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New integral inequalities and their applications to convex functions with a continuous Caputo fractional derivative

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Abstract

We say that a function $f : [a, b] \to \mathbb{R}$ is (φ, δ)-Lipschitzian, where $\delta \geqslant 0$ and $\varphi : [0, \infty) \to [0, \infty)$, if

$$|f(x) - f(y)| \leqslant \varphi(|x - y|) + \delta, \quad (x, y) \in [a, b] \times [a, b].$$

In this work, some Hadamard's type inequalities are established for the class of (φ, δ)-Lipschitzian mappings. Moreover, some applications to convex functions with a continuous Caputo fractional derivative are also discussed.

Keywords: (φ, δ)-Lipschitzian, Hadamard's type inequalities, convex function, Caputo fractional derivative, fractional mean value theorem.

2010 MSC: 26D10, 35A23, 26A33, 26A51, 26A33.

1. Introduction

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function. Then the following inequality is known as Hadamard's inequality (see [15])

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which has several applications in different branches of mathematical analysis, including convex analysis, approximation theory, numerical analysis, optimization theory, etc. For details, we refer the reader to a nice monograph by Dragomir and Pearce [10]. Hadamard's inequality has been extended to various kinds of convexity such as quasi-convexity [18, 23], s-convexity [9], η-convexity [13], φ-convexity [14], h-convexity [20], m-convexity [6, 12], etc. For further details and examples, see [4, 5, 7, 8, 11, 16, 21, 22] and the references cited therein.

In [8], Dragomir et al. established the following Hadamard's type inequalities for L-Lipschitzian mappings.

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Theorem 1.1. *Let* $f : [a, b] \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, $a < b$, *be an* L*-Lipschitzian mapping,* $L > 0$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{L}{4}(b-a)$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{f(a)+f(b)}{2} \right| \leqslant \frac{L}{3}(b-a).$$

Recall that a function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be L-Lipschitzian if $|f(x) - f(y)| \leqslant L|x - y|$, $L > 0$, $x, y \in [a, b]$. Some applications of Theorem 1.1 related to arithmetic mean, geometric mean, harmonic mean, etc., are also discussed in [8].

In this paper, we introduce a class of (φ, δ)-Lipschitzian mappings, where $\delta \geqslant 0$ and $\varphi : [0, \infty) \to [0, \infty)$, and develop Hadamard's type inequalities for these mappings. We emphasize that our results generalize the ones obtained by Dragomir et al. [8] as L-Lipschitzian mappings appear as a special case of (φ, δ)-Lipschitzian mappings with $\varphi(s) = Ls, s \geqslant 0$ and $\delta = 0$. Moreover, using a fractional version of the mean value theorem (see [17]), we present some applications to convex functions having a continuous Caputo fractional derivative.

2. Hadamard's type inequalities for (φ, δ)-Lipschitzian mappings

Definition 2.1. A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be (φ, δ)-Lipschitzian iff $|f(x) - f(y)| \leqslant \varphi(|x - y|) + \delta$, $x, y \in [a, b]$, where $\varphi : [0, \infty) \to [0, \infty)$ and $\delta \geqslant 0$.

Let us begin with the following theorem containing two Hadamard's type inequalities.

Theorem 2.2. *Let* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be a* (φ, δ)*-Lipschitzian function, where* $\varphi : [0, \infty) \to [0, \infty)$ *is continuous and* $\delta \geqslant 0$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \int_0^1 \varphi\left(\frac{(b-a)|1-2t|}{2}\right) dt + \delta \tag{2.1}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{f(a)+f(b)}{2} \right| \leqslant 2\int_0^1 t\varphi((b-a)(1-t))\, dt + \delta. \tag{2.2}$$

Proof. Let $t \in [0, 1]$. Then, for all $u, v \in [a, b]$, we have

$$\begin{aligned} &|tf(u) + (1-t)f(v) - f(tu + (1-t)v)| \\ &\quad = |t(f(u) - f(tu + (1-t)v)) + (1-t)(f(v) - f(tu + (1-t)v))| \\ &\quad \leqslant t|f(u) - f(tu + (1-t)v)| + (1-t)|f(v) - f(tu + (1-t)v)| \\ &\quad \leqslant t\left(\varphi((1-t)|u-v|) + \delta\right) + (1-t)\left(\varphi(t|u-v|) + \delta\right). \end{aligned} \tag{2.3}$$

Taking $t = \frac{1}{2}$ in (2.3), we get

$$\left| \frac{f(u)+f(v)}{2} - f\left(\frac{u+v}{2}\right) \right| \leqslant \varphi\left(\frac{|u-v|}{2}\right) + \delta. \tag{2.4}$$

Taking $u = ta + (1-t)b$ and $v = tb + (1-t)a$ for $t \in [0, 1]$ in (2.4), we get

$$\left| \frac{f(ta + (1-t)b) + f(tb + (1-t)a)}{2} - f\left(\frac{a+b}{2}\right) \right| \leqslant \varphi\left(\frac{(b-a)|1-2t|}{2}\right) + \delta.$$

Integrating the above inequality with respect to the variable t over the interval $(0,1)$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leqslant \int_0^1 \varphi\left(\frac{(b-a)|1-2t|)}{2}\right)\,dt + \delta,$$

which proves (2.1).

Now, taking $u=a, v=b$ in (2.3), we obtain

$$\begin{aligned}|tf(a)+(1-t)f(b)-f(ta+(1-t)b)| \leqslant{}& t\left(\varphi((1-t)(b-a))+\delta\right)\\ &+(1-t)\left(\varphi(t(b-a))+\delta\right), \text{for} \quad \text{all} t\in[0,1],\end{aligned}$$

which, on integrating with respect to the variable t over $(0,1)$, yields

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \leqslant 2\int_0^1 t\varphi((b-a)(1-t))\,dt + \delta.$$

This establishes (2.2). □

Remark 2.3. Under the assumptions of Theorem 2.2, note that

$$f\in L^1(a,b).$$

Indeed, for all $x\in[a,b]$, we have

$$|f(x)| \leqslant |f(x)-f(a)|+|f(a)| \leqslant \varphi(x-a)+M,$$

where $M=\delta+|f(a)|$. Since φ is continuous, $\displaystyle\int_a^b |f(x)|\,dx<\infty$.

Now, suppose that all the assumptions of Theorem 2.2 are satisfied. Moreover, it is assumed that

$$f([a,b])\subseteq[a,b].$$

In this case, we can define the function $f^2:[a,b]\to\mathbb{R}$ by

$$f^2(x)=(f\circ f)(x)=f(f(x)),\quad x\in[a,b].$$

Further suppose that $\varphi:[0,\infty)\to[0,\infty)$ is a nondecreasing function. Then, for all $x,y\in[a,b]$, we have

$$\begin{aligned}|f^2(x)-f^2(y)| &= |f(f(x))-f(f(y))|\\ &\leqslant \varphi(|f(x)|-f(y)|)+\delta\\ &\leqslant \varphi\left(\varphi(|x-y|+\delta\right)+\delta.\end{aligned}$$

Therefore, f^2 is a $(\widetilde{\varphi},\delta)$-Lipschitzian function, where $\widetilde{\varphi}:[0,\infty)\to[0,\infty)$ is defined by

$$\widetilde{\varphi}(s)=\varphi\left(\varphi(s)+\delta\right),\quad s\geqslant 0. \tag{2.5}$$

Hence, from Theorem 2.2, we deduce the following result.

Corollary 2.4. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a (φ,δ)-Lipschitzian function, where $\varphi:[0,\infty)\to[0,\infty)$ is continuous and nondecreasing, and $\delta\geqslant 0$. Suppose that $f([a,b])\subseteq[a,b]$. Then*

$$\left|\frac{1}{b-a}\int_a^b f^2(x)\,dx - f^2\left(\frac{a+b}{2}\right)\right| \leqslant \int_0^1 \widetilde{\varphi}\left(\frac{(b-a)|1-2t|)}{2}\right)\,dt + \delta$$

and

$$\left|\frac{1}{b-a}\int_a^b f^2(x)\,dx - \frac{f^2(a)+f^2(b)}{2}\right| \leqslant 2\int_0^1 t\widetilde{\varphi}((b-a)(1-t))\,dt + \delta,$$

where $\widetilde{\varphi}:[0,\infty]\to[0,\infty)$ is defined by (2.5).

In the particular case, when $\varphi(s) = Ls^p$, $p \geqslant 0$, $L > 0$, we have the following result.

Corollary 2.5. *Let* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be such that*

$$|f(x) - f(y)| \leqslant L|x - y|^p + \delta,\ x, y \in [a, b],\ p,\ \delta \geqslant 0,\ L > 0.$$

Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{L(b-a)^p}{2^p(p+1)} + \delta \tag{2.6}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| \leqslant \frac{2L(b-a)^p}{(p+1)(p+2)} + \delta. \tag{2.7}$$

Proof. Under the assumptions of Corollary 2.5, the function f is (φ, δ)-Lipschitzian, where

$$\varphi(s) = Ls^p, \quad s \geqslant 0.$$

It is easy to compute that

$$\int_0^1 \varphi\left(\frac{(b-a)|1-2t|}{2}\right)\,dt + \delta = \frac{L(b-a)^p}{2^p(p+1)} + \delta$$

and

$$2\int_0^1 t\varphi((b-a)(1-t))\,dt + \delta = \frac{2L(b-a)^p}{(p+1)(p+2)} + \delta,$$

which, together with Theorem 2.2, yield the inequalities (2.6) and (2.7). □

Remark 2.6. Taking $p = 1$ and $\delta = 0$ in Corollary 2.5, we obtain the results given by Theorem 1.1.

In case we fix $\varphi(s) = L\exp(s)$, $L > 0$, we get the following result.

Corollary 2.7. *Let* $f : [a, b] \to \mathbb{R}$, $a, b \in \mathbb{R}$ *and that* $\delta \geqslant 0$, $L > 0$. *Suppose that*

$$|f(x) - f(y)| \leqslant L\exp(|x - y|) + \delta,\ \ x, y \in [a, b].$$

Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right) - 1\right) + \delta \tag{2.8}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2} \right| \leqslant \frac{2L}{(b-a)^2}\Big(\exp(b-a) - 1 - (b-a)\Big) + \delta. \tag{2.9}$$

Proof. Obviously the function f is (φ, δ)-Lipschitzian with $\varphi(s) = L\exp(s)$, $\quad s \geqslant 0$ and that

$$\int_0^1 \varphi\left(\frac{(b-a)|1-2t|}{2}\right)\,dt + \delta = \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right) - 1\right) + \delta$$

and

$$2\int_0^1 t\varphi((b-a)(1-t))\,dt + \delta = \frac{2L}{(b-a)^2}\Big(\exp(b-a) - 1 - (b-a)\Big) + \delta.$$

Using the above values in Theorem 2.2, we deduce inequalities (2.8) and (2.9). □

Letting $\varphi(s) = \dfrac{L}{s+1}$, $L > 0$, we have the following result.

Corollary 2.8. *For $\delta \geqslant 0$, $L > 0$, let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be such that*

$$|f(x) - f(y)| \leqslant \frac{L}{|x-y|+1} + \delta, \quad x, y \in [a, b].$$

Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \right| \leqslant \frac{2L}{b-a} \ln\left(1 + \frac{b-a}{2}\right) + \delta \tag{2.10}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{f(a)+f(b)}{2} \right| \leqslant \frac{2L}{b-a}\left(1 - \left(\frac{1}{b-a} + 1\right) \ln(1+b-a)\right) + \delta. \tag{2.11}$$

Proof. Under the assumptions of Corollary 2.8, observe that the function f is (φ, δ)-Lipschitzian with $\varphi(s) = \frac{L}{s+1}$, $s \geqslant 0$. In this case, we find that

$$\int_0^1 \varphi\left(\frac{(b-a)|1-2t|)}{2}\right) dt + \delta = \frac{2L}{b-a} \ln\left(1 + \frac{b-a}{2}\right) + \delta$$

and

$$2\int_0^1 t\varphi((b-a)(1-t))\, dt + \delta = \frac{2L}{b-a}\left(1 - \left(\frac{1}{b-a} + 1\right) \ln(1+b-a)\right) + \delta.$$

Therefore, by Theorem 2.2, we obtain the inequalities (2.10) and (2.11). □

3. The mapping H

Now, let us define the function

$$H(t) = \frac{1}{b-a} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx, \quad t \in [0, 1], \tag{3.1}$$

where $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a (φ, δ)-Lipschitzian function and $\varphi : [0, \infty) \to [0, \infty)$ is continuous. Note that the mapping $H : [0, 1] \to \mathbb{R}$ is a well-defined function (see Remark 2.3).

Now we establish some properties of the mapping H defined by (3.1) in the following theorem.

Theorem 3.1. *Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a (φ, δ)-Lipschitzian function, where $\varphi : [0, \infty) \to [0, \infty)$ is continuous, and $\delta \geqslant 0$. Then*

(i) *H is a (ϕ, δ)-Lipschitzian function, where $\phi : [0, \infty) \to [0, \infty)$ is defined by*

$$\phi(s) = \frac{1}{b-a} \int_a^b \varphi\left(s\left|x - \frac{a+b}{2}\right|\right) dx, \quad s \geqslant 0. \tag{3.2}$$

(ii) *For all $t \in [0, 1]$, we have*

$$\left| H(t) - f\left(\frac{a+b}{2}\right) \right| \leqslant \phi(t) + \delta.$$

(iii) *For all $t \in [0, 1]$, we have*

$$\left| H(t) - \frac{1}{b-a} \int_a^b f(x)\, dx \right| \leqslant \phi(1-t) + \delta.$$

(iv) *For all $t \in [0, 1]$, we have*

$$\left| H(t) - \frac{t}{b-a} \int_a^b f(x)\, dx - (1-t) f\left(\frac{a+b}{2}\right) \right| \leqslant t\phi(1-t) + (1-t)\phi(t) + \delta.$$

Proof. For all $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
|H(t_1)-H(t_2)| &\leqslant \frac{1}{b-a}\int_a^b \left| f\left(t_1x+(1-t_1)\frac{a+b}{2}\right) - f\left(t_2x+(1-t_2)\frac{a+b}{2}\right)\right| dx \\
&\leqslant \frac{1}{b-a}\int_a^b \varphi\left(\left|t_1x+(1-t_1)\frac{a+b}{2} - t_2x-(1-t_2)\frac{a+b}{2}\right|\right) dx+\delta \\
&= \frac{1}{b-a}\int_a^b \varphi\left(|t_1-t_2|\left|x-\frac{a+b}{2}\right|\right) dx+\delta \\
&= \phi(|t_1-t_2|)+\delta,
\end{aligned} \tag{3.3}$$

where $\phi : [0,\infty) \to [0,\infty)$ is defined by (3.2). Therefore, (i) is accomplished.

Taking $t_1 = t, t_2 = 0$ in (3.3), we obtain (ii), while substituting $t_1 = t, t_2 = 1$ in (3.3) yields (iii). Finally, (iv) follows by multiplying (ii) and (iii) respectively by $(1-t)$ and t and then adding the resulting expressions. □

Remark 3.2. Taking $t = 1$ in (ii) and using a change of variable, we obtain (2.1).

In the particular case, when $\varphi(s) = Ls^p$, $p \geqslant 0$, $L > 0$, we have the following result.

Corollary 3.3. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a (φ,δ)-Lipschitzian function, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, and $\delta \geqslant 0$. Further, it is assumed that*

$$|f(x)-f(y)| \leqslant L|x-y|^p+\delta, \quad x,y \in [a,b].$$

Then we have

(i) $|H(t_1)-H(t_2)| \leqslant \frac{L}{p+1}\left(\frac{b-a}{2}\right)^p |t_1-t_2|^p+\delta, \ \forall t_1,t_2 \in [0,1]$;

(ii) $\left|H(t)-f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{L}{p+1}\left(\frac{b-a}{2}\right)^p t^p+\delta, \ \forall t \in [0,1]$;

(iii) $\left|H(t)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{L}{p+1}\left(\frac{b-a}{2}\right)^p (1-t)^p+\delta, \ \forall t \in [0,1]$;

(iv) $\left|H(t)-\frac{t}{b-a}\int_a^b f(x)\,dx-(1-t)f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{L}{p+1}\left(\frac{b-a}{2}\right)^p \left(t(1-t)^p+(1-t)t^p\right)+\delta, \ \forall t \in [0,1]$.

Proof. In this case, a simple computation yields

$$\phi(s) = \frac{L}{p+1}\left(\frac{b-a}{2}\right)^p s^p, \quad s \geqslant 0.$$

Therefore, by Theorem 3.1, we deduce the desired result. □

Remark 3.4. Taking $p = 1$ and $\delta = 0$ in Corollary 3.3, we obtain the results given by Theorem 3.1 in [8].

Next we take $\varphi(s) = L\exp(s)$, $L > 0$ and present the following result.

Corollary 3.5. *Let $\varphi : [0,\infty) \to [0,\infty)$ be continuous, $\delta \geqslant 0$ and that $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a (φ,δ)-Lipschitzian function satisfying the condition:*

$$|f(x)-f(y)| \leqslant L\exp(|x-y|)+\delta, \quad (x,y) \in [a,b]\times[a,b].$$

Then, $\forall t_1,t_2,t \in [0,1]$, we have

(i) $|H(t_1)-H(t_2)| \leqslant \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right)-1\right)\exp(|t_1-t_2|)+\delta$;

(ii) $\left|H(t) - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right) - 1\right)\exp(t) + \delta;$

(iii) $\left|H(t) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right) - 1\right)\exp(1-t) + \delta;$

(iv) $\left|H(t) - \frac{t}{b-a}\int_a^b f(x)\,dx - (1-t)f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right) - 1\right)\left(t\exp(1-t) + (1-t)\exp(t)\right) + \delta.$

Proof. Using

$$\phi(s) = \frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right) - 1\right)\exp(s), \quad s \geqslant 0,$$

in Theorem 3.1, we deduce the desired result. □

In the following result, we fix $\varphi(s) = \dfrac{L}{s+1}$, $L > 0$.

Corollary 3.6. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be continuous,* $\delta \geqslant 0$. *Assume that* $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ *is a* (φ,δ)-*Lipschitzian function satisfying the condition:*

$$|f(x) - f(y)| \leqslant \frac{L}{|x-y|+1} + \delta, \quad (x,y) \in [a,b]\times[a,b].$$

Then the following inequalities hold:

(i) $|H(t_1) - H(t_2)| \leqslant \frac{2L}{(b-a)}\frac{\ln\left(1+\frac{(b-a)|t_1-t_2|}{2}\right)}{|t_1-t_2|} + \delta$, $\forall t_1, t_2 \in [0,1];$

(ii) $\left|H(t) - f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{2L}{(b-a)}\frac{\ln\left(1+\frac{(b-a)t}{2}\right)}{t} + \delta$, $\forall t \in (0,1];$

(iii) $\left|H(t) - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{2L}{(b-a)}\frac{\ln\left(1+\frac{(b-a)(1-t)}{2}\right)}{(1-t)} + \delta$, $\forall t \in [0,1);$

(iv) $\left|H(t) - \frac{t}{b-a}\int_a^b f(x)\,dx - (1-t)f\left(\frac{a+b}{2}\right)\right| \leqslant \frac{2L}{(b-a)}\left(\frac{t}{1-t}\ln\left(1+\frac{(b-a)(1-t)}{2}\right) + \frac{(1-t)}{t}\ln\left(1+\frac{(b-a)t}{2}\right)\right) + \delta$, $\forall t \in (0,1)$.

Proof. It is easy to compute that

$$\phi(s) = \begin{cases} \frac{2L}{(b-a)}\frac{\ln\left(1+\frac{(b-a)s}{2}\right)}{s} & \text{if} \quad s > 0, \\ L & \text{if} \quad s = 0. \end{cases}$$

Using the above value in Theorem 3.1, we obtain the desired result. □

Theorem 3.7. *Under the assumptions of Theorem 3.1, the following inequality holds:*

$$\left|\frac{f\left(tb + (1-t)\frac{(a+b)}{2}\right) + f\left(ta + (1-t)\frac{(a+b)}{2}\right)}{2} - H(t)\right| \leqslant 2\int_0^1 s\varphi\left(t(b-a)(1-s)\right)\,ds + \delta \tag{3.4}$$

for all $t \in [0,1]$.

Proof. For $t \in [0,1]$, we have

$$H(t) = \frac{1}{u-v}\int_v^u f(x)\,dx,$$

where $u = tb + (1-t)\frac{(a+b)}{2}$ and $v = ta + (1-t)\frac{(a+b)}{2}$. Then it follows by inequality (2.2) that

$$\left|\frac{f(u)+f(v)}{2}-H(t)\right| \leqslant 2\int_0^1 s\varphi((u-v)(1-s))\,ds+\delta,$$

which yields (3.4). □

In the particular case, when $\varphi(s)=Ls^p$, $p\geqslant 0$, $L>0$, we have the following result.

Corollary 3.8. *Under the assumptions of Corollary 3.3, we have*

$$\left|\frac{f\left(tb+(1-t)\frac{(a+b)}{2}\right)+f\left(ta+(1-t)\frac{(a+b)}{2}\right)}{2}-H(t)\right| \leqslant \frac{2L(b-a)^p}{(p+1)(p+2)}t^p+\delta \tag{3.5}$$

for all $t\in[0,1]$.

Proof. In this case, a simple computation yields

$$\int_0^1 s\varphi\left(t(b-a)(1-s)\right)\,ds=\frac{L(b-a)^p}{(p+1)(p+2)}t^p.$$

Applying Theorem 3.7, we get (3.5). □

The following result is related to the case: $\varphi(s)=L\exp(s)$, $L>0$.

Corollary 3.9. *Under the assumptions of Corollary 3.5, we have*

$$\left|\frac{f\left(tb+(1-t)\frac{(a+b)}{2}\right)+f\left(ta+(1-t)\frac{(a+b)}{2}\right)}{2}-H(t)\right| \leqslant \frac{2L}{t^2(b-a)^2}\Big(\exp(t(b-a))-1-t(b-a)\Big)+\delta \tag{3.6}$$

for all $t\in(0,1]$.

Proof. In this case, we have

$$\int_0^1 s\varphi\left(t(b-a)(1-s)\right)\,ds=\frac{L}{t^2(b-a)^2}\Big(\exp(t(b-a))-1-t(b-a)\Big).$$

Then, applying Theorem 3.7, we obtain (3.6). □

Next we take $\varphi(s)=\frac{L}{s+1}$, $L>0$ and prove the following result.

Corollary 3.10. *Under the assumptions of Corollary 3.6, we have*

$$\begin{aligned}&\left|\frac{f\left(tb+(1-t)\frac{(a+b)}{2}\right)+f\left(ta+(1-t)\frac{(a+b)}{2}\right)}{2}-H(t)\right| \leqslant \\ &\qquad \frac{2L}{t(b-a)}\left(1-\left(\frac{1}{t(b-a)}+1\right)\ln\left(1+t(b-a)\right)\right)+\delta\end{aligned} \tag{3.7}$$

for all $t\in(0,1]$.

Proof. By a simple computation, we find that

$$\int_0^1 s\varphi\left(t(b-a)(1-s)\right)\,ds=\frac{L}{t(b-a)}\left(1-\left(\frac{1}{t(b-a)}+1\right)\ln\left(1+t(b-a)\right)\right),$$

which, via Theorem 3.7, yields (3.7). □

4. The mapping F

Let us define the function

$$F(t) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(tx + (1-t)y\right)\, dx\, dy, \quad t \in [0,1],$$

where $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a (φ, δ)-Lipschitzian function, and $\varphi : [0,\infty) \to [0,\infty)$ is continuous. It is easy to see that $F : [0,1] \to \mathbb{R}$ is a well-defined function (see Remark 2.3).

We enlist some properties of the mapping F in the following theorem.

Theorem 4.1. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a (φ, δ)-Lipschitzian function, where $\varphi : [0,\infty) \to [0,\infty)$ is continuous, and $\delta \geqslant 0$. Then*

(i) *F is a (μ, δ)-Lipschitzian function, where $\mu : [0,\infty) \to [0,\infty)$ is defined by*

$$\mu(s) = \frac{1}{(b-a)^2} \int_a^b \int_a^b \varphi\left(s|x-y|\right)\, dx\, dy, \quad s \geqslant 0. \tag{4.1}$$

(ii) *For all $t \in [0,1]$, we have*

(a) $\left|F(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right)\, dx\, dy\right| \leqslant \mu\left(\frac{|2t-1|}{2}\right) + \delta;$

(b) $\left|F(t) - \frac{1}{b-a} \int_a^b f(x)\, dx\right| \leqslant \mu(t) + \delta;$

(c) $|F(t) - H(t)| \leqslant \phi(1-t) + \delta.$

Proof. For all $t_1, t_2 \in [0,1]$, we have

$$\begin{aligned} |F(t_1) - F(t_2)| &\leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b \left|f\left(t_1x + (1-t_1)y\right) - f\left(t_2x + (1-t_2)y\right)\right|\, dx\, dy \\ &\leqslant \frac{1}{(b-a)^2} \int_a^b \int_a^b \varphi\left(|t_1 - t_2||x-y|\right)\, dx\, dy + \delta \\ &= \mu(|t_1 - t_2|) + \delta, \end{aligned} \tag{4.2}$$

where $\mu : [0,\infty) \to [0,\infty)$ is defined by (4.1). Therefore, (i) is proved.

Taking $t_1 = 1/2$, $t_2 = t$ in (4.2), we obtain (ii)-(a), while setting $t_1 = 0$, $t_2 = t$ in (4.2), we obtain (ii)-(b). On the other hand, since f is (μ, δ)-Lipschitzian, we have

$$\left|f(tx + (1-t)y) - f\left(tx + (1-t)\frac{(a+b)}{2}\right)\right| \leqslant \varphi\left((1-t)\left|y - \frac{a+b}{2}\right|\right) + \delta \tag{4.3}$$

for all $t \in [0,1]$ and $x, y \in [a,b]$. Integrating (4.3) over the interval $[a,b]$ with respect to x and y, we obtain (ii)-(c). □

Next we take $\varphi(s) = Ls^p$, $p \geqslant 0$, $L > 0$ and accomplish the following result.

Corollary 4.2. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ with $p, \delta \geqslant 0$ be such that*

$$|f(x) - f(y)| \leqslant L|x-y|^p + \delta, \ L > 0, \ x, y \in [a,b].$$

Then, for all $t_1, t_2, t \in [0,1]$, we have

(i) $|F(t_1) - F(t_2)| \leqslant \frac{2L(b-a)^p}{(p+1)(p+2)}|t_1 - t_2|^p + \delta;$

(ii) $\left|F(t)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\,dx\,dy\right| \leqslant \frac{L(b-a)^p}{2^{p-1}(p+1)(p+2)}|2t-1|^p+\delta$;

(iii) $\left|F(t)-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leqslant \frac{2L(b-a)^p}{(p+1)(p+2)}t^p+\delta$;

(iv) $|F(t)-H(t)| \leqslant \frac{L}{(p+1)}\left(\frac{b-a}{2}\right)^p(1-t)^p+\delta$.

Proof. A simple computation yields

$$\mu(s)=\frac{2L(b-a)^p}{(p+1)(p+2)}s^p, \quad s\geqslant 0,$$

which, by applying Theorem 4.1, yields the desired result. □

Remark 4.3. Taking $p=1$ and $\delta=0$ in Corollary 4.2, we obtain the results given by Theorem 4.1 in [8].

We develop the next result for $\varphi(s)=L\exp(s)$, $L>0$.

Corollary 4.4. *With $\delta\geqslant 0$, let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be such that*

$$|f(x)-f(y)|\leqslant L\exp(|x-y|)+\delta,\ x,y\in[a,b],\ L>0.$$

Then we have

(i) *For all $(t_1,t_2)\in[0,1]\times[0,1]$, $t_1\neq t_2$, we have*

$$|F(t_1)-F(t_2)|\leqslant\frac{2L}{(b-a)^2}\frac{\left(\exp(|t_1-t_2|(b-a))-|t_1-t_2|(b-a)-1\right)}{(t_1-t_2)^2}+\delta.$$

(ii) *For all $t\in[0,1]$, $t\neq 1/2$, we have*

$$\left|F(t)-\frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)\,dx\,dy\right|\leqslant\frac{8L}{(b-a)^2}\frac{\left(\exp\left(\frac{|2t-1|(b-a)}{2}\right)-\frac{|2t-1|(b-a)}{2}-1\right)}{(2t-1)^2}+\delta.$$

(iii) *For all $t\in(0,1]$, we have*

$$\left|F(t)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leqslant\frac{2L}{(b-a)^2}\frac{\left(\exp(t(b-a))-t(b-a)-1\right)}{t^2}+\delta.$$

(iv) *For all $t\in[0,1]$, we have*

$$|F(t)-H(t)|\leqslant\frac{2L}{b-a}\left(\exp\left(\frac{b-a}{2}\right)-1\right)\exp(1-t)+\delta.$$

Proof. In this case, from (4.1), we find that

$$\mu(s)=\begin{cases}\frac{2L\left(\exp(s(b-a))-s(b-a)-1\right)}{s^2(b-a)^2} & \text{if} \quad s>0,\\ L & \text{if} \quad s=0.\end{cases}$$

Then, via Theorem 4.1, we get the desired result. □

Next, we let $\varphi(s) = \dfrac{L}{s+1}$, $L > 0$ and establish the following result.

Corollary 4.5. *Let* $\delta \geqslant 0$. *Assume that* $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *satisfies the condition:*

$$|f(x) - f(y)| \leqslant \frac{L}{|x-y|+1} + \delta, \ \ x, y \in [a, b], \ L > 0.$$

Then

(i) *For all* $t_1, t_2 \in [0, 1]$ *with* $t_1 \neq t_2$, *we have*

$$|F(t_1) - F(t_2)| \leqslant \frac{2L\Big((|t_1 - t_2|(b-a) + 1) \ln(1 + |t_1 - t_2|(b-a)) - |t_1 - t_2|(b-a) \Big)}{(b-a)^2 (t_1 - t_2)^2} + \delta.$$

(ii) *For all* $t \in [0, 1]$, $t \neq 1/2$, *we have*

$$\begin{aligned} &\left| F(t) - \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) \, dx \, dy \right| \\ &\qquad \leqslant \frac{8L}{(b-a)^2} \frac{\left(\frac{|2t-1|(b-a)}{2} + 1\right) \ln\left(1 + \frac{|2t-1|(b-a)}{2}\right) - \frac{|2t-1|(b-a)}{2}}{(2t-1)^2}. \end{aligned}$$

(iii) *For all* $t \in (0, 1]$, *we have*

$$\left| F(t) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leqslant \frac{2L\Big((t(b-a) + 1) \ln(1 + t(b-a)) - t(b-a) \Big)}{(b-a)^2 t^2} + \delta.$$

(iv) *For all* $t \in [0, 1)$, *we have*

$$|F(t) - H(t)| \leqslant \frac{2L}{(b-a)} \frac{\ln\left(1 + \frac{(b-a)(1-t)}{2}\right)}{(1-t)}.$$

Proof. Again, using (4.1), we obtain

$$\mu(s) = \begin{cases} \frac{2L\left(s(b-a)\ln(1+s(b-a)) + \ln(1+s(b-a)) - s(b-a) \right)}{(b-a)^2 s^2} & \text{if} \quad s > 0, \\ L & \text{if} \quad s = 0. \end{cases}$$

As before, we get the desired result by means of Theorem 4.1. □

5. Applications to convex functions with a continuous Caputo fractional derivative

Recently, fractional calculus received a great attention form many researchers in different disciplines. In fact, it was discovered that in many situations, physical problems can be modeled more adequately using fractional derivatives rather than ordinary derivatives. For more details on fractional calculus and its applications, we refer the reader to [1–3, 19, 24], and the references therein.

In this section, using a fractional version of the mean value theorem, we obtain Hadamard's type inequalities for convex functions with a continuous Caputo fractional derivative. For more details on fractional calculus, we refer the reader to [19].

Definition 5.1. Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a given function. The Caputo fractional derivative of f of order $\alpha \in (0, 1]$ is given by

$$(D_a^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t)}{(x-t)^\alpha}\, dt, \quad x \in [a, b],$$

provided the above integral exists.

Now we state the fractional version of the mean value theorem.

Lemma 5.2 ([17]). *Suppose that* $f \in C[a, b]$ *and* $D_a^\alpha f \in C(a, b]$, *for some* $\alpha \in (0, 1]$. *Then we have*

$$f(x) = f(a) + \frac{1}{\Gamma(\alpha+1)} (D_a^\alpha f)(c)(x-a)^\alpha, \quad x \in [a, b],$$

where $c \in [a, x]$.

Lemma 5.3. *Let* $\alpha \in (0, 1]$, *and let* $a, b \in \mathbb{R}$ *with* $0 < a < b$. *Then*

$$|x^\alpha - y^\alpha| \leqslant \alpha a^{\alpha-1} |x - y|, \quad (x, y) \in [a, b] \times [a, b].$$

Proof. Let

$$h(x) = x^\alpha, \quad x \in [a, b].$$

By the classical mean value theorem, we have

$$h(x) - h(y) = (x - y) h'(c), \quad x, y \in [a, b],$$

where c is between x and y. Therefore,

$$|x^\alpha - y^\alpha| = \alpha |x - y| c^{\alpha-1} \leqslant \alpha a^{\alpha-1} |x - y|, \quad x, y \in [a, b],$$

which proves the desired inequality. □

As an application of Theorem 2.2, we have the following Hadamard's type inequalities for convex functions with a continuous Caputo fractional derivative.

Theorem 5.4. *Let* $f : [0, \infty) \to \mathbb{R}$ *be a convex function, and let* $a, b \in \mathbb{R}$ *with* $0 < a < b$. *Suppose that* $f \in C[0, b]$ *and* $D_0^\alpha f \in C(0, b]$, *for some* $\alpha \in (0, 1]$. *Then*

$$0 \leqslant \frac{1}{b-a} \int_a^b f(x)\, dx - f\left(\frac{a+b}{2}\right) \leqslant \frac{M}{4\Gamma(\alpha+1)} \left[\alpha a^{\alpha-1}(b-a) + 8b^\alpha\right]$$

and

$$0 \leqslant \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx \leqslant \frac{M}{3\Gamma(\alpha+1)} \left[\alpha a^{\alpha-1}(b-a) + 6b^\alpha\right],$$

where $M = \max_{a \leqslant s \leqslant b} |(D_0^\alpha f)(s)|$.

Proof. For all $x, y \in [a, b]$, using Lemma 5.2, we have

$$f(x) = f(0) + \frac{(D_0^\alpha f)(c_1)}{\Gamma(\alpha+1)} x^\alpha,$$

$$f(y) = f(0) + \frac{(D_0^\alpha f)(c_2)}{\Gamma(\alpha+1)} y^\alpha,$$

where $c_1 \in [0, x]$ and $c_2 \in [0, y]$. Therefore, for all $x, y \in [a, b]$, we have

$$\begin{aligned}|f(x) - f(y)| &= \frac{1}{\Gamma(\alpha+1)} |(D_0^\alpha f)(c_1) x^\alpha - (D_0^\alpha f)(c_2) y^\alpha| \\ &= \frac{1}{\Gamma(\alpha+1)} |(D_0^\alpha f)(c_1)(x^\alpha - y^\alpha) + ((D_0^\alpha f)(c_1) - (D_0^\alpha f)(c_2)) y^\alpha| \\ &\leqslant \frac{1}{\Gamma(\alpha+1)} |(D_0^\alpha f)(c_1)||x^\alpha - y^\alpha| + \frac{|(D_0^\alpha f)(c_1) - (D_0^\alpha f)(c_2)|}{\Gamma(\alpha+1)} y^\alpha \\ &\leqslant \frac{M}{\Gamma(\alpha+1)} |x^\alpha - y^\alpha| + \frac{2Mb^\alpha}{\Gamma(\alpha+1)}.\end{aligned}$$

By lemma 5.3, we obtain

$$|f(x) - f(y)| \leqslant \frac{\alpha a^{\alpha-1} M}{\Gamma(\alpha+1)} |x - y| + \frac{2Mb^\alpha}{\Gamma(\alpha+1)} \quad x, y \in [a, b].$$

Hence, f is a (φ, δ)-Lipschitzian function, with

$$\varphi(s) = \frac{\alpha a^{\alpha-1} M}{\Gamma(\alpha+1)} s, \quad s \geqslant 0$$

and

$$\delta = \frac{2Mb^\alpha}{\Gamma(\alpha+1)}.$$

Therefore, by Theorem 2.2 (or Corollary 2.5), we obtain the desired inequalities. □

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