Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 11 (2018), 575–590
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.isr-publications.com/jnsa

Fourier series of sums of products of r-derangement functions

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Communicated by N. Shahzad

Abstract

A derangement is a permutation that has no fixed point and the derangement number d_m is the number of fixed point-free permutations on an m element set. A generalization of the derangement numbers are the r-derangement numbers and their natural companions are the r-derangement polynomials. In this paper we will study three types of sums of products of r-derangement functions and find Fourier series expansions of them. In addition, we will express them in terms of Bernoulli functions. As immediate corollaries to this, we will be able to express the corresponding three types of polynomials as linear combinations of Bernoulli polynomials.

Keywords: Fourier series, r-derangement polynomials.

2010 MSC: 11B83, 11B68, 42A16.

1. Introduction

A derangement is a permutation that has no fixed points. The problem of counting derangements was begun in 1708 by Pierre Rémond de Montmort (see [3]).

The derangement number d_m is the number of fixed point-free permutations on an m element set (see [1, 2, 6]).

The first few terms of the derangement number sequence $\{d_m\}_{m=0}^{\infty}$ are

$$d_0 = 1, d_1 = 0, d_2 = 1, d_3 = 2, d_4 = 9, d_5 = 44,$$
$$d_6 = 265, d_7 = 1854, d_8 = 14833, d_9 = 133496, d_{10} = 1334961, \dots .$$

It is well known that d_m is in fact given by the closed form formula

$$d_m = m! \sum_{k=0}^{m} \frac{(-1)^k}{k!}. \tag{1.1}$$

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doi: 10.22436/jnsa.011.04.12

Received: 2017-12-01 *Revised:* 2017-12-20 *Accepted:* 2017-12-22

We can easily deduce the exponential generating function for the derangement numbers d_m from (1.1) which is given by

$$\frac{1}{1-t}e^{-t} = \sum_{m=0}^{\infty} d_m \frac{t^m}{m!}.$$

As a natural companion to derangement numbers d_m, the following derangement polynomials $d_m(x)$ are defined by

$$\frac{1}{1-t}e^{(x-1)t} = \sum_{m=0}^{\infty} d_m(x)\frac{t^m}{m!}. \tag{1.2}$$

We observe here that $d_m(x+1)$, viewed as a function of the real variable x, is the same as the function $D_m(x) = m!\sum_{i=0}^{m}\frac{x^i}{i!}$, which is defined in [4, p.6]. Clearly $d_m(0) = d_m$.

The notion of derangement numbers was further generalized in [5, 7, 9] to r-derangement numbers by imposing some restrictions. The r-derangement number $d_r(m)$ $(0 \leqslant r \leqslant m)$ is the number of derangements on an $m+r$ element set such that the first r elements appear in distinct cycles in its cycle decomposition. We note here that $d_0(m) = d_m$, and $d_r(m) = 0$, if $m < r$.

A closed form formula exists for $d_r(m)$, which is given by

$$d_r(m) = \sum_{j=1}^{m}\binom{j}{r}\frac{m!}{(m-j)!}(-1)^{m-j}, \ (0 \leqslant r \leqslant m). \tag{1.3}$$

The exponential generating function for the r-derangement numbers $d_r(m)$ can be easily deduced from (1.3) as follows:

$$\frac{t^r}{(1-t)^{r+1}}e^{-t} = \sum_{m=0}^{\infty} d_r(m)\frac{t^m}{m!}. \tag{1.4}$$

The reader should refer to [7–9] for further details about r-derangement numbers.

The r-derangement polynomials $d_r(m,x)$ are defined by

$$\frac{t^r}{(1-t)^{r+1}}e^{(x-1)t} = \sum_{m=0}^{\infty} d_r(m,x)\frac{t^m}{m!}. \tag{1.5}$$

Clearly, from (1.2), (1.4), and (1.5), we see that

$$d_r(m,0) = d_r(m), \ \ d_0(m,x) = d_m(x), \ \ d_r(m,x) = 0, \text{ for } m < r, \ \ d_r(r,x) = r!.$$

Further, we see from (1.5) that

$$\sum_{m=0}^{\infty} d_r(m,1)\frac{t^m}{m!} = \frac{t^r}{(1-t)^{r+1}} = \sum_{m=r}^{\infty} m!\binom{m}{r}\frac{t^m}{m!}.$$

Then we have shown that

$$d_r(m,1) = \begin{cases} m!\binom{m}{r}, & \text{for } m \geqslant r, \\ 0, & \text{for } 0 \leqslant m < r. \end{cases} \tag{1.6}$$

As we can see, $d_r(m,x)$ are Appell polynomials and in particular we have

$$\frac{d}{dx}d_r(m,x) = md_r(m-1,x), \ (m \geqslant 1).$$

For any real number x, the fractional part of x is denoted by

$$< x >= x - \lfloor x \rfloor \in [0, 1).$$

The Bernoulli polynomials $B_m(x)$ are given by

$$\frac{t}{e^t - 1} e^{xt} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

We need the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geqslant 2$,

$$B_m(< x >) = -m! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m};$$

(b) for $m = 1$,

$$-\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}$$

Here in this paper we will study the following three types of sums of products of r-derangement functions $\alpha_m(< x >), \beta_m(< x >)$, and $\gamma_m(< x >)$, and find Fourier series expansions of them. In addition, we will express them in terms of Bernoulli functions. As immediate corollaries to this, we will be able to express the corresponding three types of polynomials as linear combinations of Bernoulli polynomials.

Throughout this paper, we let r and s be fixed nonnegative integers.

(a) $\alpha_m(< x >) = \sum_{k=0}^{m} d_r(k, < x >) d_s(m-k, < x >)$, $(m > r + s)$;
(b) $\beta_m(< x >) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} d_r(k, < x >) d_s(m-k, < x >)$, $(m > r + s)$;
(c) $\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} d_r(k, < x >) d_s(m-k, < x >)$, $(m > r + s,\ \text{if } r + s \geqslant 1;\ m > 1,\ \text{if } r + s = 0)$.

2. Fourier series expansion of $\alpha_m(< x >)$

Let $\alpha_m(x) = \sum_{k=0}^{m} d_r(k, x) d_s(m-k, x)$, $(m \geqslant 0)$. We note here that $\alpha_m(x) = 0$ for $m < r + s$, since $d_r(m, x) = 0$ for $m < r$. Moreover,

$$\alpha_m(x) = \sum_{k=r}^{m-s} d_r(k, x) d_s(m-k, x), \ (m \geqslant r + s), \tag{2.1}$$

and hence, in particular, we have

$$\alpha_{r+s}(x) = d_r(r, x) d_s(s, x) = r!s!.$$

Now, we will consider the function

$$\alpha_m(< x >) = \sum_{k=0}^{m} d_r(k, < x >) d_s(m-k, < x >), \ (m > r + s),$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\alpha_m(<x>)$ is $\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}$, where

$$A_n^{(m)} = \int_0^1 \alpha_m(<x>)e^{-2\pi i n x}dx = \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx.$$

Before proceeding any further, we need to observe the following:

$$\begin{aligned}
\frac{d}{dx}\alpha_m(x) &= \sum_{k=0}^{m}\Big(kd_r(k-1,x)d_s(m-k,x) + (m-k)d_r(k,x)d_s(m-k-1,x)\Big)\\
&= \sum_{k=1}^{m} kd_r(k-1,x)d_s(m-k,x) + \sum_{k=0}^{m-1}(m-k)d_r(k,x)d_s(m-k-1,x)\\
&= \sum_{k=0}^{m-1}(k+1)d_r(k,x)d_s(m-1-k,x) + \sum_{k=0}^{m-1}(m-k)d_r(k,x)d_s(m-1-k,x)\\
&= (m+1)\alpha_{m-1}(x).
\end{aligned}$$

This implies that

$$\frac{d}{dx}\left(\frac{\alpha_{m+1}(x)}{m+2}\right) = \alpha_m(x) \quad \text{and} \quad \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\big(\alpha_{m+1}(1) - \alpha_{m+1}(0)\big).$$

For $m > r+s$, we put

$$\begin{aligned}
\Delta_m = \alpha_m(1) - \alpha_m(0) &= \sum_{k=r}^{m-s} d_r(k,1)d_s(m-k,1) - \sum_{k=r}^{m-s} d_r(k)d_s(m-k)\\
&= \sum_{k=r}^{m-s} k!\binom{k}{r}(m-k)!\binom{m-k}{s} - \sum_{k=r}^{m-s} d_r(k)d_s(m-k)\\
&= m!\sum_{k=r}^{m-s}\binom{k}{r}\binom{m-k}{s}\binom{m}{k}^{-1} - \sum_{k=r}^{m-s} d_r(k)d_s(m-k), \quad \text{(cf. (1.6) and (2.1))}.
\end{aligned}$$

We now note that

$$\alpha_m(0) = \alpha_m(1) \Leftrightarrow \Delta_m = 0 \quad \text{and} \quad \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

We are going to determine the Fourier coefficients $A_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
A_n^{(m)} = \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx &= -\frac{1}{2\pi i n}[\alpha_m(x)e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1\left(\frac{d}{dx}\alpha_m(x)\right)e^{-2\pi i n x}dx\\
&= -\frac{1}{2\pi i n}\big(\alpha_m(1) - \alpha_m(0)\big) + \frac{m+1}{2\pi i n}\int_0^1 \alpha_{m-1}(x)e^{-2\pi i n x}dx\\
&= \frac{m+1}{2\pi i n}A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m.
\end{aligned}$$

Thus we have shown that

$$A_n^{(m)} = \frac{m+1}{2\pi i n}A_n^{(m-1)} - \frac{1}{2\pi i n}\Delta_m. \tag{2.2}$$

Noting that $A_n^{(r+s)} = r!s!\int_0^1 e^{-2\pi i n x} = 0$, and proceeding by induction on m, from (2.2), we easily derive that

$$A_n^{(m)} = -\sum_{j=1}^{m-(r+s)} \frac{(m+1)_{j-1}}{(2\pi i n)^j}\Delta_{m-j+1} = -\frac{1}{m+2}\sum_{j=1}^{m-(r+s)} \frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1},\ (m > r+s).$$

Case 2: $n = 0$.

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx = \frac{1}{m+2}\Delta_{m+1}.$$

$\alpha_m(< x >)$, $(m > r+s)$ is piecewise C^∞. Moreover $\alpha_m(< x >)$ is continuous for those integers $m > r+s$ with $\Delta_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m > r+s$ with $\Delta_m \neq 0$.

Assume first that $\Delta_m = 0$ for an integer $m > r+s$. Then $\alpha_m(0) = \alpha_m(1)$. Hence $\alpha_m(< x >)$ is piecewise C^∞, and continuous. The Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned}
\alpha_m(< x >) &= \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-(r+s)}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=1}^{m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}\left(-j!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right)\\
&= \frac{1}{m+2}\Delta_{m+1} + \frac{1}{m+2}\sum_{j=2}^{m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >)\\
&\quad + \Delta_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z},\\ 0, & \text{for } x \in \mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we can state our first result.

Theorem 2.1. *For each integer $l > r+s$, we let*

$$\Delta_l = l!\sum_{k=r}^{l-s}\binom{k}{r}\binom{l-k}{s}\binom{l}{k}^{-1} - \sum_{k=r}^{l-s} d_r(k)d_s(l-k).$$

Assume that $\Delta_m = 0$ for an integer $m > r+s$. Then we have the following.

(a) $\sum_{k=0}^{m} d_r(k,< x >)d_s(m-k,< x >)$ *has the Fourier series expansion*

$$\sum_{k=0}^{m} d_r(k,< x >)d_s(m-k,< x >) = \frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\frac{1}{m+2}\sum_{j=1}^{m-(r+s)}\frac{(m+2)_j}{(2\pi i n)^j}\Delta_{m-j+1}\right)e^{2\pi i n x}$$

for all $x \in \mathbb{R}$. Here the convergence is uniform.

(b)

$$\sum_{k=0}^{m} d_r(k,< x >)d_s(m-k,< x >) = \frac{1}{m+2}\sum_{\substack{j=0\\ j\neq 1}}^{m-(r+s)}\binom{m+2}{j}\Delta_{m-j+1}B_j(< x >)$$

for all $x \in \mathbb{R}$.

Next, we assume that $\Delta_m \neq 0$ for an integer $m > r+s$. Then $\alpha_m(0) \neq \alpha_m(1)$. So $\alpha_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\alpha_m(< x >)$ converges pointwise to $\alpha_m(< x >)$ for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\big(\alpha_m(0)+\alpha_m(1)\big) = \alpha_m(0) + \frac{1}{2}\Delta_m$$

for $x \in \mathbb{Z}$.

Next, we can state our second result.

Theorem 2.2. *For each integer $l > r+s$, we let*

$$\Delta_l = l! \sum_{k=r}^{l-s} \binom{k}{r}\binom{l-k}{s}\binom{l}{k}^{-1} - \sum_{k=r}^{l-s} d_r(k)d_s(l-k).$$

Assume that $\Delta_m \neq 0$ for an integer $m > r+s$. Then we have the following.

(a)

$$\frac{1}{m+2}\Delta_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\frac{1}{m+2} \sum_{j=1}^{m-(r+s)} \frac{(m+2)_j}{(2\pi i n)^j} \Delta_{m-j+1} \right) e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m} d_r(k, < x >) d_s(m-k, < x >), & \text{for } x \notin \mathbb{Z}, \\ \sum_{k=0}^{m} d_r(k) d_s(m-k) + \frac{1}{2}\Delta_m, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$\frac{1}{m+2} \sum_{j=0}^{m-(r+s)} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) = \sum_{k=0}^{m} d_r(k, < x >) d_s(m-k, < x >) \ \text{for } x \notin \mathbb{Z},$$

$$\frac{1}{m+2} \sum_{\substack{j=0 \\ j\neq 1}}^{m-(r+s)} \binom{m+2}{j} \Delta_{m-j+1} B_j(< x >) = \sum_{k=0}^{m} d_r(k) d_s(m-k) + \frac{1}{2}\Delta_m \ \text{for } x \in \mathbb{Z}.$$

Corollary 2.3. *For each integer $l > r+s$, we let*

$$\Delta_l = l! \sum_{k=r}^{l-s} \binom{k}{r}\binom{l-k}{s}\binom{l}{k}^{-1} - \sum_{k=r}^{l-s} d_r(k)d_s(l-k).$$

Then we have the following polynomial identity.

$$\sum_{k=0}^{m} d_r(k,x) d_s(m-k,x) = \frac{1}{m+2} \sum_{j=0}^{m-(r+s)} \binom{m+2}{j} \Delta_{m-j+1} B_j(x), \ (m > r+s).$$

3. Fourier series expansion of $\beta_m(< x >)$

Let $\beta_m(x) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} d_r(k,x) d_s(m-k,x)$, $(m \geqslant 0)$. As in the case of $\alpha_m(x)$, we note that

$$\beta_m(x) = 0 \text{ for } m < r+s, \ \beta_{r+s}(x) = 1,$$
$$\beta_m(x) = \sum_{k=r}^{m-s} \frac{1}{k!(m-k)!} d_r(k,x) d_s(m-k,x), \ (m \geqslant r+s). \tag{3.1}$$

Now, we will consider the function

$$\beta_m(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!} d_r(k,<x>) d_s(m-k,<x>), \ (m > r+s)$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier seires of $\beta_m(<x>)$ is $\sum_{n=-\infty}^{\infty} B_n^{(m)} e^{2\pi i n x}$, where

$$B_n^{(m)} = \int_0^1 \beta_m(<x>) e^{-2\pi i n x} dx = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx.$$

To proceed further, we observe the following

$$\begin{aligned}
\frac{d}{dx}\beta_m(x) &= \sum_{k=0}^{m} \left(\frac{k}{k!(m-k)!} d_r(k-1,x) d_s(m-k,x) + \frac{m-k}{k!(m-k)!} d_r(k,x) d_s(m-k-1,x) \right) \\
&= \sum_{k=1}^{m} \frac{1}{(k-1)!(m-k)!} d_r(k-1,x) d_s(m-k,x) \\
&\quad + \sum_{k=0}^{m-1} \frac{1}{k!(m-k-1)!} d_r(k,x) d_s(m-k-1,x) \Big) \\
&= \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} d_r(k,x) d_s(m-1-k,x) \\
&\quad + \sum_{k=0}^{m-1} \frac{1}{k!(m-1-k)!} d_r(k,x) d_s(m-1-k,x) \Big) \\
&= 2\beta_{m-1}(x).
\end{aligned}$$

These imply that

$$\frac{d}{dx}\left(\frac{1}{2}\beta_{m+1}(x)\right) = \beta_m(x) \quad \text{and} \quad \int_0^1 \beta_m(x) dx = \frac{1}{2}\Big(\beta_{m+1}(1) - \beta_{m+1}(0)\Big).$$

For $m > r+s$, we put

$$\begin{aligned}
\Omega_m = \beta_m(1) - \beta_m(0) &= \sum_{k=r}^{m-s} \frac{1}{k!(m-k)!} d_r(k,1) d_s(m-k,1) - \sum_{k=r}^{m-s} \frac{1}{k!(m-k)!} d_r(k) d_s(m-k) \\
&= \sum_{k=r}^{m-s} \binom{k}{r}\binom{m-k}{s} - \sum_{k=r}^{m-s} \frac{1}{k!(m-k)!} d_r(k) d_s(m-k), \quad \text{(cf. (1.6), (3.1)).}
\end{aligned}$$

We observe now that

$$\beta_m(0) = \beta_m(1) \Leftrightarrow \Omega_m = 0 \quad \text{and} \quad \int_0^1 \beta_m(x) dx = \frac{1}{2}\Omega_{m+1}.$$

Our next task is to determine the Fourier coefficients $B_n^{(m)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
B_n^{(m)} = \int_0^1 \beta_m(x) e^{-2\pi i n x} dx &= -\frac{1}{2\pi i n}[\beta_m(x) e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1 \left(\frac{d}{dx}\beta_m(x)\right) e^{-2\pi i n x} dx \\
&= -\frac{1}{2\pi i n}\Big(\beta_m(1) - \beta_m(0)\Big) + \frac{2}{2\pi i n}\int_0^1 \beta_{m-1}(x) e^{-2\pi i n x} dx \\
&= \frac{2}{2\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n}\Omega_m,
\end{aligned}$$

from which by induction on m we can show that

$$B_n^{(m)} = -\sum_{j=1}^{m-(r+s)} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}, \ (m > r+s).$$

Case 2: $n = 0$.

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx = \frac{1}{2}\Omega_{m+1}.$$

$\beta_m(< x >)$, $(m > r+s)$ is piecewise C^∞. Moreover $\beta_m(< x >)$ is continuous for those integers $m > r+s$ with $\Omega_m = 0$, and discontinuous with jump discontinuities at integers for those integers $m > r+s$ with $\Omega_m \neq 0$.

Assume first that $\Omega_m = 0$ for an integer $m > r+s$. Then $\beta_m(0) = \beta_m(1)$. Then $\beta_m(< x >)$ is piecewise C^∞, and continuous. The Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned}
\beta_m(< x >) &= \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\sum_{j=1}^{m-(r+s)} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \right) e^{2\pi i n x} \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=1}^{m-(r+s)} \frac{2^{j-1}}{j!}\Omega_{m-j+1} \left(-j! \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^j} \right) \\
&= \frac{1}{2}\Omega_{m+1} + \sum_{j=2}^{m-(r+s)} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >) + \Omega_m \times \begin{cases} B_1(< x >), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases}
\end{aligned}$$

Now we are ready to state our first result.

Theorem 3.1. *For each integer $l > r+s$, we let*

$$\Omega_l = \sum_{k=r}^{l-s} \binom{k}{r}\binom{l-k}{s} - \sum_{k=r}^{l-s} \frac{1}{k!(l-k)!} d_r(k) d_s(l-k).$$

Assume that $\Omega_m = 0$, for an integer $m > r+s$. Then we have the following.

(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} d_r(k,< x >) d_s(m-k,< x >)$ *has the Fourier series expansion*

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} d_r(k,< x >) d_s(m-k,< x >) = \frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(-\sum_{j=1}^{m-(r+s)} \frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1} \right) e^{2\pi i n x}$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} d_r(k,< x >) d_s(m-k,< x >) = \sum_{\substack{j=0 \\ j\neq 1}}^{m-(r+s)} \frac{2^{j-1}}{j!}\Omega_{m-j+1} B_j(< x >)$$

for all $x \in \mathbb{R}$.

Assume next that $\Omega_m \neq 0$ for an integer $m > r+s$. Thus $\beta_m(0) \neq \beta_m(1)$. Then $\beta_m(< x >)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. The Fourier series of $\beta_m(< x >)$ converges pointwise to $\beta_m(< x >)$ for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\big(\beta_m(0)+\beta_m(1)\big) = \beta_m(0) + \frac{1}{2}\Omega_m$$

for $x \in \mathbb{Z}$.

We are now ready to state our second result.

Theorem 3.2. *For each integer* $l > r+s$, *we let*

$$\Omega_l = \sum_{k=r}^{l-s}\binom{k}{r}\binom{l-k}{s} - \sum_{k=r}^{l-s}\frac{1}{k!(l-k)!}d_r(k)d_s(l-k).$$

Assume that $\Omega_m \neq 0$, *for an integer* $m > r+s$. *Then we have the following.*

(a)

$$\frac{1}{2}\Omega_{m+1} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\left(-\sum_{j=1}^{m-(r+s)}\frac{2^{j-1}}{(2\pi i n)^j}\Omega_{m-j+1}\right)e^{2\pi i n x}$$
$$= \begin{cases} \sum_{k=0}^{m}\frac{1}{k!(m-k)!}d_r(k,< x >)d_s(m-k,< x >), & \textit{for } x \notin \mathbb{Z},\\ \sum_{k=0}^{m}\frac{1}{k!(m-k)!}d_r(k)d_s(m-k) + \frac{1}{2}\Omega_m, & \textit{for } x \in \mathbb{Z}.\end{cases}$$

(b)

$$\sum_{j=0}^{m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}d_r(k,< x >)d_s(m-k,< x >) \quad \textit{for } x \notin \mathbb{Z},$$

$$\sum_{\substack{j=0\\ j\neq 1}}^{m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(< x >) = \sum_{k=0}^{m}\frac{1}{k!(m-k)!}d_r(k)d_s(m-k) + \frac{1}{2}\Omega_m \quad \textit{for } x \in \mathbb{Z}.$$

Corollary 3.3. *For each integer* $l > r+s$, *we let*

$$\Omega_l = \sum_{k=r}^{l-s}\binom{k}{r}\binom{l-k}{s} - \sum_{k=r}^{l-s}\frac{1}{k!(l-k)!}d_r(k)d_s(l-k).$$

Then we have the following polynomial identity.

$$\sum_{k=0}^{m}\frac{1}{k!(m-k)!}d_r(k,x)d_s(m-k,x) = \sum_{j=0}^{m-(r+s)}\frac{2^{j-1}}{j!}\Omega_{m-j+1}B_j(x), \ (m > r+s).$$

4. Fourier series expansion of $\gamma_m(< x >)$

Let $\gamma_m(x) = \sum_{k=1}^{m-1}\frac{1}{k(m-k)}d_r(k,x)d_s(m-k,x)$, $(m \geqslant 2)$. Before proceeding further, we remark the following, where there are several cases to consider.

(a) $r \geqslant 1$, $s \geqslant 1$.

$$\gamma_m(x) = 0 \text{ for } m < r+s, \ \gamma_{r+s}(x) = (r-1)!(s-1)!, \tag{4.1}$$

$$\gamma_m(x) = \sum_{k=r}^{m-s}\frac{1}{k(m-k)}d_r(k,x)d_s(m-k,x), \ (m \geqslant r+s). \tag{4.2}$$

(b) $r \geqslant 1$, $s = 0$.

$$\gamma_m(x) = 0 \text{ for } m \leqslant r+s, \tag{4.3}$$

$$\gamma_m(x) = \sum_{k=r}^{m-1} \frac{1}{k(m-k)} d_r(k,x) d_{m-k}(x), \ (m > r = r+s). \tag{4.4}$$

(c) $r = 0$, $s \geqslant 1$.

$$\gamma_m(x) = 0 \text{ for } m \leqslant r+s, \tag{4.5}$$

$$\gamma_m(x) = \sum_{k=1}^{m-s} \frac{1}{k(m-k)} d_k(x) d_s(m-k,x). \tag{4.6}$$

(d) $r = 0$, $s = 0$.

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} d_k(x) d_{m-k}(x), \ (m \geqslant 2). \tag{4.7}$$

Throughout the rest of our discussion, in view of (4.1)-(4.7) we assume that $m > r+s$, if $r+s \geqslant 1$, and that $m > 1$, if $r+s = 0$. Then we consider the function

$$\gamma_m(< x >) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} d_r(k, < x >) d_s(m-k, < x >),$$

$(m > r+s$, if $r+s \geqslant 1$; $m > 1$, if $r+s = 0)$, defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier seires of $\gamma_m(< x >)$ is $\sum_{n=-\infty}^{\infty} C_n^{(m)} e^{2\pi i n x}$, where

$$C_n^{(m)} = \int_0^1 \gamma_m(< x >) e^{-2\pi i n x} dx = \int_0^1 \gamma_m(x) e^{-2\pi i n x} dx.$$

From (1.5), we immediately see that

$$d_r(0,x) = \delta_{0,r},$$

where $\delta_{0,r}$ is the Kronecker's delta. Now,

$$\begin{aligned} \frac{d}{dx}\gamma_m(x) &= \sum_{k=1}^{m-1} \frac{1}{m-k} d_r(k-1,x) d_s(m-k,x) + \sum_{k=1}^{m-1} \frac{1}{k} d_r(k,x) d_s(m-k-1,x) \\ &= \sum_{k=1}^{m-2} \left(\frac{1}{m-1-k} + \frac{1}{k} \right) d_r(k,x) d_s(m-1-k,x) \\ &\quad + \frac{1}{m-1} \Big(d_r(0,x) d_s(m-1,x) + d_r(m-1,x) d_s(0,x) \Big) \\ &= (m-1)\gamma_{m-1}(x) + \frac{1}{m-1} \Big(\delta_{0,r} d_s(m-1,x) + \delta_{0,s} d_r(m-1,x) \Big). \end{aligned} \tag{4.8}$$

From (4.8), we note that

$$\frac{d}{dx}\frac{1}{m}\Big(\gamma_{m+1}(x) - \frac{1}{m(m+1)}\delta_{0,r} d_s(m+1,x) - \frac{1}{m(m+1)}\delta_{0,s} d_r(m+1,x) \Big) = \gamma_m(x),$$

and

$$\begin{aligned}\int_0^1 \gamma_m(x)dx &= \frac{1}{m}\Big[\gamma_{m+1}(x) - \frac{1}{m(m+1)}\delta_{0,r}d_s(m+1,x) - \frac{1}{m(m+1)}\delta_{0,s}d_r(m+1,x)\Big]_0^1 \\ &= \frac{1}{m}\Big\{\gamma_{m+1}(1) - \gamma_{m+1}(0) - \frac{1}{m(m+1)}\delta_{0,r}\Big((m+1)!\binom{m+1}{s} - d_s(m+1)\Big) \\ &\quad - \frac{1}{m(m+1)}\delta_{0,s}\Big((m+1)!\binom{m+1}{r} - d_r(m+1)\Big)\Big\} \\ &= \frac{1}{m}\Big\{\gamma_{m+1}(1) - \gamma_{m+1}(0) - (m-1)!\Big(\binom{m+1}{s}\delta_{0,r} + \binom{m+1}{r}\delta_{0,s}\Big) \\ &\quad + \frac{1}{m(m+1)}\Big(d_s(m+1)\delta_{0,r} + d_r(m+1)\delta_{0,s}\Big)\Big\}.\end{aligned}$$

For $m > 1$, we let

$$\begin{aligned}\Lambda_m = \gamma_m(1) - \gamma_m(0) &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\Big(d_r(k,1)d_s(m-k,1) - d_r(k,0)d_s(m-k,0)\Big) \\ &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\Big(k!\binom{k}{r}(m-k)!\binom{m-k}{s} - d_r(k)d_s(m-k)\Big).\end{aligned} \tag{4.9}$$

We note that, from (4.1), (4.3), and (4.5), or from (4.9), $\Lambda_2 = \cdots = \Lambda_{r+s} = 0$, for $r+s \geqslant 1$. For convenience, we also let, for $r+s \geqslant 1$, and also for $r+s=0$,

$$\Lambda_1 = 0. \tag{4.10}$$

Obviously, we have

$$\gamma_m(0) = \gamma_m(1) \Leftrightarrow \Lambda_m = 0,$$

and

$$\begin{aligned}\int_0^1 \gamma_m(x)dx = \frac{1}{m}\Big\{&\Lambda_{m+1} - (m-1)!\Big(\binom{m+1}{s}\delta_{0,r} + \binom{m+1}{r}\delta_{0,s}\Big) \\ &+ \frac{1}{m(m+1)}\Big(d_s(m+1)\delta_{0,r} + d_r(m+1)\delta_{0,s}\Big)\Big\}.\end{aligned}$$

Next, we would like to determine the Fourier coefficients $C_n^{(m)}$.
Case 1: $n \neq 0$.

$$\begin{aligned}C_n^{(m)} &= \int_0^1 \gamma_m(x)e^{-2\pi i n x}dx \\ &= -\frac{1}{2\pi i n}[\gamma_m(x)e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n}\int_0^1\Big(\frac{d}{dx}\gamma_m(x)\Big)e^{-2\pi i n x}dx \\ &= -\frac{1}{2\pi i n}\Big(\gamma_m(1) - \gamma_m(0)\Big) + \frac{1}{2\pi i n}\int_0^1\Big\{(m-1)\gamma_{m-1}(x) \\ &\quad + \frac{1}{m-1}\Big(\delta_{0,r}d_s(m-1,x) + \delta_{0,s}d_r(m-1,x)\Big)\Big\}e^{-2\pi i n x}dx \\ &= \frac{m-1}{2\pi i n}C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m + \frac{1}{2\pi i n(m-1)}\delta_{0,r}\int_0^1 d_s(m-1,x)e^{-2\pi i n x}dx\end{aligned} \tag{4.11}$$

$$+\frac{1}{2\pi i n(m-1)}\delta_{0,s}\int_0^1 d_r(m-1,x)e^{-2\pi i n x}dx.$$

By integrating by parts and induction on m, we can easily deduce that

$$\int_0^1 d_r(m,x)e^{-2\pi i n x} = \begin{cases} -\sum_{j=1}^{m-r}\frac{(m)_{k-1}}{(2\pi i n)^k}\Big((m-k+1)!\binom{m-k+1}{r}-d_r(m-k+1)\Big), & \text{for } n\neq 0,\\ \frac{1}{m+1}\Big((m+1)!\binom{m+1}{r}-d_r(m+1)\Big), & \text{for } n=0.\end{cases} \tag{4.12}$$

Combining (4.11) and (4.12), we have shown that

$$C_n^{(m)} = \frac{m-1}{2\pi i n}C_n^{(m-1)} - \frac{1}{2\pi i n}\Lambda_m - \frac{1}{2\pi i n(m-1)}\Big(\delta_{0,r}\Phi_n^{(m,s)} + \delta_{0,s}\Phi^{(m,r)}\Big), \tag{4.13}$$

where

$$\Phi^{(m,r)} = \sum_{k=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi i n)^k}\Big((m-k)!\binom{m-k}{r}-d_r(m-k)\Big).$$

An application of induction on m to (4.13) gives us the following expression

$$C_n^{(m)} = -\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi i n)^j}\Lambda_{m-j+1} - \sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi i n)^j(m-j)}\Big(\delta_{0,r}\Phi_n^{(m-j+1,s)} + \delta_{0,s}\Phi_n^{(m-j+1,r)}\Big). \tag{4.14}$$

Note here that we used the fact $\Phi_n^{(m,r)} = 0$ for $m \leqslant r$.

In order to find more explicit expressions, we note the following.

$$\begin{aligned}
&\sum_{j=1}^{m-1}\frac{(m-1)_{j-1}}{(2\pi i n)^j(m-j)}\Phi_n^{(m-j+1,r)}\\
&= \sum_{j=1}^{m-1}\frac{(m-1)_{k-1}}{(2\pi i n)^j(m-j)}\sum_{k=1}^{m-j}\frac{(m-j)_{k-1}}{(2\pi i n)^k}\\
&\quad\times\Big((m-j-k+1)!\binom{m-j-k+1}{r}-d_r(m-j-k+1)\Big)\\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=1}^{m-j}\frac{(m-1)_{j+k-2}}{(2\pi i n)^{j+k}}\times\Big((m-j-k+1)!\binom{m-j-k+1}{r}-d_r(m-j-k+1)\Big)\\
&= \sum_{j=1}^{m-1}\frac{1}{m-j}\sum_{k=j+1}^{m}\frac{(m-1)_{k-2}}{(2\pi i n)^k}\times\Big((m-k+1)!\binom{m-k+1}{r}-d_r(m-k+1)\Big)\\
&= \sum_{j=1}^{m}\frac{(m-1)_{k-2}}{(2\pi i n)^k}\Big(H_{m-1}-H_{m-k}\Big)\times\Big((m-k+1)!\binom{m-k+1}{r}-d_r(m-k+1)\Big)\\
&= \frac{1}{m}\sum_{j=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\Big(H_{m-1}-H_{m-k}\Big)\times\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big),
\end{aligned} \tag{4.15}$$

where $H_m = \sum_{j=1}^m \frac{1}{j}$, $(m \geqslant 1)$ are the harmonic numbers.

From (4.14) and (4.15), and recalling that $\Lambda_1 = 0$, by convention (cf. (4.10)), we can conclude the following:

$$C_n^{(m)} = -\sum_{k=1}^{m}\frac{(m-1)_{k-1}}{(2\pi i n)^k}\Lambda_{m-k+1} - \frac{1}{m}\sum_{j=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\Big(H_{m-1}-H_{m-k}\Big)$$

$$\times \Big\{\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big) + \delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big\}.$$

Case 2: $n=0$.

$$C_0^{(m)}=\int_0^1\gamma_m(x)dx=\frac{1}{m}\Big\{\Lambda_{m+1}-(m-1)!\Big(\binom{m+1}{s}\delta_{0,r}+\binom{m+1}{r}\delta_{0,s}\Big) + \frac{1}{m(m+1)}\Big(d_s(m+1)\delta_{0,r}+d_r(m+1)\delta_{0,s}\Big)\Big\}.$$

As we mentioned earlier, $m>r+s$, if $r+s\geqslant 1$, and $m>1$, if $r+s=0$. $\gamma_m(<x>)$ is piecewise C^∞. Moreover $\gamma_m(<x>)$ is continuous for those integers m with $\Lambda_m=0$, and discontinuous with jump discontinuities at integers for those integers m with $\Lambda_m\neq 0$.

Assume first that $\Lambda_m=0$ for an integer m. Then $\gamma_m(0)=\gamma_m(1)$, and hence $\gamma_m(<x>)$ is piecewise C^∞, and continuous. Thus Fourier series of $\gamma_m(<x>)$ converges uniformly to $\gamma_m(<x>)$, and

$$\begin{aligned}
\gamma_m(<x>) &= \frac{1}{m}\Big\{\Lambda_{m+1}-(m-1)!\Big(\binom{m+1}{s}\delta_{0,r}+\binom{m+1}{r}\delta_{0,s}\Big) \\
&\quad + \frac{1}{m(m+1)}\Big(d_s(m+1)\delta_{0,r}+d_r(m+1)\delta_{0,s}\Big)\Big\} \\
&\quad - \frac{1}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\Big(\Lambda_{m-k+1}+(H_{m-1}-H_{m-k}) \\
&\quad \times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big) \\
&\quad + \delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)\Big\}e^{2\pi i n x} \\
&= \frac{1}{m}\Big\{\Lambda_{m+1}-(m-1)!\Big(\binom{m+1}{s}\delta_{0,r}+\binom{m+1}{r}\delta_{0,s}\Big) \\
&\quad + \frac{1}{m(m+1)}\Big(d_s(m+1)\delta_{0,r}+d_r(m+1)\delta_{0,s}\Big)\Big\} \\
&\quad + \frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}\Big(\Lambda_{m-k+1}+(H_{m-1}-H_{m-k}) \\
&\quad \times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big) \\
&\quad + \delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)\Big(-k!\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^k}\Big) \\
&= \frac{1}{m}\sum_{\substack{k=0\\ k\neq 1}}^{m}\binom{m}{k}\Big(\Lambda_{m-k+1}+(H_{m-1}-H_{m-k}) \\
&\quad \times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big) \\
&\quad + \delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)B_k(<x>) \\
&\quad + \Lambda_m\times\begin{cases} B_1(<x>), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

Now, we are going to state our first result.

Theorem 4.1. *For each integer* $l > 1$, *we let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)} \Big(k! \binom{k}{r} (l-k)! \binom{l-k}{s} - d_r(k) d_s(l-k) \Big),$$

with $\Lambda_1 = 0$. *Assume that* $\Lambda_m = 0$ *for some integer* $m > r+s$, $(r+s \geqslant 1)$, *or for some integer* $m > 1$, $(r+s=0)$. *Then we have the following.*

(a) $\sum_{k=1}^{m-1} \frac{1}{k(m-k)} d_r(k, <x>) d_s(m-k, <x>)$ *has the Fourier series expansion*

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} d_r(k, <x>) d_s(m-k, <x>) \\
&= \frac{1}{m} \Big\{ \Lambda_{m+1} - (m-1)! \Big(\binom{m+1}{s} \delta_{0,r} + \binom{m+1}{r} \delta_{0,s} \Big) \\
&+ \frac{1}{m(m+1)} \Big(d_s(m+1)\delta_{0,r} + d_r(m+1)\delta_{0,s} \Big) \Big\}. \\
&- \frac{1}{m} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \Big\{ \sum_{k=1}^{m} \frac{(m)_k}{(2\pi i n)^k} \Big(\Lambda_{m-k+1} + \big(H_{m-1} - H_{m-k} \big) \\
&\times \Big(\delta_{0,s} \Big((m-k)! \binom{m-k+1}{r} - \frac{d_r(m-k+1)}{m-k+1} \Big) \\
&+ \delta_{0,r} \Big((m-k)! \binom{m-k+1}{s} - \frac{d_s(m-k+1)}{m-k+1} \Big) \Big) \Big) \Big\} e^{2\pi i n x}
\end{aligned}$$

for all $x \in \mathbb{R}$, *where the convergence is uniform.*

(b)

$$\begin{aligned}
&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} d_r(k, <x>) d_s(m-k, <x>) \\
&= \frac{1}{m} \sum_{\substack{k=0 \\ k \neq 1}}^{m} \binom{m}{k} \Big(\Lambda_{m-k+1} + \big(H_{m-1} - H_{m-k} \big) \times \Big(\delta_{0,s} \Big((m-k)! \binom{m-k+1}{r} - \frac{d_r(m-k+1)}{m-k+1} \Big) \\
&+ \delta_{0,r} \Big((m-k)! \binom{m-k+1}{s} - \frac{d_s(m-k+1)}{m-k+1} \Big) \Big) \Big) B_k(<x>)
\end{aligned}$$

for all $x \in \mathbb{R}$.

Assume that $\Lambda_m \neq 0$, for an integer $m > r+s$, $(r+s \geqslant 1)$, or for some integer $m > 1$, $(r+s=0)$. Then $\gamma_m(0) \neq \gamma_m(1)$, and hence $\gamma_m(<x>)$ is piecewise C^∞, and discontinuous with jump discontinuities at integers. Hence the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$ for $x \notin \mathbb{Z}$, and converges to

$$\frac{1}{2}\big(\gamma_m(0) + \gamma_m(1)\big) = \gamma_m(0) + \frac{1}{2}\Lambda_m$$

for $x \in \mathbb{Z}$. Next, we are going to state our second result.

Theorem 4.2. *For each integer* $l > 1$, *we let*

$$\Lambda_l = \sum_{k=1}^{l-1} \frac{1}{k(l-k)} \Big(k! \binom{k}{r} (l-k)! \binom{l-k}{s} - d_r(k) d_s(l-k) \Big)$$

with $\Lambda_1 = 0$. *Assume that* $\Lambda_m \neq 0$ *for some integer* $m > r+s$, $(r+s \geqslant 1)$, *or for some integer* $m > 1$, $(r+s=0)$.

Then we have the following.

(a)

$$\begin{aligned}
&\frac{1}{m}\Big\{\Lambda_{m+1}-(m-1)!\Big(\binom{m+1}{s}\delta_{0,r}+\binom{m+1}{r}\delta_{0,s}\Big)\\
&\quad+\frac{1}{m(m+1)}\big(d_s(m+1)\delta_{0,r}+d_r(m+1)\delta_{0,s}\big)\Big\}\\
&\quad-\frac{1}{m}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Big\{\sum_{k=1}^{m}\frac{(m)_k}{(2\pi i n)^k}\Big(\Lambda_{m-k+1}+\big(H_{m-1}-H_{m-k}\big)\\
&\quad\times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big)\\
&\quad+\delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)\Big\}e^{2\pi i n x}\\
&=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}d_r(k,<x>)d_s(m-k,<x>), & \textit{for } x\notin\mathbb{Z},\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}d_r(k)d_s(m-k)+\frac{1}{2}\Lambda_m. & \textit{for } x\in\mathbb{Z}.\end{cases}
\end{aligned}$$

(b)

$$\begin{aligned}
&\frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\Big(\Lambda_{m-k+1}+\big(H_{m-1}-H_{m-k}\big)\times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big)\\
&\quad+\delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)B_k(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}d_r(k,<x>)d_s(m-k,<x>),\ \textit{for } x\notin\mathbb{Z},\\
&\frac{1}{m}\sum_{\substack{k=0\\ k\neq 1}}^{m}\binom{m}{k}\Big(\Lambda_{m-k+1}+\big(H_{m-1}-H_{m-k}\big)\times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big)\\
&\quad+\delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)B_k(<x>)\\
&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}d_r(k)d_s(m-k)+\frac{1}{2}\Lambda_m,\ \textit{for } x\in\mathbb{Z}.
\end{aligned}$$

Corollary 4.3. *For each integer $l>1$, we let*

$$\Lambda_l=\sum_{k=1}^{l-1}\frac{1}{k(l-k)}\Big(k!\binom{k}{r}(l-k)!\binom{l-k}{s}-d_r(k)d_s(l-k)\Big)$$

with $\Lambda_1=0$. Then we have the following polynomial identity:

$$\begin{aligned}
\sum_{k=1}^{m-1}\frac{1}{k(m-k)}d_r(k,x)d_s(m-k,x)&=\frac{1}{m}\sum_{k=0}^{m}\binom{m}{k}\Big(\Lambda_{m-k+1}+\big(H_{m-1}-H_{m-k}\big)\\
&\quad\times\Big(\delta_{0,s}\Big((m-k)!\binom{m-k+1}{r}-\frac{d_r(m-k+1)}{m-k+1}\Big)\\
&\quad+\delta_{0,r}\Big((m-k)!\binom{m-k+1}{s}-\frac{d_s(m-k+1)}{m-k+1}\Big)\Big)\Big)B_k(x),
\end{aligned}$$

($m>r+s$, if $r+s\geqslant 1$, and $m>1$, if $r+s=0$).

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