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A note on a singular coupled Burgers equation and double Laplace transform method

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Abstract

In this paper, modification of double Laplace decomposition method is proposed for the analytical approximation solution of a coupled system of Burgers equation with appropriate initial conditions. Some examples are given to support the validity and applicability of the presented method.

Keywords: Double Laplace transform, inverse Laplace transform, singular Burgers equation, coupled Burgers equation, single Laplace transform, decomposition methods.

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1. Introduction

In general, some of the nonlinear models of real-life problems are still very difficult to solve either numerically or theoretically. Burgers equation is considered as a model equation that describes the interaction of convection and diffusion. It occurs in several areas of applied mathematics, such as heat conduction, phenomena of turbulence, and flow through a shock wave traveling in a viscous fluid such as modeling of dynamics. Recently many authors have proposed analytical solution to one dimensional coupled Burgers equation, e.g., [7, 9] using Adomian decomposition method and in [5, 13], the homotopy perturbation method has been used to obtain the exact solution of nonlinear Burgers' equation. In [3] the author has used Laplace transform and homotopy perturbation method to obtain approximate solutions of homogeneous and nonhomogeneous coupled Burgers' equations. Authors in [10, 14] have obtained approximate solution of the viscous coupled Burgers equation using cubic and cubic B-spline collocation method. The convergence of Adomian's method has been studied by several authors [1, 2, 4, 6]. In this work, modified double Laplace decomposition method and the self-canceling noise-terms phenomenon will be employed in the treatments of these models. The main aim of this method is that it can be used

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directly without using restrictive assumptions or linearization. Now, we recall the following definitions which are given by [8, 11, 12]. The double Laplace transform is defined as

$$L_x L_t\left[f(x,t)\right] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{-st} f(x,t)dt\,dx, \tag{1.1}$$

where $x, t > 0$ and p, s are complex values, and further double Laplace transform of the first order partial derivatives is given by

$$L_x L_t\left[\frac{\partial u(x,t)}{\partial x}\right] = pU(p,s) - U(0,s). \tag{1.2}$$

Similarly, the double Laplace transform for second partial derivative with respect to x and t are defined as follows

$$\begin{aligned} L_x L_t\left[\frac{\partial^2 u(x,t)}{\partial^2 x}\right] &= p^2 U(p,s) - pU(0,s) - \frac{\partial U(0,s)}{\partial x}, \\ L_x L_t\left[\frac{\partial^2 u(x,t)}{\partial^2 t}\right] &= s^2 U(p,s) - sU(p,0) - \frac{\partial U(p,0)}{\partial t}. \end{aligned} \tag{1.3}$$

The inverse double Laplace transform $L_p^{-1} L_s^{-1}\left[F(p,s)\right] = f(x,t)$ is defined as in [8, 11] by the complex double integral formula

$$L_p^{-1} L_s^{-1}\left[F(p,s)\right] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} dp \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st} ds,$$

where $F(p,s)$ must be an analytic function for all p and s in the region defined by the inequalities $\mathrm{Re}(p) \geqslant c$ and $\mathrm{Re}(s) \geqslant d$, where c and d are real constants to be chosen suitably.

The following basic lemma of the double Laplace transform is given and shall be used in this paper.

Lemma 1.1. *Double Laplace transform of the non constant coefficient second order partial derivative* $x^r \frac{\partial^2 u}{\partial t^2}$ *and the function* $x^r f(x,t)$ *is given by*

$$L_x L_t\left(x^r \frac{\partial^2 u}{\partial t^2}\right) = (-1)^r \frac{d^r}{dp^r}\left[s^2 U(p,s) - sU(p,0) - \frac{\partial U(p,0)}{\partial t}\right],$$

and

$$L_x L_t\left(x^r f(x,t)\right) = (-1)^r \frac{d^r}{dp^r}\left[L_x L_t\left(f(x,t)\right)\right] = (-1)^r \frac{d^r F(p,s)}{dp^r},$$

where $r = 1, 2, 3, \ldots$

One can prove this lemma by using the definition of double Laplace transform in Eqs. (1.1), (1.2), and (1.3).

2. Singular one dimensional Burgers' equations

The main aim of this section is to discuss the use of modified double Laplace decomposition method for solving singular one dimensional Burgers' equation. We consider a singular one dimensional Burgers' equation with initial condition in the form:

$$\frac{\partial u}{\partial t} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + u\frac{\partial u}{\partial x} = f(x,t), \quad t > 0 \tag{2.1}$$

with initial condition

$$u(x,0) = f_1(x),$$

where $\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)$ is the Bessel operator and $f(x,t)$, $f_1(x)$ are known functions. In order to obtain the solution of Eq. (2.1), we use modified double Laplace decomposition methods as follows.

Step 1: Multiply both sides of Eq. (2.1) by x.

Step 2: Using Lemma 1.1 and definition of the double Laplace transform of partial derivatives for equations in Step 1 and single Laplace transform for initial condition, we get

$$\frac{dU(p,s)}{dp}=\frac{1}{s}\frac{dF_1(p)}{dp}+\frac{1}{s}\frac{dF(p,s)}{dp}-\frac{1}{s}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)-xu\frac{\partial u}{\partial x}\right]. \tag{2.2}$$

Step 3: By integrating both sides of Eq. (2.2) from 0 to p with respect to p, we have

$$U(p,s)=\frac{F_1(p)}{s}+\frac{1}{s}\int_0^p dF(p,s)-\frac{1}{s}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)-xN_1\right]dp, \tag{2.3}$$

where $N_1=u\frac{\partial u}{\partial x}$ and $F_1(p)$, $F(p,s)$ are single and double Laplace transforms of $f_1(x)$ and $f(x,t)$, respectively.

Step 4: Using double Laplace Adomian decomposition methods to define the solution of the system as $u(x,t)$ by the infinite series

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t). \tag{2.4}$$

The nonlinear operators can be defined as

$$N_1=\sum_{n=0}^{\infty}A_n,$$

where A_n is given by

$$A_n=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\sum_{i=0}^{\infty}\left(\lambda^i u_i\right)\right]\right)_{\lambda=0}.$$

Here, Adomian's polynomials A_n are given by

$$\begin{aligned}
A_0&=u_0u_{0x},\\
A_1&=u_0u_{1x}+u_1u_{0x}\\
A_2&=u_0u_{2x}+u_1u_{1x}+u_2u_{0x},\\
A_3&=u_0u_{3x}+u_1u_{2x}+u_2u_{1x}+u_3u_{0x},\\
A_4&=u_0u_{4x}+u_1u_{3x}+u_2u_{2x}+u_3u_{1x}+u_4u_{0x}.
\end{aligned} \tag{2.5}$$

Step 5: Operating the inverse double Laplace transform on both sides of Eq. (2.3) and using Eq. (2.4), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}u_n(x,t)=f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dF(p,s)\right]-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n\right)\right]dp\right]\\
+L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\int_0^p\left(x\sum_{n=0}^{\infty}A_n\right)dp\right]\right],
\end{aligned}$$

we define the following recursively formula:

$$u_0=f_1(x)+L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dF(p,s)\right], \tag{2.6}$$

and the rest terms can be written as

$$u_{n+1} = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n\right)\right]dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\int_0^p\left(x\sum_{n=0}^{\infty}A_n\right)dp\right]\right], \tag{2.7}$$

where L_xL_t is the double Laplace transform with respect to x, t and double inverse Laplace transform denoted by $L_p^{-1}L_s^{-1}$ with respect to p, s. Here we provide double inverse Laplace transform with respect to p and s exists for each terms in the right hand side of Eqs. (2.6) and (2.7). To confirm our method for solving the singular one dimensional Burgers equations, we consider the following example.

Example 2.1. Consider the following nonhomogeneous form of a singular one dimensional Burgers equation:

$$\frac{\partial u}{\partial t} - xt\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + \frac{2}{x}u\frac{\partial u}{\partial x} = x^2, \qquad t > 0,$$

subject to the initial condition

$$u(x,0) = 0.$$

According to the above steps, we have

$$u(x,t) = x^2t - L_p^{-1}L_s^{-1}\left[\int_0^p \frac{1}{s}L_xL_t\left[x^2t\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n\right)\right]dp\right] + L_p^{-1}L_s^{-1}\left[\int_0^p \frac{1}{s}L_xL_t\left[2\sum_{n=0}^{\infty}A_n\right]dp\right],$$

where A_n is given by Eq. (2.5). Using equations analogous to Eqs. (2.6) and (2.7), we obtain

$$u_0 = x^2t, \qquad\qquad u_1 = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[x^2t\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}u_0\right) - (2A_0)\right]dp\right].$$

Therefore we have

$$u_1 = 0.$$

In the same manner, we obtain that

$$u_2 = 0.$$

It is obvious that the self-canceling some terms appear between various components and connected by coming terms, we have

$$u(x,t) = u_0 + u_1 + \cdots.$$

Therefore, the exact solution is given by

$$u(x,t) = x^2t.$$

3. Modified double Laplace decomposition method applied to coupled Burgers' equation

In this section, we discuss the solutions of two problems by applying Modified double Laplace decomposition method.

Problem 3.1 (The first problem)**.** Regular burgers coupled equation is given by

$$u_t - u_{xx} + \eta uu_x + \alpha(uv)_x = f(x,t), \quad v_t - v_{xx} + \eta vv_x + \beta(uv)_x = g(x,t), \tag{3.1}$$

subject to

$$u(x,0) = f_1(x), \qquad v(x,0) = g_1(x) \tag{3.2}$$

for $t > 0$. Here, $f(x,t)$, $g(x,t)$, $f_1(x)$, and $g_1(x)$ are given functions, η is a real constant, α and β are arbitrary constants depending on the system parameters such as Peclet number, Stokes velocity of particles due to gravity and Brownian diffusivity [15]. By taking double Laplace transform for both sides of (3.1) and single Laplace transform for (3.2) we obtain

$$U(p,s) = \frac{F_1(p)}{s} + \frac{F(p,s)}{s} + \frac{1}{s}L_xL_t\left[u_{xx} - \eta u u_x - \alpha\left(uv\right)_x\right], \tag{3.3}$$

and

$$V(p,s) = \frac{G_1(p)}{s} + \frac{G(p,s)}{s} + \frac{1}{s}L_xL_t\left[v_{xx} - \eta v v_x - \beta\left(uv\right)_x\right]. \tag{3.4}$$

The modified double Laplace decomposition method (MDLDM) defines the solution of regular burgers coupled equation as $u(x,t)$ and $v(x,t)$ by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \quad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t). \tag{3.5}$$

We can give Adomian's polynomials A_n, B_n, and C_n respectively as follows

$$A_n = \sum_{n=0}^{\infty} u_n u_{xn}, \quad B_n = \sum_{n=0}^{\infty} v_n v_{xn}, \quad \text{and} \quad C_n = \sum_{n=0}^{\infty} u_n v_n. \tag{3.6}$$

The Adomian polynomials for the nonlinear term uu_x are given by Eq. (2.5), and for the nonlinear terms, vv_x and uv are given by

$$\begin{aligned}
B_0 &= v_0 v_{0x},\\
B_1 &= v_0 v_{1x} + v_1 v_{0x},\\
B_2 &= v_0 v_{2x} + v_1 v_{1x} + v_2 v_{0x},\\
B_3 &= v_0 v_{3x} + v_1 v_{2x} + v_2 v_{1x} + v_3 v_{0x},\\
B_4 &= v_0 v_{4x} + v_1 v_{3x} + v_2 v_{2x} + v_3 v_{1x} + v_4 v_{0x}.
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
C_0 &= u_0 v_0,\\
C_1 &= u_0 v_1 + u_1 v_0\\
C_2 &= u_0 v_2 + u_1 v_1 + u_2 v_0.\\
C_3 &= u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0,\\
C_3 &= u_0 v_4 + u_1 v_3 + u_2 v_2 + u_3 v_1 + u_4 v_0.
\end{aligned} \tag{3.8}$$

By applying inverse double Laplace transform on both sides of (3.3) and (3.4) and using (3.6), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) &+ L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s}\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2}{\partial x^2}u_n\right]\right]\\
&- L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\eta A_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\alpha\left(C_n\right)_x\right]\right],
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\sum_{n=0}^{\infty} v_n(x,t) = g_1(x) &+ L_p^{-1}L_s^{-1}\left[\frac{G(p,s)}{s}\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2}{\partial x^2}v_n\right]\right]\\
&- L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\eta B_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\beta\left(C_n\right)_x\right]\right].
\end{aligned} \tag{3.10}$$

On comparing both sides of Eqs. (3.9) and (3.10) we have

$$u_0 = f_1(x) + L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s}\right], \qquad v_0 = g_1(x) + L_p^{-1}L_s^{-1}\left[\frac{G(p,s)}{s}\right]. \tag{3.11}$$

In general, the recursive relation is given by

$$u_{n+1} = L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2}{\partial x^2}u_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\eta A_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\alpha\left(C_n\right)_x\right]\right], \tag{3.12}$$

and

$$v_{n+1} = L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2}{\partial x^2}v_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\eta B_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\beta\left(C_n\right)_x\right]\right]. \tag{3.13}$$

Here, we provide double inverse Laplace transform with respect to p and s exists for each terms in the right hand side of above equations. To illustrate this method for one dimensional coupled Burgers' equations we take the following example.

Example 3.2. Consider the following homogeneous form of a coupled Burgers equation ([3])

$$u_t - u_{xx} - 2uu_x + (uv)_x = 0, \qquad v_t - v_{xx} - 2vv_x + (uv)_x = 0,$$

with initial conditions

$$u(x,0) = \sin x, \quad v(x,0) = \sin x.$$

By using Eqs. (3.11), (3.12), and (3.13) we have

$$\begin{aligned}
u_0 &= \sin x, \quad v_0 = \sin x,\\
u_1 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2 u_0}{\partial x^2} + 2u_0u_{0x} - (u_0v_0)_x\right]\right] = -t\sin x,\\
v_1 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2 v_0}{\partial x^2} + 2v_0v_{0x} - (u_0v_0)_x\right]\right] = -t\sin x\\
u_2 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2 u_1}{\partial x^2} + 2\left(u_0u_{1x} + u_1u_{0x}\right) - (u_0v_1 + u_1v_0)_x\right]\right] = \frac{t^2}{2}\sin x,\\
v_2 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2 v_1}{\partial x^2} + 2\left(v_0v_{1x} + v_1v_{0x}\right) - (u_0v_1 + u_1v_0)_x\right]\right] = \frac{t^2}{2}\sin x,
\end{aligned}$$

and

$$\begin{aligned}
u_3 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2 u_2}{\partial x^2} + 2\left(u_0u_{2x} + u_1u_{1x} + u_2u_{0x}\right)\right]\right]\\
&\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[(u_0v_2 + u_1v_1 + u_2v_0)_x\right]\right] = -\frac{t^3}{6}\sin x,\\
v_3 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[\frac{\partial^2 v_2}{\partial x^2} + 2\left(v_0v_{2x} + v_1v_{1x} + v_2v_{0x}\right)\right]\right]\\
&\quad - L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_xL_t\left[(u_0v_2 + u_1v_1 + u_2v_0)_x\right]\right] = -\frac{t^3}{6}\sin x,
\end{aligned}$$

and so on for other components. Using Eq. (3.5), the series solutions are therefore given by

$$u(x,t) = u_0 + u_2 + u_3 + \cdots = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\sin x,$$

$$v(x,t) = v_0 + v_2 + v_3 + \cdots = \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right)\sin x,$$

and hence the exact solutions become

$$u(x,t) = e^{-t}\sin x, \qquad v(x,t) = e^{-t}\sin x.$$

Problem 3.3 (The second problem)**.** Singular one dimensional coupled Burgers equations with Bessel operator are given by

$$u_t - \frac{1}{x}(xu_x)_x + \eta u u_x + \alpha (uv)_x = f(x,t), \qquad v_t - \frac{1}{x}(xv_x)_x + \eta v v_x + \beta (uv)_x = g(x,t), \tag{3.14}$$

with initial conditions

$$u(x,0) = f_1(x), \qquad v(x,0) = g_1(x),$$

where the linear term $\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right)$ is the called Bessel operator, and α, β, and η are real constants. In order to obtain the solution of Eq. (3.14), applying the above steps, we get

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) = f_1(x) &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dF(p,s)\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x L_t\left[\left(x\frac{\partial}{\partial x}u_n\right)_x\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p (\eta x A_n)\,dp\right]\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p (\alpha x (C_n)_x)\,dp\right]\right],\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} v_n(x,t) = g_1(x) &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dG(p,s)\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x L_t\left[\left(x\frac{\partial}{\partial x}v_n\right)_x\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p (\eta x B_n)\,dp\right]\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p (\beta x (C_n)_x)\,dp\right]\right].\end{aligned}$$

The first few components can be written as

$$u_0 = f_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dF(p,s)\right], \qquad v_0 = g_1(x) + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p dG(p,s)\right], \tag{3.15}$$

and

$$\begin{aligned}u_{n+1}(x,t) = &-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x L_t\left[\left(x\frac{\partial}{\partial x}\sum_{n=0}^{\infty}u_n\right)_x\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p \left(\eta x \sum_{n=0}^{\infty} A_n\right)dp\right]\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p \left(\alpha x \left(\sum_{n=0}^{\infty} C_n\right)_x\right)dp\right]\right],\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}v_{n+1}(x,t) = &-L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_x L_t\left[\left(x\frac{\partial}{\partial x}v_n\right)_x\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p (\eta x B_n)\,dp\right]\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s}L_x L_t\left[\int_0^p (\beta x (C_n)_x)\,dp\right]\right].\end{aligned} \tag{3.17}$$

Here we provide double inverse Laplace transform with respect to p and s which exist for each terms in the right hand side of of Eqs. (3.15), (3.16), and (3.17).

Example 3.4. Consider the following non homogeneous form of a coupled Burgers equation

$$u_t - \frac{1}{x}(xu_x)_x - 2uu_x + (uv)_x = -x^2e^{-t} - 4e^{-t}, \quad v_t - \frac{1}{x}(xv_x)_x - 2vv_x + (uv)_x = -x^2e^{-t} - 4e^{-t},$$

subject to

$$u(x,0) = x^2, \qquad v(x,0) = x^2.$$

By applying the above steps, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} u_n(x,t) = x^2e^{-t} + 4e^{-t} - 4 - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[\left(x\frac{\partial}{\partial x}u_n\right)_x\right]dp\right] \\ - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[2x(A_n)\right]dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[2x(C_n)_x\right]dp\right]\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} v_n(x,t) = x^2e^{-t} + 4e^{-t} - 4 - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[\left(x\frac{\partial}{\partial x}v_n\right)_x\right]dp\right] \\ - L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[2x(B_n)\right]dp\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[2x(C_n)_x\right]dp\right],\end{aligned}$$

where A_n, B_n, and C_n are defined in Eqs. (2.5), (3.7), and (3.8), respectively. On using Eqs. (3.15), (3.16), and (3.17) the components are given by

$$\begin{aligned}&u_0 = x^2e^{-t} + 4e^{-t} - 4, \quad v_0 = x^2e^{-t} + 4e^{-t} - 4, \\ &u_1 = -L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[(xu_{0x})_x + 2xu_0u_{0x} - x(u_0v_0)_x\right]dp\right], \\ &u_1 = 4 - 4e^{-t}, \\ &v_1 = L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[(xv_{0x})_x + 2xv_0v_{0x} - x(u_0v_0)_x\right]dp\right], \\ &v_1 = 4 - 4e^{-t}.\end{aligned}$$

In the same manner, we obtain that

$$\begin{aligned}u_2 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[(xu_{1x})_x + 2x(u_0u_{1x} + u_1u_{0x}) - x(u_0v_1 + u_1v_0)_x\right]dp\right] = 0, \\ v_2 &= L_p^{-1}L_s^{-1}\left[\frac{1}{s}\int_0^p L_xL_t\left[(xv_{1x})_x + 2x(v_0v_{1x} + v_1v_{0x}) - x(u_0v_1 + u_1v_0)_x\right]dp\right] = 0.\end{aligned}$$

It is obvious that some self-canceling terms appear between various components and the connected by coming terms, then we have,

$$u(x,t) = u_0 + u_1 + u_2 + \cdots, \quad v(x,t) = v_0 + v_1 + v_2 + \cdots.$$

Therefore, the exact solution is given by

$$u(x,t) = x^2e^{-t} \text{ and } v(x,t) = x^2e^{-t}.$$

4. conclusion

In this paper, we have proposed new modified double Laplace decomposition methods to solve singular Burgers equation and coupled Burgers equations. The efficiency and accuracy of the presented scheme are validated through examples. This method can be applied to many complicated linear and non-linear PDEs and also for system of PDEs on which linearization is not required.

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