha^{n-k} + \bar{\alpha}^{n-k})\ln|V_k|\right)\\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \sum_{k=0}^{n-1}\mathrm{Re}(\alpha^{n-k})\ln|V_k|\right)\\ &= \exp\left(\alpha^n c_1 + \bar{\alpha}^n c_2 + \sum_{k=0}^{n-1}\cos\left(\frac{(n-k)\pi}{2}\right)\ln|V_k|\right). \end{aligned}$$

Setting $n := 4n + j$ and $H_j = \alpha^j c_1 + \bar{\alpha}^j c_2$, we obtain

$$|u_{4n+j}| = \exp\left(H_j + \sum_{k=0}^{4n+j-1}(\cos\left(\frac{(j-k)\pi}{2}\right))\ln|V_k|\right). \tag{2.8}$$

For $j = 0$, (2.8) becomes

$$|u_{4n}| = \exp(H_0)\exp(\ln|V_0| - \ln|V_2| + \ln|V_4| - \ln|V_6| + \cdots + \ln|V_{4n-4}| - \ln|V_{4n-2}|).$$

However, using (2.8) with $j = 1, n = 0$, $|u_0| = \exp(H_0)$. It can be shown that we do not need the absolute values, thus

$$u_{4n} = u_0\prod_{s=0}^{n-1}\frac{V_{4s}}{V_{4s+2}}.$$

Using (2.7), where $n := 2s$ and $i = 0$ for V_{4s}, and $n := 2s+1$ and $i = 0$ for V_{4s+2}, we have

$$\begin{aligned} V_{4s} = V_0\prod_{k_1=0}^{2s-1}A_{2k_1} + \sum_{l=0}^{2s-1}B_{2l}\prod_{k_2=l+1}^{2s-1}A_{2k_2} &= V_0\left(\prod_{k_1=0}^{2s-1}A_{2k_1} + \frac{1}{V_0}\sum_{l=0}^{2s-1}B_{2l}\prod_{k_2=l+1}^{2s-1}A_{2k_2}\right)\\ &= \frac{1}{u_0u_2}\left(\prod_{k_1=0}^{2s-1}A_{2k_1} + u_0u_2\sum_{l=0}^{2s-1}B_{2l}\prod_{k_2=l+1}^{2s-1}A_{2k_2}\right), \end{aligned}$$

and

$$\begin{aligned} V_{4s+2} = V_0\prod_{k_1=0}^{2s}A_{3k_1} + \sum_{l=0}^{2s}B_{3l}\prod_{k_2=l+1}^{2s}A_{3k_2} &= V_0\left(\prod_{k_1=0}^{2s}A_{2k_1} + \frac{1}{V_0}\sum_{l=0}^{2s}B_{2l}\prod_{k_2=l+1}^{2s}A_{2k_2}\right)\\ &= \frac{1}{u_0u_2}\left(\prod_{k_1=0}^{2s}A_{2k_1} + u_0u_2\sum_{l=0}^{2s}B_{2l}\prod_{k_2=l+1}^{2s}A_{2k_2}\right). \end{aligned}$$

Thus

$$u_{4n} = u_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} A_{2k_1} + u_0u_2 \sum_{l=0}^{2s-1} B_{2l} \prod_{k_2=l+1}^{2s-1} A_{2k_2}}{\prod_{k_1=0}^{2s} A_{2k_1} + u_0u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2}},$$

which implies that

$$x_{4n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{2k_1} + x_{-3}x_{-1} \sum_{l=0}^{2s-1} b_{2l} \prod_{k_2=l+1}^{2s-1} a_{2k_2}}{\prod_{k_1=0}^{2s} a_{2k_1} + x_{-3}x_{-1} \sum_{l=0}^{2s} b_{2l} \prod_{k_2=l+1}^{2s} a_{2k_2}}. \tag{2.9}$$

For $j = 1$, (2.8) becomes

$$|u_{4n+1}| = \exp(H_1)\exp(\ln|V_1| - \ln|V_3| + \ln|V_5| - \ln|V_7| + \ldots + \ln|V_{4n-3}| - \ln|V_{4n-1}|).$$

However, using (2.8) with $j = 1, n = 0$, $|u_1| = \exp(H_1)$. It can be shown that we do not need the absolute values, thus

$$u_{4n+1} = u_1 \prod_{s=0}^{n-1} \frac{V_{4s+1}}{V_{4s+3}}.$$

By (2.7), we have

$$\begin{aligned}
V_{4s+1} &= V_1 \prod_{k_1=0}^{2s-1} A_{2k_1+1} + \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1} \\
&= V_1 \left(\prod_{k_1=0}^{2s-1} A_{2k_1+1} + \frac{1}{V_1} \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2} \right) \\
&= \frac{1}{u_1u_3} \left(\prod_{k_1=0}^{2s-1} A_{2k_1+1} + u_1u_3 \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1} \right),
\end{aligned}$$

and similarly,

$$V_{4s+3} = \frac{1}{u_1u_5} \left(\prod_{k_1=0}^{2s} A_{2k_1+1} + u_1u_3 \sum_{l=0}^{2s} B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1} \right).$$

Now

$$u_{4n+1} = u_1 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} A_{2k_1+1} + u_1u_3 \sum_{l=0}^{2s-1} B_{2l+1} \prod_{k_2=l+1}^{2s-1} A_{2k_2+1}}{\prod_{k_1=0}^{2s} A_{2k_1+1} + u_1u_3 \sum_{l=0}^{2s} B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}},$$

which implies that

$$x_{4n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s-1} a_{2k_1+1} + x_{-2}x_0 \sum_{l=0}^{2s-1} b_{2l+1} \prod_{k_2=l+1}^{2s-1} a_{2k_2+1}}{\prod_{k_1=0}^{2s} a_{2k_1+1} + x_{-2}x_0 \sum_{l=0}^{2s} b_{2l+1} \prod_{k_2=l+1}^{2s} a_{2k_2+1}}. \tag{2.10}$$

For $j = 2$, we find that (2.8) becomes

$$|u_{4n+2}| = \exp(H_2)\exp(-\ln|V_0| + \ln|V_2| - \ln|V_4| + \cdots + \ln|V_{4n-2}| - \ln|V_{4n}|).$$

Similar to the earlier cases, setting $n = 0$ and $j = 2$ yields the equation $|u_2| = \exp(H_2)\frac{1}{|V_0|}$. So we have

$$u_{4n+2} = u_2 \prod_{s=0}^{n-1} \frac{V_{4s+2}}{V_{4s+4}}.$$

The expressions for V_{4s+2} and V_{4s+4} are obtained from (2.7), by setting $n = 2s+1, i = 0$ and $n = 2s+2, i = 0$, respectively. They are as follows:

$$\begin{aligned} V_{4s+2} = V_0 \prod_{k_1=0}^{2s} A_{2k_1} + \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} &= V_0 \left(\prod_{k_1=0}^{2s} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \right) \\ &= \frac{1}{u_0 u_2} \left(\prod_{k_1=0}^{2s} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2} \right), \end{aligned}$$

and

$$\begin{aligned} V_{4s+4} = V_0 \prod_{k_1=0}^{2s+1} A_{2k_1} + \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2} &= V_0 \left(\prod_{k_1=0}^{2s+1} A_{2k_1} + \frac{1}{V_0} \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s=1} A_{2k_2} \right) \\ &= \frac{1}{u_0 u_2} \left(\prod_{k_1=0}^{2s+1} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2} \right). \end{aligned}$$

Hence

$$u_{4n+2} = u_2 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s} B_{2l} \prod_{k_2=l+1}^{2s} A_{2k_2}}{\prod_{k_1=0}^{2s+1} A_{2k_1} + u_0 u_2 \sum_{l=0}^{2s+1} B_{2l} \prod_{k_2=l+1}^{2s+1} A_{2k_2}},$$

which gives

$$x_{4n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{2k_1} + x_{-3} x_{-1} \sum_{l=0}^{2s} b_{2l} \prod_{k_2=l+1}^{2s} a_{2k_2}}{\prod_{k_1=0}^{2s+1} a_{2k_1} + x_{-3} x_{-1} \sum_{l=0}^{2s+1} b_{2l} \prod_{k_2=l+1}^{2s+1} a_{2k_2}}. \tag{2.11}$$

For $j = 3$, (2.8) becomes

$$|u_{4n+3}| = \exp(H_3) \exp(-\ln|V_1| + \ln|V_3| - \ln|V_5| + \cdots + \ln|V_{4n-1}| - \ln|V_{4n+1}|).$$

Setting $n = 0$ and $j = 3$, we find that $|u_3| = \exp(H_3)\frac{1}{|V_1|}$. Hence $u_{4n+3} = u_3 \prod_{s=1}^{n-1} \frac{V_{4s+3}}{V_{4s+5}}$. Following a similar approach as was done in the earlier cases ($j = 0, 1, 2$), the reader can verify that

$$u_{4n+3} = u_3 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s} B_{2l+1} \prod_{k_2=l+1}^{2s} A_{2k_2+1}}{\prod_{k_1=0}^{2s+1} A_{2k_1+1} + u_1 u_3 \sum_{l=0}^{2s+1} B_{2l+1} \prod_{k_2=l+1}^{2s+1} A_{2k_2+1}}.$$

Thus

$$x_{4n} = x_0 \prod_{s=0}^{n-1} \frac{\prod_{k_1=0}^{2s} a_{2k_1+1} + x_{-2} x_0 \sum_{l=0}^{2s} b_{2l+1} \prod_{k_2=l+1}^{2s} a_{2k_2+1}}{\prod_{k_1=0}^{2s+1} a_{2k_1+1} + x_{-2} x_0 \sum_{l=0}^{2s+1} b_{2l+1} \prod_{k_2=l+1}^{2s+1} a_{2k_2+1}}. \tag{2.12}$$

Therefore, the solution $\{x_n\}$ to the equation

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1} x_{n-3}}$$

satisfies equations (2.9), (2.10), (2.11), and (2.12), as long as the denominators do not vanish.

3. The case when a_j and b_j are 2-periodic sequences

We assume that $\{a_j\}_{j\geqslant 0} = \{a_0, a_1, a_0, \ldots\}$ where $a_0 \neq a_1$, and $\{b_j\}_{j\geqslant 0} = \{b_0, b_1, b_0, b_1, \ldots\}$ where $b_0 \neq b_1$.

3.1. $a_0 \neq 1, a_1 \neq 1$

Then after substitution, (2.9), (2.10), (2.11), and (2.12) become

$$x_{4n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{a_0^{2s} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}, \qquad x_{4n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{a_1^{2s} + x_{-2}x_0 b_1\frac{1-a_1^{2s}}{1-a_1}}{a_1^{2s+1} + x_{-2}x_0 b_1\frac{1-a_1^{2s+1}}{1-a_1}},$$

$$x_{4n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+2}}{1-a_0}}, \qquad x_{4n} = x_0 \prod_{s=0}^{n-1} \frac{a_1^{2s+1} + x_{-2}x_0 b_1\frac{1-a_1^{2s+1}}{1-a_1}}{a_1^{2s+2} + x_{-2}x_0 b_1\frac{1-a_1^{2s+2}}{1-a_1}},$$

where

$$\prod_{i=1}^{2} \left(\frac{b_0(1-a_0^{2s+i})}{1-a_0} x_{-3}x_{-1} + a_0^{2s+i} \right) \left(\frac{b_1(1-a_1^{2s+i})}{1-a_1} x_{-2}x_0 + a_1^{2s+i} \right) \neq 0$$

for all $s = 1, 2, \ldots, n-1$.

4. The case when a_j and b_j are 1-periodic

In this case, replace a_1 and b_1, in the above section, by a_0 and b_0, respectively.

4.1. $a_0 \neq 1$

In this case, the solution equations are given by

$$x_{4n-3} = x_{-3} \prod_{s=0}^{n-1} \frac{a_0^{2s} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}, \qquad x_{4n-2} = x_{-2} \prod_{s=0}^{n-1} \frac{a_0^{2s} + x_{-2}x_0 b_0\frac{1-a_0^{2s}}{1-a_0}}{a_0^{2s+1} + x_{-2}x_0 b_0\frac{1-a_0^{2s+1}}{1-a_0}},$$

$$x_{4n-1} = x_{-1} \prod_{s=0}^{n-1} \frac{a_0^{2s+1} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2} + x_{-3}x_{-1}b_0\frac{1-a_0^{2s+2}}{1-a_0}}, \qquad x_{4n} = x_0 \prod_{s=0}^{n-1} \frac{a_0^{2s+1} + x_{-2}x_0 b_0\frac{1-a_0^{2s+1}}{1-a_0}}{a_0^{2s+2} + x_{-2}x_0 b_0\frac{1-a_0^{2s+2}}{1-a_0}},$$

where

$$\prod_{i=1}^{2} \left(\frac{b_0(1-a_0^{2s+i})}{1-a_0} x_{-3}x_{-1} + a_0^{2s+i} \right) \left(\frac{b_0(1-a_0^{2s+i})}{1-a_0} x_{-2}x_0 + a_0^{2s+i} \right) \neq 0$$

for all $s = 1, 2, \ldots, n-1$.

4.1.1. $a_0 = -1, b_0 = 1$

The solution is given by the following equations, and it appears in [9, Theorem 6].

$$x_{4n-3} = x_{-3}(-1 + x_{-3}x_{-1})^{-n}, \qquad x_{4n-2} = x_{-2}(-1 + x_{-2}x_0)^{-n},$$
$$x_{4n-1} = x_{-1}(-1 + x_{-3}x_{-1})^{n}, \qquad x_{4n} = x_0(-1 + x_{-2}x_0)^{n},$$

where $x_{-2}x_0 \neq 1$ and $x_{-3}x_{-1} \neq 1$.

4.1.2. $a_0 = -1, b_0 = -1$

The solution is given by the following equations, and it appears in [9, Theorem 9].

$$x_{4n-3} = (-1)^n x_{-3}(1 + x_{-3}x_{-1})^{-n}, \qquad x_{4n-2} = (-1)^n x_{-2}(1 + x_{-2}x_0)^{-n},$$
$$x_{4n-1} = (-1)^n x_{-1}(1 + x_{-3}x_{-1})^{n}, \qquad x_{4n} = (-1)^n x_0(1 + x_{-2}x_0)^{n},$$

where $x_{-2}x_0 \neq -1$ and $x_{-3}x_{-1} \neq -1$.

4.2. $a_0 = 1$

The solution is given by the following equations:

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1+2sb_0x_{-3}x_{-1}}{1+(2s+1)b_0x_{-3}x_{-1}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1+2sb_0x_{-2}x_0}{1+(2s+1)b_0x_{-2}x_0},$$
$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1+(2s+1)b_0x_{-3}x_{-1}}{1+(2s+2)b_0x_{-3}x_{-1}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1+(2s+1)b_0x_{-2}x_0}{1+(2s+2)b_0x_{-2}x_0},$$

where $jb_0x_{-3}x_{-1} \neq -1$ and $jb_0x_{-2}x_0 \neq -1$ for all $j = 1, 2, \ldots, 2n$.

4.2.1. $a_0 = 1, b_0 = 1$

The solution is given by the following equations, and it appears in [9, Theorem 1]:

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1+2sx_{-3}x_{-1}}{1+(2s+1)x_{-3}x_{-1}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1+2sx_{-2}x_0}{1+(2s+1)x_{-2}x_0},$$
$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1+(2s+1)x_{-3}x_{-1}}{1+(2s+2)x_{-3}x_{-1}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1+(2s+1)x_{-2}x_0}{1+(2s+2)x_{-2}x_0},$$

where $jx_{-3}x_{-1} \neq -1$ and $jx_{-2}x_0 \neq -1$ for all $j = 1, 2, \ldots, 2n$.

4.2.2. $a_0 = 1, b_0 = -1$

The solution is given by the following equations, and it appears in [9, Theorem 4]:

$$x_{4n-3} = x_{-3}\prod_{s=0}^{n-1}\frac{1-2sx_{-3}x_{-1}}{1-(2s+1)x_{-3}x_{-1}}, \qquad x_{4n-2} = x_{-2}\prod_{s=0}^{n-1}\frac{1-2sx_{-2}x_0}{1-(2s+1)x_{-2}x_0},$$
$$x_{4n-1} = x_{-1}\prod_{s=0}^{n-1}\frac{1-(2s+1)x_{-3}x_{-1}}{1-(2s+2)x_{-3}x_{-1}}, \qquad x_{4n} = x_0\prod_{s=0}^{n-1}\frac{1-(2s+1)x_{-2}x_0}{1-(2s+2)x_{-2}x_0},$$

where $jx_{-3}x_{-1} \neq 1$ and $jx_{-2}x_0 \neq 1$ for all $j = 1, 2, \ldots, 2n$.

5. Conclusion

We obtained two non-trivial symmetry generators of shifted equation and used one of these generators to obtain exact solutions to difference equations of the form

$$x_{n+1} = \frac{x_{n-3}}{a_n + b_n x_{n-1} x_{n-3}},$$

where (a_n) and (b_n) are arbitrary sequences of real numbers. In particular, our work generalized a result of Elsayed [9].

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