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Approximation of solutions to a general system of variational inclusions in Banach spaces and applications

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Hongbo Liu, Qiang Long, Yi Li*

School of Science, Southwest University of Science and Technology, Mianyang, Sichuan 621010, China.

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Abstract

In this paper, a general system of variational inclusions in Banach Spaces is introduced. An iterative method for finding solutions of a general system of variational inclusions with inverse-strongly accretive mappings and common set of fixed points for a λ-strict pseudocontraction is established. Under certain conditions, by forward-backward splitting method, we prove strong convergence theorems in uniformly convex and 2-uniformly smooth Banach spaces. The results presented in the paper improve and extend various results in the existing literatures. Moreover, some applications to monotone variational inequality problem and convex minimization problem are presented.

Keywords: General system of variational inclusions, forward-backward splitting method, 2-uniformly smooth Banach spaces, resolvent operator, strictly pseudocontractive.

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1. Introduction

Let E be a real Banach space. We study the following variational inclusion problem: find $x^* \in E$ such that

$$0 \in Ax^* + Mx^*, \tag{1.1}$$

where $A : E \to E$ is an operator and $M : E \to 2^E$ is a set-valued operator. This problem includes, as special cases, convex programming, variational inequalities, split feasibility problem, and minimization problem. To be more precise, some concrete problems in machine learning, image processing, and linear inverse problem can be modeled mathematically as this form (1.1) (see examples in [4]).

A classical method for solving the problem (1.1) is the forward-backward splitting method [8, 10, 16] which is defined by the following manner: for any fixed $x_1 \in E$ and for $r > 0$,

$$x_{n+1} = (I + rM)^{-1}(x_n - rAx_n), \quad \forall n > 0.$$

We see that each step of the iteration involves only with A as the forward step and M as the backward

*Corresponding author
Email addresses: liuhongbo@swust.edu.cn (Hongbo Liu), longqiang@swust.edu.cn (Qiang Long), liyi@swust.edu.cn (Yi Li)

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step, but not the sum of M. In fact, this method includes, in particular, the proximal point algorithm [5] and the gradient method.

In 2015, Cholamjiak [6] introduced the following Halpern-type forward-backward method: $x_1 \in X$ and

$$x_{n+1} = \alpha_n u + \lambda_n x_n + \delta_n J_{r_n}^M (x_n - r_n A x_n) + e_n, \quad n \geqslant 1, \tag{1.2}$$

where $J_{r_n}^M = (I + r_n M)^{-1}$, X is a uniformly convex and q-uniformly smooth Banach space, $A : X \to X$ and $M : X \to 2^X$ are nonlinear mappings such that $\Omega := (A + M)^{-1} \neq \emptyset$. He prove that sequence $\{x_n\}$ generated by (1.2) strongly converges to a zero point of the sum of A and M under some appropriate conditions. There have been many works concerning the problem of finding zero points of the sum of two monotone operators (in Hilbert spaces) and accretive operators (in Banach spaces) (see[7, 15]).

In 2010, Qin et al. [11] considered the following system of variational inclusions. Find $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} 0 \in x^* - y^* + \rho_1 (A_1 y^* + M_1 x^*), \\ 0 \in y^* - x^* + \rho_2 (A_2 x^* + M_2 y^*), \end{cases} \tag{1.3}$$

where $A_i : X \to X$ and $M_i : X \to 2^X$ are nonlinear mappings for each $i = 1, 2$.

Obviously, problem (1.1) is special case of problem (1.3).

Next, we consider a general system of variational inclusions.

Find $(u_1, u_2, \ldots, u_l) \in E \times E \times \cdots \times E := E^l$ such that

$$\begin{cases} 0 \in u_1 - u_2 + \rho_1 (A_1 u_2 + M_1 u_1), \\ 0 \in u_2 - u_3 + \rho_2 (A_2 u_3 + M_2 u_2), \\ \quad \vdots \\ 0 \in u_{l-1} - u_l + \rho_{l-1} (A_{l-1} u_l + M_{l-1} u_{l-1}), \\ 0 \in u_l - u_1 + \rho_l (A_l u_1 + M_l u_l), \end{cases} \tag{1.4}$$

where $A_i : E \to E$ and $M_i : E \to 2^E$ are nonlinear mappings for each $i = 1, 2, \ldots, l$.

Obviously, problem (1.1) and (1.3) are special cases of problem (1.4).

Qin et al. [11] introduced an iterative method for finding common elements of the set of solutions to a general system of variational inclusions with inverse-strongly accretive mappings and common set of fixed points for a λ-strict pseudocontraction. Strong convergence theorems are established in uniformly convex and 2-uniformly smooth Banach spaces.

Theorem 1.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant k. Let $M_i : E \to 2^E$ be a maximal monotone mapping and $A_i : E \to E$ be a γ_i-inverse-strongly accretive mapping, respectively, for each $i = 1, 2$. Let $T : E \to E$ be a λ-strict pseudocontraction with a fixed point. Define a mapping S by $Sx = (1 - \frac{\lambda}{k^2})x + \frac{\lambda}{k^2} Tx$ for all $x \in E$. Assume that $\Omega = F(T) \cap F(Q)$, where Q is defined as Lemma 2.11. For an arbitrary initial point $x_1 = u \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_{\rho_2}^{M_2} (x_n - \rho_2 A_2 x_n), \\ y_n = J_{\rho_1}^{M_1} (z_n - \rho_1 A_1 z_n), \\ x_{n+1} = \alpha_n x_n + \beta_n x_n + (1 - \alpha_n - \beta_n)(\mu S x_n + (1 - \mu) y_n), \quad n \geqslant 0, \end{cases}$$

where $\mu \in (0, 1)$, $\rho_1 \in (0, \frac{\gamma_1}{k^2})$, $\rho_2 \in (0, \frac{\gamma_2}{k^2})$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. If $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$, *then* $\{x_n\}$ *converges strongly to* $x^* = P_\Omega u$, *where* P_Ω *is the sunny nonexpansive retraction from* E *onto* Ω, *and* (x^*, y^*) *is a solution to problem* (1.3).

In this paper, motivated by Qin et al. [11], Takahashi et al. [15], Chang et al. [4], and Combettes et al. [7], a general system of variational inclusions in Banach Spaces is introduced. A relaxed extragradient-type iterative method for finding solutions of a general system of variational inclusions with inverse-strongly accretive mappings and common set of fixed points for a λ-strict pseudocontraction is established. Under certain conditions, by forward-backward splitting method, we prove strong convergence theorems in uniformly convex and 2-uniformly smooth Banach spaces. The results presented in the paper extend and improve some recent results announced in the current literatures. Moreover, some applications to monotone variational inequality problem and convex minimization problem are presented.

2. Preliminaries

In order to prove the main results of the paper, we need the following basic concepts and lemmas.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geqslant \epsilon \quad \text{implies} \quad \|\frac{x+y}{2}\| \leqslant 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space E is said to be smooth if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for all $x, y \in U$. The norm of E is said to be Fréchet differentiable if, for any $x \in U$, the above limit is attained uniformly for all $y \in U$. The modulus of smoothness of E is defined by

$$\rho(\tau) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in E, \quad \|x\| = 1, \|y\| = \tau\right\},$$

where $\rho : [0, +\infty) \to [0, +\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leqslant 2$. A Banach space E is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leqslant c\tau^q$ for all $\tau > 0$.

In what follows, we always assume that E is a uniformly convex and q-uniformly smooth Banach space for some $q \in (1, 2]$.

Recall that the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{j_q(x) \in E^* : \langle j_q(x), x\rangle = \|x\|.\|j_q(x)\|, \|j_q(x)\| = \|x\|^{q-1}\right\}, \tag{2.1}$$

and the following subdifferential inequality holds: for any $x, y \in E$,

$$\|x + y\|^q \leqslant \|x\|^q + q\langle y, j_q(x + y)\rangle, \quad j_q(x + y) \in J_q(x + y).$$

In particular, $J = J_2$ is called the normalized duality mapping. If E is a Hilbert space then J is the identity mapping I.

Next, we assume that E is a smooth Banach space. Let T be a mapping from E into itself. In this paper, we use $F(T)$ to denote the set of fixed points of the mapping T.

Recall that the mapping T is said to be nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in E.$$

The mapping T is said to be λ-strictly pseudocontractive if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, J(x - y)\rangle \leqslant \|x - y\|^2 - \lambda\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in E.$$

Recall that an operator A of E into itself is said to be accretive if

$$\langle Ax - Ay, J(x-y)\rangle \geqslant 0, \quad \forall x, y \in E, \tag{2.2}$$

and, for any $\alpha > 0$, an operator A of E into itself is said to be α-inverse strongly accretive if

$$\langle Ax - Ay, J(x-y)\rangle \geqslant \alpha\|Ax - Ay\|, \quad \forall x, y \in E.$$

Definition 2.1 ([11, page 4]). Let D be a subset of C and P a mapping of C into D. Then P is said to be sunny if

$$P(Px + t(x - Px)) = Px,$$

whenever $Px + t(x - Px) \in C$ for $x \in C$ and $t \geqslant 0$. A mapping P of C into itself is called a retraction if $P^2 = P$. If a mapping P of C into itself is a retraction, then $Pz = z$ for all $z \in R(P)$, where $R(P)$ is the range of P. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

In 2006, Aoyama et al. [1] considered the following problem.
Find $x^* \in C$ such that

$$\langle Ax^*, J(y - x^*)\rangle \geqslant 0, \quad \forall y \in C. \tag{2.3}$$

They proved that the variational inequality (2.3) is equivalent to a fixed point problem, that is, the element $x^* \in C$ is a solution of the variational inequality (2.3) if and only if $x^* \in C$ satisfies the following equation:

$$x^* = P_C(x^* - \lambda Ax^*),$$

where $\lambda > 0$ is a constant and P_C is a sunny nonexpansive retraction from E onto C.

Lemma 2.2 ([12]). *Let E be a smooth Banach space and C a nonempty subset of E. Let $P : E \to C$ be a retraction and J the normalized duality mapping on E. Then the following are equivalent:*

(1) *P is sunny and nonexpansive;*
(2) *$\langle x - Px, J(y - Px)\rangle \leqslant 0$ for all $x \in E$, $y \in C$;*

Lemma 2.3 ([9]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and T a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then the set $F(T)$ is a sunny nonexpansive retract of C.*

Definition 2.4 ([20]). Let $M : E \to 2^E$ be a multivalued maximal accretive mapping. The single valued mapping $J^M_\rho : E \to E$ defined by

$$J^M_\rho x = (I + \rho M)^{-1}x, \quad \forall x \in E$$

is called the resolvent operator associated with M, where ρ is any positive number and I is the identity mapping.

Lemmas 2.5-2.6 can be obtained from Zhang [20], see also Aoyama et al. [2].

Lemma 2.5 ([2, 20]). *The resolvent operator J^M_ρ associated with M is single valued and nonexpansive for all $\rho > 0$.*

Lemma 2.6 ([2, 20]). *$u \in E$ is a solution of variational inclusion (1.1) if and only if $u = J^M_\rho(u - \rho Au)$ for all $\rho > 0$, that is*

$$VI(E, A, M) = F(J^M_\rho(I - \rho A)),$$

where $VI(E, A, M)$ denotes the set of solutions to problem (1.1).

In order to prove the main results, we need the following lemmas.

Lemma 2.7 ([14]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \geqslant 1$ and $\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leqslant 0$, then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.8 ([17]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 2.9 ([21]). *Let E be a real 2-uniformly smooth Banach space and $T : E \to E$ a λ-strict pseudocontraction. Then $S := (1-\frac{\lambda}{K^2})I + \frac{\lambda}{K^2}T$ is nonexpansive and $F(T) = F(S)$.*

Lemma 2.10 ([11, Lemma 1.8]). *Let E be a strictly convex Banach space. Let T_1 and T_2 be two nonexpansive mappings from E into itself with a common fixed point. Define a mapping $S : E \to E$ by*

$$Sx = \lambda T_1 x + (1-\lambda)T_2 x, \quad x \in E,$$

where λ is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1)\bigcap F(T_2)$.

Lemma 2.11. *Let $A_i : E \to E$ and $M_i : E \to 2^E$ be nonlinear mappings for each $i = 1, 2, \dots, l$. For any $(u_1, u_2, \dots, u_l) \in E \times E \times \cdots \times E := E^l$, $(u_1, u_2, \cdots, u_l)$ is a solution of problem (1.4) if and only if u_1 is a fixed point of the mapping Q defined by*

$$Q(x) = T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_l}^{(A_l,M_l)} x,$$

where $T_{\rho_i}^{(A_i,M_i)} := J_{\rho_i}^{M_i}(I - \rho_i A_i)$.

Proof. note that

$$\begin{cases} 0 \in u_1 - u_2 + \rho_1(A_1u_2 + M_1u_1), \\ 0 \in u_2 - u_3 + \rho_2(A_2u_3 + M_2u_2), \\ \quad\vdots \\ 0 \in u_{l-1} - u_l + \rho_{l-1}(A_{l-1}u_l + M_{l-1}u_{l-1}), \\ 0 \in u_l - u_1 + \rho_l(A_lu_1 + M_lu_l), \end{cases} \Leftrightarrow \begin{cases} u_1 = T_{\rho_1}^{(A_1,M_1)}u_2, \\ u_2 = T_{\rho_2}^{(A_2,M_2)}u_3, \\ \quad\vdots \\ u_{l-1} = T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})}u_l, \\ u_l = T_{\rho_l}^{(A_l,M_l)}u_1, \end{cases}$$

$$\Leftrightarrow$$

$$\begin{aligned} Q(u_1) &= T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})} \circ T_{\rho_l}^{(A_l,M_l)} u_1 \\ &= T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})} u_l \\ &\;\;\vdots \\ &= T_{\rho_1}^{(A_1,M_1)} u_2 = u_1. \end{aligned}$$

This completes the proof. □

Lemma 2.12 ([18, Lemma 2.1]). *Let $\{a_n\}$ be sequences of nonnegative numbers such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n, \quad \forall n \geqslant 1,$$

where sequences $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy following properties

(1) $\{\gamma_n\} \subset (0,1)$ *and* $\{\delta_n\} \subset \mathbb{R}$;
(2) $\sum_{n=1}^{+\infty} \gamma_n = +\infty$;
(3) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leqslant 0$ *or* $\sum_{n=1}^{+\infty} |\delta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Now, we are ready to give our main results in this paper.

Theorem 3.1. *Let* E *be a uniformly convex and* 2*-uniformly smooth Banach space with the smooth constant* K*. Let* $M_i : E \to 2^E$ *be a maximal monotone mapping,* $A_i : E \to E$ *a* γ_i*-inverse-strongly accretive mapping and* $T_{\rho_i}^{(A_i,M_i)} := J_{\rho_i}^{M_i}(I - \rho_i A_i)$*, respectively, for each* $i = 1, 2, \ldots, l$*. Let* $Q := T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_l}^{(A_l,M_l)}$*. Let* $T : E \to E$ *be a* λ*-strict pseudocontraction with a fixed point. Define a mapping* S *by*

$$Sx = (1 - \frac{\lambda}{K^2})x + \frac{\lambda}{K^2}Tx, \quad \forall x \in E.$$

Assume that $\mathcal{F} = F(T) \bigcap F(Q) \neq \emptyset$*, for an arbitrary initial point* $u \in E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})} \circ T_{\rho_l}^{(A_l,M_l)} x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)(\mu S x_n + (1 - \mu) y_n), \quad n \geqslant 0, \end{cases}$$

where $\mu \in (0, 1)$*,* $\rho_i \in (0, \frac{\gamma_i}{K^2})$ $(i = 1, 2, \ldots, l)$*, and sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following conditions:*

(i) $\{\alpha_n\} \subset (0, 1)$*,* $\lim_{n\to\infty} \alpha_n = 0$*, and* $\sum_{n=1}^{+\infty} \alpha_n = +\infty$*;*
(ii) $\{\beta_n\} \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$*,*

then $\{x_n\}$ *converges strongly to* $u_1^* = P_{\mathcal{F}}u$*, where* $P_{\mathcal{F}}$ *is the sunny nonexpansive retraction from* E *onto* $\mathcal{F}$ *and* $(u_1^*, u_2^*, \cdots, u_l^*) \in E^l$*, where*

$$\begin{cases} u_1^* = T_{\rho_1}^{(A_1,M_1)} u_2^*, \\ u_2^* = T_{\rho_2}^{(A_2,M_2)} u_3^*, \\ \quad \vdots \\ u_{l-1}^* = T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})} u_l^*, \\ u_l^* = T_{\rho_l}^{(A_l,M_l)} u_1^* \end{cases}$$

is a solution to problem (1.4)*.*

Proof.

(I). We prove that the mappings $T_{\rho_i}^{(A_i,M_i)} := J_{\rho_i}^{M_i}(I - \rho_i A_i)$ for all $i = 1, 2, \ldots, l$ are nonexpansive.
In fact, for any $x, y \in E$, by Lemma 2.8 and the condition $\rho_i \in (0, \frac{\gamma_i}{K^2})$, we have that

$$\begin{aligned} \|(I - \rho_i A_i)x - (I - \rho_i A_i)y\|^2 &= \|(x - y) - \rho_i(A_i x - A_i y\|^2 \\ &\leqslant \|(x - y)\|^2 - 2\rho_i \langle A_i x - A_i y, J(x - y)\rangle + 2K^2\rho_i^2 \|A_i x - A_i y\|^2 \\ &\leqslant \|(x - y)\|^2 - 2\rho_i\gamma_i \|A_i x - A_i y\|^2 + 2K^2\rho_i^2 \|A_i x - A_i y\|^2 \\ &= \|(x - y)\|^2 - 2\rho_i(\gamma_i - K^2\rho_i)\|A_i x - A_i y\|^2 \\ &\leqslant \|(x - y)\|^2, \end{aligned}$$

which implies the mappings $I - \rho_i A_i$ for all $i = 1, 2, \ldots, l$ are nonexpansive. By Lemma 2.5, we have that the mappings $T_{\rho_i}^{(A_i,M_i)} := J_{\rho_i}^{M_i}(I - \rho_i A_i)$ for all $i = 1, 2, \ldots, l$ are nonexpansive.

(II). We show that the sequence $\{x_n\}$ is bounded.
First, taking $p \in \mathcal{F}$, that is $p \in F(Q)$, one has

$$p = T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})} \circ T_{\rho_l}^{(A_l,M_l)} p.$$

$$\begin{aligned}\|y_n-p\| &= \|T_{\rho_1}^{(A_1,M_1)}\circ T_{\rho_2}^{(A_2,M_2)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}x_n - T_{\rho_1}^{(A_1,M_1)}\circ T_{\rho_2}^{(A_2,M_2)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}p\|\\ &\leqslant \|T_{\rho_2}^{(A_2,M_2)}\circ T_{\rho_3}^{(A_3,M_3)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}x_n - T_{\rho_1}^{(A_2,M_2)}\circ T_{\rho_3}^{(A_3,M_3)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}p\|\\ &\ \ \vdots\\ &\leqslant \|T_{\rho_l}^{(A_l,M_l)}x_n - T_{\rho_l}^{(A_l,M_l)}p\|\\ &\leqslant \|x_n-p\|.\end{aligned}\tag{3.1}$$

Letting $z_n = \mu Sx_n + (1-\mu)y_n$. It follows from Lemma 2.9 that S is nonexpansive and $p \in F(T)\bigcap F(S)$. From (3.1), we have

$$\begin{aligned}\|z_n-p\| &= \|\mu Sx_n + (1-\mu)y_n - p\|\\ &= \|\mu(Sx_n - Sp) + (1-\mu)(y_n-p)\|\\ &\leqslant \mu\|Sx_n - Sp\| + (1-\mu)\|y_n-p\|\\ &\leqslant \mu\|x_n-p\| + (1-\mu)\|x_n-p\|\\ &= \|x_n-p\|.\end{aligned}\tag{3.2}$$

From (3.2), it follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n u + \beta_n x_n + (1-\alpha_n-\beta_n)z_n - p\|\\ &\leqslant \alpha_n\|u-p\| + \beta_n\|x_n-p\| + (1-\alpha_n-\beta_n)\|z_n-p\|\\ &\leqslant \alpha_n\|u-p\| + (1-\alpha_n)\|x_n-p\|\\ &\leqslant \max\{\|u-p\|, \|x_1-p\|\}.\end{aligned}$$

this shows that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

(III). We show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$.

In fact, from (I), we know that the mappings $T_{\rho_i}^{(A_i,M_i)}$ for all $i = 1, 2, \ldots, l$ are nonexpansive. One sees that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|T_{\rho_1}^{(A_1,M_1)}\circ T_{\rho_2}^{(A_2,M_2)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}x_{n+1} - T_{\rho_1}^{(A_1,M_1)}\circ T_{\rho_2}^{(A_2,M_2)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}x_n\|\\ &\leqslant \|T_{\rho_2}^{(A_2,M_2)}\circ T_{\rho_3}^{(A_3,M_3)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}x_{n+1} - T_{\rho_1}^{(A_2,M_2)}\circ T_{\rho_3}^{(A_3,M_3)}\cdots\circ T_{\rho_l}^{(A_l,M_l)}x_n\|\\ &\ \ \vdots\\ &\leqslant \|T_{\rho_l}^{(A_l,M_l)}x_{n+1} - T_{\rho_l}^{(A_l,M_l)}x_n\|\\ &\leqslant \|x_{n+1}-x_n\|.\end{aligned}$$

This implies that

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|(\mu Sx_{n+1} + (1-\mu)y_{n+1}) - (\mu Sx_n + (1-\mu)y_n)\|\\ &\leqslant \mu\|Sx_{n+1} - Sx_n\| + (1-\mu)\|y_{n+1}-y_n\|\\ &\leqslant \mu\|x_{n+1}-x_n\| + (1-\mu)\|y_{n+1}-y_n\|\\ &\leqslant \|x_{n+1}-x_n\|.\end{aligned}\tag{3.3}$$

Setting

$$x_{n+1} = (1-\beta_n)w_n + \beta_n x_n,\tag{3.4}$$

where $w_n := \frac{\alpha_n u + (1-\alpha_n-\beta_n)z_n}{1-\beta_n}$. From (3.3), one sees that

$$\|w_{n+1}-w_n\| = \|\frac{\alpha_{n+1}u + (1-\alpha_{n+1}-\beta_{n+1})z_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n u + (1-\alpha_n-\beta_n)z_n}{1-\beta_n}\|$$

$$= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(u-z_{n+1}) - \frac{\alpha_n}{1-\beta_n}(u-z_n) + z_{n+1} - z_n\|$$
$$\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-z_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-z_n\| + \|z_{n+1}-z_n\|$$
$$\leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-z_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-z_n\| + \|x_{n+1}-x_n\|,$$

this implies that

$$\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\| \leqslant \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-z_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u-z_n\|.$$

It follows from the conditions (i), (ii), and (II) that

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\|) \leqslant 0.$$

Hence, from lemma 2.7, it follows that

$$\lim_{n\to\infty}\|w_n - x_n\| = 0. \tag{3.5}$$

Since, from (3.4), it follows that

$$\|x_{n+1}-x_n\| = (1-\beta_n)\|w_n - x_n\|.$$

By virtue of the conditions (i) and (3.5), we get

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.6}$$

(IV). We show that $\limsup_{n\to\infty}\langle u-u_1^*, J(x_n-u_1^*)\rangle \leqslant 0$, where $u_1^* \in P_{\mathcal{F}}$, and $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$.

First, one has

$$x_{n+1}-x_n = \alpha_n(u-z_n) + (1-\beta_n)(z_n-x_n).$$

It follows that

$$\|z_n - x_n\| \leqslant \frac{\|x_{n+1}-x_n\| + \alpha_n\|u-z_n\|}{1-\beta_n}.$$

From conditions (i), (ii), and (3.6),

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.7}$$

Define a mapping $\mathcal{W}$ by

$$\mathcal{W}x = \mu Sx + (1-\mu)T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_l}^{(A_l,M_l)}x, \quad \forall y \in E.$$

In view of Lemmas 2.11 and 2.2, we see that $\mathcal{W}$ is nonexpansive such that

$$F(\mathcal{W}) = F(S)\bigcap F(T_{\rho_1}^{(A_1,M_1)} \circ T_{\rho_2}^{(A_2,M_2)} \circ \cdots \circ T_{\rho_l}^{(A_l,M_l)}).$$

From (3.7), it follows that

$$\lim_{n\to\infty}\|\mathcal{W}x_n - x_n\| = 0. \tag{3.8}$$

Let z_t be the fixed point of the contraction $z \mapsto tu + (1-t)\mathcal{W}z$, where $t \in (0,1)$. That is, $z_t = tu + (1-t)\mathcal{W}z_t$. It follows that

$$\|z_t - x_n\| = \|(1-t)(\mathcal{W}z_t - x_n) + t(u - x_n)\|.$$

On the other hand, we have

$$\begin{aligned}\|z_t - x_n\|^2 &= \|(1-t)(\mathcal{W}z_t - x_n) + t(u - x_n)\|^2 \\ &\leqslant (1-t)^2\|z_t - x_n\|^2 + 2t\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \\ &\quad + \|\mathcal{W}x_n - x_n\|^2 + 2\|z_t - x_n\|\|\mathcal{W}x_n - x_n\|.\end{aligned}$$

It follows that

$$\langle z_t - u, J(z_t - x_n)\rangle \leqslant \frac{t}{2}\|z_t - x_n\|^2 + \frac{1}{2t}(\|\mathcal{W}x_n - x_n\| + 2\|z_t - x_n\|)\|\mathcal{W}x_n - x_n\|.$$

Since, by (3.8), one sees that

$$\lim_{n\to\infty}[(\|\mathcal{W}x_n - x_n\| + 2\|z_t - x_n\|)\|\mathcal{W}x_n - x_n\|] = 0,$$

hence, we arrive at

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leqslant \frac{t}{2}M, \tag{3.9}$$

where $M > 0$ is an appropriate constant such that $\|z_t - x_n\|^2 \leqslant M$ for all $t \in (0,1)$ and $n \geqslant 1$. Letting $t \to 0$ in (3.9), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leqslant 0.$$

So, for any $\varepsilon > 0$, there exists a positive number $\delta_1 > 0$ with $t \in (0, \delta_1)$ such that

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \leqslant \frac{\varepsilon}{2}. \tag{3.10}$$

On the other hand, we see that $P_{F(\mathcal{W})}u = \lim_{t\to 0} z_t$ and $F(\mathcal{W}) = \mathcal{F}$. It follows that $\lim_{t\to 0} z_t = u_1^* = P_{\mathcal{F}}$. There exists a positive number $\delta_2 > 0$ with $t \in (0, \delta_2)$ such that

$$\begin{aligned}&|\langle u - u_1^*, J(x_n - u_1^*)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\leqslant |\langle u - u_1^*, J(x_n - u_1^*)\rangle - \langle u - u_1^*, J(x_n - z_t)\rangle| \\ &\quad + |\langle u - u_1^*, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &= |\langle u - u_1^*, J(x_n - u_1^*) - J(x_n - z_t)\rangle| + |\langle z_t - u_1^*, J(x_n - z_t)\rangle| \\ &\leqslant \|u - u_1^*\|\|J(x_n - u_1^*) - J(x_n - z_t)\| + \|z_t - u_1^*\|\|J(x_n - z_t)\| \leqslant \frac{\varepsilon}{2}.\end{aligned} \tag{3.11}$$

From (3.11), choosing $\delta \leqslant \min\{\delta_1, \delta_2\}$, it follows that, for each $t \in (0, \delta)$,

$$\langle u - u_1^*, J(x_n - u_1^*)\rangle \leqslant \langle z_t - u, J(z_t - x_n)\rangle + \frac{\varepsilon}{2},$$

which implies that

$$\limsup_{n\to\infty}\langle u - u_1^*, J(x_n - u_1^*)\rangle \leqslant \limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \frac{\varepsilon}{2}.$$

It follows from (3.10) that

$$\limsup_{n\to\infty}\langle u-u_1^*, J(x_n-u_1^*)\rangle \leqslant \varepsilon.$$

Since $\varepsilon > 0$ is chosen arbitrarily, we have

$$\limsup_{n\to\infty}\langle u-u_1^*, J(x_n-u_1^*)\rangle \leqslant 0. \tag{3.12}$$

(V). We show that $\lim_{n\to\infty} x_n = u_1^*$. In fact, from (2.1) and (3.2), we have

$$\begin{aligned}
\|x_{n+1}-u_1^*\|^2 &= \langle x_{n+1}-u_1^*, J(x_{n+1}-u_1^*)\rangle \\
&= \langle \alpha_n u+\beta_n x_n+(1-\alpha_n-\beta_n)z_n-u_1^*, J(x_{n+1}-u_1^*)\rangle \\
&= \alpha_n\langle u-u_1^*, J(x_{n+1}-u_1^*)\rangle+\beta_n\langle x_n-u_1^*, J(x_{n+1}-u_1^*)\rangle \\
&\quad +(1-\alpha_n-\beta_n)\langle z_n-u_1^*, J(x_{n+1}-u_1^*)\rangle \\
&\leqslant \alpha_n\langle u-u_1^*, J(x_{n+1}-u_1^*)\rangle+\beta_n\|x_n-u_1^*\|\|J(x_{n+1}-u_1^*)\| \\
&\quad +(1-\alpha_n-\beta_n)\|z_n-u_1^*\|\|J(x_{n+1}-u_1^*)\| \\
&\leqslant \alpha_n\langle u-u_1^*, J(x_{n+1}-u_1^*)\rangle+\beta_n\|x_n-u_1^*\|\|x_{n+1}-u_1^*\| \\
&\quad +(1-\alpha_n-\beta_n)\|x_n-u_1^*\|\|x_{n+1}-u_1^*\| \\
&= \alpha_n\langle u-u_1^*, J(x_{n+1}-u_1^*)\rangle+(1-\alpha_n)\|x_n-u_1^*\|\|x_{n+1}-u_1^*\| \\
&\leqslant \alpha_n\langle u-u_1^*, J(x_{n+1}-u_1^*)\rangle+\frac{1-\alpha_n}{2}(\|x_n-u_1^*\|^2+\|x_{n+1}-u_1^*\|^2),
\end{aligned}$$

which implies that

$$\|x_{n+1}-u_1^*\|^2 \leqslant (1-\alpha_n)\|x_n-u_1^*\|^2+2\alpha_n\langle u-u_1^*, J(x_{n+1}-u_1^*)\rangle.$$

Hence, by condition (i), (3.12), and Lemma 2.12, we have

$$\lim_{n\to\infty} x_n = u_1^*,$$

where $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $P_{\mathcal{F}}$ and $(u_1^*, u_2^*, \ldots, u_l^*) \in E^l$, where

$$\begin{cases}
u_1^* = T_{\rho_1}^{(A_1,M_1)}u_2^*,\\
u_2^* = T_{\rho_2}^{(A_2,M_2)}u_3^*,\\
\quad\vdots\\
u_{l-1}^* = T_{\rho_{l-1}}^{(A_{l-1},M_{l-1})}u_l^*,\\
u_l^* = T_{\rho_l}^{(A_l,M_l)}u_1^*,
\end{cases}$$

is a solution to problem (1.4). This completes the proof of Theorem 3.1. □

Remark 3.2. Theorem 3.1 which includes Qin et al. [11], Ceng et al. [3], Yao et al. [19], and Zhang et al. [20] as special cases, mainly improves Qin et al. [11] in the following respects:

(1) from a single variational inclusion to a system of variational inclusions;
(2) from a system of variational inclusions to a general system of variational inclusions;
(3) from nonexpansive mappings to strict pseudocontractions.

Remark 3.3.

(1) As special cases of problem (1.4), we have the following. If $l = 2$ in problem (1.4), then problem (1.4)

is reduced to the following. Find $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} 0 \in x^* - y^* + \rho_1(A_1 y^* + M_1 x^*), \\ 0 \in y^* - x^* + \rho_2(A_2 x^* + M_2 y^*). \end{cases} \tag{3.13}$$

(2) If $A_1 = A_2 = A$ and $M_1 = M_2 = M$ in problem (3.13), then problem (1.4) is reduced to the following. Find $(x^*, y^*) \in X \times X$ such that

$$\begin{cases} 0 \in x^* - y^* + \rho_1(A y^* + M x^*), \\ 0 \in y^* - x^* + \rho_2(A x^* + M y^*). \end{cases} \tag{3.14}$$

(3) If $x^* = y^*$ in problem (3.13), then problem (1.4) is reduced to the following. Find $x^* \in X$ such that

$$0 \in Ax^* + Mx^*. \tag{3.15}$$

From Remark 3.3, as some applications of Theorem 3.1, we have the following results.

Theorem 3.4. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $M : E \to 2^E$ be a maximal monotone mapping and $A : E \to E$ a γ-inverse-strongly accretive mapping, respectively. Let $T : E \to E$ be a λ-strict pseudocontraction with a fixed point. Define a mapping S by*

$$Sx = (1 - \frac{\lambda}{K^2})x + \frac{\lambda}{K^2}Tx, \quad \forall x \in E.$$

Assume that $\mathcal{F} = F(T) \bigcap F(T_{\rho_1}^{(A,M)} \circ T_{\rho_2}^{(A,M)}) \neq \emptyset$. For an arbitrary initial point $u \in E$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)(\mu S x_n + (1-\mu) T_{\rho_1}^{(A,M)} \circ T_{\rho_2}^{(A,M)} x_n, \quad n \geqslant 1,$$

where $\mu \in (0,1)$, $\rho_i \in (0, \frac{\gamma}{K^2})$ $(i = 1, 2)$, and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) *$\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;*
(ii) *$\{\beta_n\} \subset (0,1)$ and $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$,*

then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}} u$, where $\mathcal{F}$ is the sunny nonexpansive retraction from E onto $P_{\mathcal{F}}$ and $(x^*, y^*) \in E \times E$, where $y^* = T_{\rho_2}^{(A,M)} x^*$ is a solution to problem (3.14).

Theorem 3.5. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $M : E \to 2^E$ be a maximal monotone mapping and $A : E \to E$ a γ-inverse-strongly accretive mapping, respectively. Let $T : E \to E$ be a λ-strict pseudocontraction with a fixed point. Define a mapping S by*

$$Sx = (1 - \frac{\lambda}{K^2})x + \frac{\lambda}{K^2}Tx, \quad \forall x \in E.$$

Assume that $\mathcal{F} = F(T) \bigcap F(Q) \neq \emptyset$, where $Q = (A + M)^{-1}$. For an arbitrary initial point $u \in E$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)(\mu S x_n + (1-\mu) T_{\rho}^{(A,M)} x_n), \quad n \geqslant 1,$$

where $\mu \in (0,1)$, $\rho \in (0, \frac{\gamma}{K^2})$, and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) *$\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;*
(ii) *$\{\beta_n\} \subset (0,1)$ and $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$,*

then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}} u$, where $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$ and $x^* \in E$ is a solution to problem (1.1).

4. Application

In this section, we shall utilize our results in the paper to study the monotone variational inequality problem, convex minimization problem, and convexly constrained linear inverse problem. Throughout this section, let C be a nonempty closed and convex subset of Banach space E.

4.1. *Application to monotone variational inequality problem*

First, we present an example of monotone variational inequality problems.

Example 4.1 ([4])**.** If $M = \partial\phi : H \to 2^H$, where $\phi : H \to (-\infty, +\infty]$ is a proper convex and lower semi-continuous function, and $\partial\phi$ is the sub-differential of ϕ, then problem (1.1) is equivalent to find $x^* \in H$ such that

$$\langle Ax^*, v - x^* \rangle + \phi(v) - \phi(x^*) \geqslant 0, \quad \forall v \in H, \tag{4.1}$$

which is said to be the mixed quasi-variational inequality. If ϕ is the indicator function of C, that is, if $x \in C$, then $\phi(x) = 0$, and if $x \overline{\in} C$, then $\phi(x) = +\infty$. Then problem (4.1) is equivalent to the classical variational inequality problem, that is, find $x^* \in C$ such that

$$\langle Ax^*, v - x^* \rangle \geqslant 0, \quad \forall v \in C. \tag{4.2}$$

It is easy to see that problem (4.2) is equivalent to finding a point $x^* \in C$ such that

$$0 \in (A + M)x^*,$$

where M is the subdifferential of the indicator of C, and it is a maximal monotone operator.

By Theorem 3 in [13], the resolvent of M is nothing but the projection operator P_C. Therefore, the following result can be obtained from Theorem 3.5 immediately.

Theorem 4.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $M : E \to 2^E$ be the subdifferential of the indicator of C witch is a maximal monotone operator, and $A : E \to E$ be a γ-inverse-strongly accretive mapping, respectively. Let $T : E \to E$ be a λ-strict pseudocontraction with a fixed point. Define a mapping S by*

$$Sx = (1 - \frac{\lambda}{K^2})x + \frac{\lambda}{K^2}Tx, \quad \forall x \in E.$$

Assume that $\mathcal{F} = F(T) \bigcap F((A + M)^{-1}0) \neq \emptyset$. For an arbitrary initial point $u \in E$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)(\mu S x_n + (1 - \mu)P_C(x_n - \rho A x_n)), \quad n \geqslant 1,$$

where $\mu \in (0, 1)$, $\rho \in (0, \frac{\gamma}{K^2})$, and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
(ii) $\{\beta_n\} \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}u$, where $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$ and $x^* \in E$ is a solution to problem (4.2).

4.2. *Application to convex minimization problem*

Let $\psi : H \to R$ be a convex smooth function and $\phi : H \to R$ be a proper convex and lower-semicontinuous function. We consider the following convex minimization problem of finding $x^* \in H$ such that

$$\psi(x^*) + \phi(x^*) = \min_{x \in H}\{\psi(x) + \phi(x)\}. \tag{4.3}$$

This Problem (4.3) is equivalent to the problem of finding $x^* \in H$ such that

$$0 \in \nabla\psi(x^*) + \partial\phi(x^*), \tag{4.4}$$

where $\nabla\psi$ is a gradient of ψ and $\partial\phi$ is a subdifferential of ϕ. Set $A = \nabla\psi$ and $M = \partial\phi$ in Theorem 3.5. If $\nabla\psi$ is $(\frac{1}{L})$-Lipschitz continuous, then it is L-inverse strongly monotone. Moreover, $\partial\phi$ is maximal monotone. Hence, from Theorem 3.5, we have the following result.

Theorem 4.3. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the smooth constant K. Let $\psi : E \to R$ be a convex smooth function and $\phi : E \to R$ be a proper convex and lower-semicontinuous function. $\nabla\psi$ is a gradient of ψ and $\partial\phi$ is a subdifferential of ϕ. If $\nabla\psi$ is $(\frac{1}{L})$-Lipschitz continuous and $\partial\phi$ is maximal monotone, then let $T : E \to E$ be a L-strict pseudocontraction with a fixed point. Define a mapping S by*

$$Sx = (1 - \frac{L}{K^2})x + \frac{L}{K^2}Tx, \quad \forall x \in E.$$

Assume that $\mathcal{F} = F(T)\bigcap F((\nabla\psi + \partial\phi)^{-1}0) \neq \emptyset$. For an arbitrary initial point $u \in E$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n)(\mu S x_n + (1-\mu)J_\rho^{\partial\phi}(x_n - \rho\nabla\psi x_n)), \quad n \geqslant 1,$$

where $\mu \in (0, 1)$, $\rho \in (0, \frac{\gamma}{K^2})$, and $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{+\infty} \alpha_n = +\infty$;
(ii) $\{\beta_n\} \subset (0, 1)$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$,

then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}}u$, where $P_{\mathcal{F}}$ is the sunny nonexpansive retraction from E onto $\mathcal{F}$ and $x^* \in E$ is a solution to problem (4.4).

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