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On certain classes of bi-univalent functions related to m-fold symmetry

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Abstract

In our present investigation, we introduce two new subclasses $S_{\Sigma_m}(\alpha,\lambda,\mu)$ and $S_{\Sigma_m}(\beta,\lambda,\mu)$ of analytic and m-fold symmetric bi-univalent functions in the open unit disk E. Results concerning coefficient estimates for the functions of these classes are derived. Many interesting new and already existing corollaries are also presented.

Keywords: m-Fold symmetry, bi-univalent functions, coefficient estimates.

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1. Introduction

Let $\mathcal{A}$ denotes the class of all functions $f(z)$ which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and has the Taylor series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function is said to be univalent if it never takes the same value twice. By $\mathcal{S}$ we mean the subclass of $\mathcal{A}$ consisting of univalent functions. Every univalent function $f \in \mathcal{S}$ has an inverse f^{-1} which is defined as:

$$f^{-1}(f(z)) = z, \quad z \in E,$$

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and

$$f(f^{-1}(w)) = w,\ |w| < r_0(f),\ r_0(f) \geqslant \frac{1}{4},$$

where

$$g_1(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots . \tag{1.1}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in E if both f and f^{-1} are univalent in E. Let Σ denotes the class of analytic and bi-univalent functions in E. Few examples of functions in class Σ are

$$h_1(z) = \frac{z}{1-z},\ h_2(z) = -\log(1-z),\ h_3(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), z \in E.$$

For $f \in \Sigma$, Lewin [20], showed that $|a_2| < 1.5$. For more work on bi-univalent one can refer to [1, 3–13, 15, 16, 18, 19, 21–26, 28–31].

Let m be a positive integer. A domain E is said to be m-fold symmetric if

$$f\left(e^{i\frac{2\pi}{m}}z\right) = e^{i\frac{2\pi}{m}}f(z),\ z \in E,\ f \in \mathcal{A}.$$

For each function $f \in \mathcal{S}$, the function

$$h(z) = \sqrt[m]{f(z^m)}$$

is univalent and maps the unit disk E into a region with m-fold symmetry.

We denote by $\mathcal{S}^m$ the class of m-fold symmetric univalent functions in E and clearly $S^1 = S$. Let $f \in \mathcal{S}^m$ has a series expansion of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1}. \tag{1.2}$$

Srivastava et al. [25], introduced a natural extensions of m-fold symmetric univalent functions and defined the class Σ_m of symmetric bi-univalent functions. They obtained the series expansion for $g = f^{-1}$ as:

$$g(w) = f^{-1}(w) = \begin{cases} w - a_{m+1}w^{m+1} + ((m+1)a_{m+1}^2 - a_{2m+1})w^{2m+1}, \\ -\left\{\begin{array}{c} \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 \\ -(3m+2)a_{m+1}a_{2m+1} + a_{3m+1}) \end{array}\right\} w^{3m+1} + \cdots , \end{cases} \tag{1.3}$$

where f is given by (1.2). For $m = 1$, the equation (1.3) coincides with the equation (1.1) of the class Σ. Under the following assumptions, $z, w \in E$, $f^{-1} = g$, $m \in N$, $0 < \alpha \leqslant 1$, $0 \leqslant \beta < 1$, $0 \leqslant \mu$, and $0 \leqslant \lambda \leqslant 1$, we introduced new subclasses of m-fold symmetric bi-univalent functions and derive initial coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for these classes.

1.1. *The class* $\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$

Definition 1.1. A function $f \in \Sigma_m$, is said to be in class $\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$ if the following conditions are satisfied

$$\left|\arg\left[(1-\lambda)\left(\frac{z^{1-\mu}f'(z)}{[f(z)]^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu}f''(z)}{[zf'(z)]^{1-\mu}}\right)\right]\right| < \frac{\alpha\pi}{2},$$

and

$$\left|\arg\left[(1-\lambda)\left(\frac{w^{1-\mu}g'(w)}{[g(w)]^{1-\mu}}\right) + \lambda\left(1 + \frac{w^{2-\mu}g''(w)}{[wg'(w)]^{1-\mu}}\right)\right]\right| < \frac{\alpha\pi}{2}.$$

Remark 1.2. On specializing the parameter λ, μ, m one can state the various new as well as known subclasses of analytic bi-univalent functions studied earlier in the literature.

(i) For $m = 1$, we obtain new class of bi-univalent function.

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{S}_{\Sigma}(\alpha,\lambda,\mu).$$

(ii) For $\lambda = 0$, we obtain new class which consists m-fold symmetric bi starlike function.

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{R}_{\Sigma_m}(\alpha,\mu).$$

(iii) For $\lambda = 1$, we obtain a new class which consists m-fold symmetric convex bi-univalent function.

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{C}_{\Sigma_m}(\alpha,\mu).$$

(iv) For $\lambda = 0$, and $\mu = 0$, we obtain class which consists m-fold symmetric bi-univalent function [2].

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{S}^{\alpha}_{\Sigma,m}.$$

(v) For $\lambda = 0, m = 1$ and $\mu = 0$, we obtain class of bi-univalent function introduced by Brannan and Taha [5].

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{S}^{*}_{\Sigma}(\alpha).$$

(vi) For $\lambda = 0$ and $\mu = 1$, we obtain class which consists m-fold symmetric bi-univalent function introduced by Srivastava et al. [27].

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{H}_{\Sigma,m}(\alpha).$$

(vii) For $\lambda = 0$, $m = 1$, and $\mu = 1$, we obtain class of bi-univalent function introduced by Srivastava et al. [26].

$$\mathcal{S}_{\Sigma_m}(\alpha,\lambda,\mu) = \mathcal{H}_{\Sigma}(\alpha).$$

1.2. The class $\mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu)$

Definition 1.3. A function $f \in \Sigma_m$ is said to be in class $S_{\Sigma_m}(\beta,\lambda,\mu)$ if the following conditions are satisfied

$$\operatorname{Re}\left[(1-\lambda)\left(\frac{z^{1-\mu}f'(z)}{[f(z)]^{1-\mu}}\right)+\lambda\left(1+\frac{z^{2-\mu}f''(z)}{[zf'(z)]^{1-\mu}}\right)\right] > \beta,$$

and

$$\operatorname{Re}\left[(1-\lambda)\left(\frac{w^{1-\mu}g'(w)}{[g(w)]^{1-\mu}}\right)+\lambda\left(1+\frac{w^{2-\mu}g''(w)}{[wg'(w)]^{1-\mu}}\right)\right] > \beta.$$

Remark 1.4. On specializing the parameters λ, μ, m one can state the various new as well as known subclasses of analytic bi-univalent functions studied earlier in the literature.

(i) For $m = 1$, we obtain new class of bi-univalent function.

$$\mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu) = \mathcal{S}_{\Sigma}(\beta,\lambda,\mu).$$

(ii) For $\lambda = 0$, we obtain new class which consists m-fold symmetric bi starlike function.

$$\mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu) = \mathcal{R}_{\Sigma_m}(\beta,\mu).$$

(iii) For $\lambda = 1$, we obtain new class which consists m-fold symmetric convex bi-univalent function.

$$\mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu) = \mathcal{C}_{\Sigma_m}(\beta,\mu).$$

(iv) For $\lambda = 0$, and $\mu = 0$, we obtain class which consists m-fold symmetric bi-univalent function [17].

$$\mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu) = \mathcal{N}^{0}_{\Sigma,m}(\beta,1).$$

(v) For $\lambda = 0$, $m = 1$ and $\mu = 0$, we obtain class of bi-univalent function introduced by Brannan and Taha [5].
$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{S}^*_{\Sigma}(\beta).$$

(vi) For $\lambda = 0$ and $\mu = 1$, we obtain class which consists m-fold symmetric bi-univalent function introduced by Srivastava et al. [27].
$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{H}_{\Sigma,m}(\beta).$$

(vii) For $\lambda = 0$, $m = 1$, and $\mu = 1$, we obtain of bi-univalent function introduced by Srivastava et al. [26]
$$\mathcal{S}_{\Sigma_m}(\beta, \lambda, \mu) = \mathcal{H}_{\Sigma}(\beta).$$

In order to establish our main results, we shall required the following lemma.

Lemma 1.5 ([14])**.** $p \in P$, $|c_n| \leqslant 2$, $n \in N$, *where the Caratheodory class* P *is the family of all functions* p *analytic in* E *for which* $\operatorname{Re}(p(z)) > 0$,
$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots .$$

2. Main results

Theorem 2.1. *Let* f *given by* (1.2) *is in the class* $\mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$*, then*
$$|a_{m+1}| \leqslant \frac{2\alpha}{\sqrt{\alpha Q_1(\lambda, \mu, m) - (\alpha Q_2(\lambda, \mu, m) + Q_3(\lambda, \mu, m))}},$$
and
$$|a_{2m+1}| \leqslant \frac{2\alpha}{[(2m+\mu) + \lambda(4m^2 - \mu)]} + \frac{2\alpha^2\,[Q_1(\lambda, \mu, m) + Q_2(\lambda, \mu, m)]}{[(2m+\mu) + \lambda(4m^2 - \mu)]\,[(m+\mu) + \lambda(m^2 - \mu)]^2},$$
where $Q_1(\lambda, \mu, m)$*,* $Q_2(\lambda, \mu, m)$*,* $Q_3(\lambda, \mu, m)$ *are given by* (2.9)*,* (2.10)*,* (2.12)*, respectively.*

Proof. Let $f \in \mathcal{S}_{\Sigma_m}(\alpha, \lambda, \mu)$, then
$$(1-\lambda)\left(\frac{z^{1-\mu} f'(z)}{[f(z)]^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu} f''(z)}{[zf'(z)]^{1-\mu}}\right) = [p(z)]^{\alpha}, \tag{2.1}$$
and for its inverse map $g = f^{-1}$, we have
$$(1-\lambda)\left(\frac{w^{1-\mu} g'(w)}{[g(w)]^{1-\mu}}\right) + \lambda\left(1 + \frac{w^{2-\mu} g''(w)}{[wg'(w)]^{1-\mu}}\right) = [q(w)]^{\alpha}, \tag{2.2}$$
where $p(z)$ and $q(w)$ have the following forms:
$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots,$$
and
$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots .$$
Now equating the coefficients in (2.1) and (2.2) we obtain
$$(m+\mu) + \lambda(m^2 - \mu) a_{m+1} = \alpha p_m, \tag{2.3}$$
$$\left\{\begin{array}{c} (2m+\mu) + \lambda(4m^2 - \mu) a_{2m+1} \\ -Q_2(\lambda, \mu, m) a_{m+1}^2 \end{array}\right\} = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2, \tag{2.4}$$
$$-(m+\mu) + \lambda(m^2 - \mu) a_{m+1} = \alpha q_m, \tag{2.5}$$

$$\left\{ \begin{array}{c} Q_1(\lambda,\mu,m)a_{m+1}^2 \\ -(2m+\mu)+\lambda(4m^2-\mu)a_{2m+1} \end{array} \right\} = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \tag{2.6}$$

From (2.3) and (2.5) we obtain

$$p_m = -q_m, \tag{2.7}$$

and

$$\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2 2a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.8}$$

Also form (2.4), (2.6), and (2.8) we have

$$\begin{aligned} \{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)\}\, a_{m+1}^2 &= \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2+q_m^2) \\ &= \left\{ \begin{array}{c} \alpha(p_{2m}+q_{2m}) \\ +\frac{(\alpha-1)}{\alpha}\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2 a_{m+1}^2 \end{array} \right\}, \end{aligned}$$

where

$$Q_1(\lambda,\mu,m) = \left[(1-\lambda)(1+2m)(m+\mu)+\lambda m(m+1)\{3m+\mu(m+1)+1)\}\right], \tag{2.9}$$

$$Q_2(\lambda,\mu,m) = \frac{(1-\mu)}{2!}\left\{(1-\lambda)(\mu+2m)+2\lambda m(m+1)^2\right\}. \tag{2.10}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{\alpha Q_1(\lambda,\mu,m)-[\alpha Q_2(\lambda,\mu,m)+Q_3(\lambda,\mu,m)]}, \tag{2.11}$$

where

$$Q_3(\lambda,\mu,m) = (\alpha-1)\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2. \tag{2.12}$$

Applying Lemma 1.5, on equation (2.11) for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leqslant \frac{2\alpha}{\sqrt{\alpha Q_1(\lambda,\mu,m)-[\alpha Q_2(\lambda,\mu,m)+Q_3(\lambda,\mu,m)]}}.$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (2.6) from (2.4), we obtain

$$\left[\begin{array}{c} 2\left\{(2m+\mu)+\lambda(4m^2-\mu)\right\} a_{2m+1} \\ -\{Q_2(\lambda,\mu,m)+Q_1(\lambda,\mu,m)\}\, a_{m+1}^2 \end{array} \right] = \alpha(p_{2m}-q_{2m}) + \frac{\alpha(\alpha-1)}{2}(p_m^2-q_m^2). \tag{2.13}$$

Then, in view of (2.7) and (2.8), and applying Lemma 1.5, on (2.13) for the coefficients p_{2m}, q_{2m}, p_m and q_m we have

$$|a_{2m+1}| \leqslant \frac{2\alpha}{[(2m+\mu)+\lambda(4m^2-\mu)]} + \frac{2\alpha^2\left[Q_1(\lambda,\mu,m)+Q_2(\lambda,\mu,m)\right]}{[(2m+\mu)+\lambda(4m^2-\mu)]\left[(m+\mu)+\lambda(m^2-\mu)\right]^2},$$

which completes the proof of Theorem 2.1. □

For $m = 1$, in Theorem 2.1, we have the following Corollary.

Corollary 2.2. *Let* f *given by* (1.2) *is in the class* $\mathcal{S}_\Sigma(\alpha,\lambda,\mu)$, *then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{\alpha Q_{10}(\lambda,\mu)-(\alpha Q_{11}(\lambda,\mu)+Q_{12}(\lambda,\mu))}},$$

and

$$|a_3| \leqslant \frac{2\alpha}{(2+\mu)+\lambda(4-\mu)} + \frac{2\alpha^2\left[Q_{10}(\lambda,\mu)+Q_{11}(\lambda,\mu)\right]}{[(2+\mu)+\lambda(4-\mu)]\left[(1+\mu)+\lambda(1-\mu)\right]^2},$$

where

$$Q_{10}(\lambda,\mu) = 3(1-\lambda)(1+\mu) + 4\lambda\,(2+\mu)\,, \tag{2.14}$$

$$Q_{11}(\lambda,\mu) = \frac{(1-\mu)}{2!}\{(1-\lambda)(\mu+2)+8\lambda\}, \tag{2.15}$$

$$Q_{12}(\lambda,\mu) = (\alpha-1)\{(1+\mu)+\lambda(1-\mu)\}^2.$$

For $\lambda = 0$ in Theorem 2.1, we have the following Corollary.

Corollary 2.3. *Let* f *given by* (1.2) *is in the class* $\mathcal{R}_{\Sigma_m}(\alpha,\mu)$, *then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{\sqrt{\alpha Q_4(\mu,m)-(\alpha Q_5(\mu,m)+Q_6(\mu,m))}},$$

and

$$|a_{2m+1}| \leqslant \frac{2\alpha}{(2m+\mu)} + \frac{2\alpha^2\,[Q_4(\mu,m)+Q_5(\mu,m)]}{(2m+\mu)(m+\mu)^2},$$

where

$$Q_4(\mu,m) = (1+2m)(m+\mu), \tag{2.16}$$

$$Q_5(\mu,m) = \frac{(1-\mu)}{2!}(\mu+2m), \tag{2.17}$$

$$Q_6(\mu,m) = (\alpha-1)(m+\mu)^2.$$

For $\lambda = 1$ in Theorem 2.1 we have the following Corollary.

Corollary 2.4. *Let* f *given by* (1.2) *is in the class* $\mathcal{C}_{\Sigma_m}(\alpha,\mu)$, *then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{\sqrt{\alpha Q_7(\mu,m)-(\alpha Q_8(\mu,m)+Q_9(\mu,m))}},$$

and

$$|a_{2m+1}| \leqslant \frac{2\alpha}{[(2m+\mu)+(4m^2-\mu)]} + \frac{2\alpha^2\,[Q_7(\mu,m)+Q_8(\mu,m)]}{[(2m+\mu)+(4m^2-\mu)]\,[(m+\mu)+(m^2-\mu)]^2},$$

where

$$Q_7(\mu,m) = m(m+1)\{3m+\mu(m+1)+1)\}, \tag{2.18}$$

$$Q_8(\mu,m) = \frac{(1-\mu)}{2!}\left\{2m(m+1)^2\right\}, \tag{2.19}$$

$$Q_9(\mu,m) = (\alpha-1)\left\{(m+\mu)+(m^2-\mu)\right\}^2.$$

For $\lambda = 0$ and $\mu = 1$ in Theorem 2.1, we obtain the following Corollary.

Corollary 2.5 ([27])**.** *Let* f *given by* (1.2) *be in the class* $\mathcal{H}^{\alpha}_{\Sigma,m}$, *then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{\sqrt{(1+m)(1+m+2\alpha)}}, \quad \textit{and} \quad |a_{2m+1}| \leqslant \frac{2\alpha}{(2m+1)} + \frac{2\alpha^2}{(m+1)}.$$

For $\lambda = 0$, $m = 1$, and $\mu = 1$, in Theorem 2.1, we obtain the following Corollary.

Corollary 2.6 ([26])**.** *Let* f *given by* (1.2) *be in the class* H^{α}_{Σ}, *then*

$$|a_2| \leqslant \frac{\alpha}{\sqrt{(1+\alpha)}}, \quad \textit{and} \quad |a_3| \leqslant \frac{2\alpha}{3} + \frac{\alpha^2}{1}.$$

For $\lambda = 0$ and $\mu = 0$ in Theorem 2.1, we obtain the following Corollary.

Corollary 2.7 ([2]). *Let* f *given by* (1.2) *be in the class* $\mathcal{S}^{\alpha}_{\Sigma,m}$, *then*

$$|a_{m+1}| \leqslant \frac{2\alpha}{m\sqrt{\alpha+1}}, \quad \textit{and} \quad |a_{2m+1}| \leqslant \frac{\alpha}{m} + \frac{2\alpha^2\left[(1+m)\right]}{m^2}.$$

For $\lambda = 0,\ m = 1$, and $\mu = 0$, in Theorem 2.1, we obtain the following Corollary.

Corollary 2.8 ([2]). *Let* f *given by* (1.2) *be in the class* $\mathcal{S}^{*}_{\Sigma}(\alpha)$, *then*

$$|a_2| \leqslant \frac{2\alpha}{\sqrt{\alpha+1}}, \quad \textit{and} \quad |a_3| \leqslant 4\alpha^2 + \alpha.$$

Theorem 2.9. *Let* f *given by* (1.2) *is in the class* $\mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu)$, *then*

$$|a_{m+1}| \leqslant \frac{2\sqrt{(1-\beta)}}{\sqrt{\{Q_1(\lambda,\mu,m) - Q_2(\lambda,\mu,m)\}}},$$

and

$$|a_{2m+1}| \leqslant \frac{2(1-\beta)}{\{(2m+\mu)+\lambda(4m^2-\mu)\}} + \frac{2\{Q_1(\lambda,\mu,m)+Q_2(\lambda,\mu,m)\}(1-\beta)^2}{\{(2m+\mu)+\lambda(4m^2-\mu)\}\{(m+\mu)+\lambda(m^2-\mu)\}^2},$$

where $Q_1(\lambda,\mu,m)$ *and* $Q_2(\lambda,\mu,m)$ *are given by* (2.9) *and* (2.10), *respectively.*

Proof. Let $f \in \mathcal{S}_{\Sigma_m}(\beta,\lambda,\mu)$, then

$$(1-\lambda)\left(\frac{z^{1-\mu}f'(z)}{[f(z)]^{1-\mu}}\right) + \lambda\left(1 + \frac{z^{2-\mu}f''(z)}{[zf'(z)]^{1-\mu}}\right) = \beta + (1-\beta)p(z), \tag{2.20}$$

and for its inverse map $g = f^{-1}$, we have

$$(1-\lambda)\left(\frac{w^{1-\mu}g'(w)}{[g(w)]^{1-\mu}}\right) + \lambda\left(1 + \frac{w^{2-\mu}g''(w)}{[wg'(w)]^{1-\mu}}\right) = \beta + (1-\beta)q(w), \tag{2.21}$$

where $p,\ q \in P$ and $g = f^{-1}$. Now, equating the coefficients in (2.20) and (2.21), we obtain

$$(m+\mu)+\lambda(m^2-\mu)a_{m+1} = (1-\beta)p_m, \tag{2.22}$$
$$\left\{(2m+\mu)+\lambda(4m^2-\mu)\right\}a_{2m+1} - Q_2(\lambda,\mu,m)a^2_{m+1} = (1-\beta)p_{2m}, \tag{2.23}$$
$$-(m+\mu)+\lambda(m^2-\mu)a_{m+1} = (1-\beta)q_m, \tag{2.24}$$
$$Q_1(\lambda,\mu,m)a^2_{m+1} - \left\{(2m+\mu)+\lambda(4m^2-\mu)\right\}a_{2m+1} = (1-\beta)q_{2m}. \tag{2.25}$$

From (2.22) and (2.24) we obtain

$$p_m = -q_m, \tag{2.26}$$

and

$$2\left\{(m+\mu)+\lambda(m^2-\mu)\right\}^2 a^2_{m+1} = (1-\beta)^2(p^2_m + q^2_m). \tag{2.27}$$

Adding (2.23) and (2.25), we have

$$\{Q_1(\lambda,\mu,m) - Q_2(\lambda,\mu,m)\}\,a^2_{m+1} = (1-\beta)(p_{2m}+q_{2m}),$$

therefore, we have

$$a^2_{m+1} = \frac{(1-\beta)(p_{2m}+q_{2m})}{Q_1(\lambda,\mu,m) - Q_2(\lambda,\mu,m)}. \tag{2.28}$$

Applying Lemma 1.5, on equation (2.28) for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leqslant \frac{2\sqrt{(1-\beta)}}{\sqrt{Q_1(\lambda,\mu,m)-Q_2(\lambda,\mu,m)}}.$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (2.25) from (2.23), we obtain

$$\begin{bmatrix} 2\left\{(2m+\mu)+\lambda(4m^2-\mu)\right\} a_{2m+1} \\ -\{Q_1(\lambda,\mu,m)+Q_2(\lambda,\mu,m)\} a_{m+1}^2 \end{bmatrix} = (1-\beta)(p_{2m}-q_{2m}), \tag{2.29}$$

then, in view of (2.26) and (2.27), and applying Lemma 1.5, on equation (2.29) for the coefficients p_{2m}, q_{2m}, p_m, and q_m we have

$$|a_{2m+1}| \leqslant \frac{2(1-\beta)}{(2m+\mu)+\lambda(4m^2-\mu)} + \frac{2\{Q_1(\lambda,\mu,m)+Q_2(\lambda,\mu,m)\}(1-\beta)^2}{\{(2m+\mu)+\lambda(4m^2-\mu)\}\{(m+\mu)+\lambda(m^2-\mu)\}^2},$$

which completes the proof of Theorem 2.9. □

For $m = 1$ in Theorem 2.9, we have the following Corollary.

Corollary 2.10. *Let* f *given by* (1.2) *is in the class* $\mathcal{S}_\Sigma(\beta,\lambda,\mu)$, *then*

$$|a_2| \leqslant \frac{2\sqrt{(1-\beta)}}{\sqrt{\{Q_{10}(\lambda,\mu)-Q_{11}(\lambda,\mu)\}}},$$

and

$$|a_3| \leqslant \frac{2(1-\beta)}{\{(2+\mu)+\lambda(4-\mu)\}} + \frac{2\{Q_{10}(\lambda,\mu)+Q_{11}(\lambda,\mu)\}(1-\beta)^2}{\{(2+\mu)+\lambda(4-\mu)\}\{(1+\mu)+\lambda(1-\mu)\}^2},$$

where $Q_{10}(\lambda,\mu)$ *and* $Q_{11}(\lambda,\mu)$ *are given by* (2.14) *and* (2.15).

For $\lambda = 0$, in Theorem 2.9, we have the following Corollary.

Corollary 2.11. *Let* f *given by* (1.2) *is in the class* $\mathcal{R}_{\Sigma_m}(\beta,\mu)$, *then*

$$|a_{m+1}| \leqslant \frac{2\sqrt{(1-\beta)}}{\sqrt{Q_4(\mu,m)-Q_5(\mu,m)}},$$

and

$$|a_{2m+1}| \leqslant \frac{2(1-\beta)}{(2m+\mu)} + \frac{2\{Q_4(\mu,m)+Q_5(\mu,m)\}(1-\beta)^2}{(2m+\mu)(m+\mu)^2},$$

where $Q_4(\mu,m)$ *and* $Q_5(\mu,m)$ *are given by* (2.16) *and* (2.17), *respectively.*

For $\lambda = 1$, in Theorem 2.9, we have the following Corollary.

Corollary 2.12. *Let* f *given by* (1.2) *is in the class* $\mathcal{C}_{\Sigma_m}(\beta,\mu)$, *then*

$$|a_{m+1}| \leqslant \frac{2\sqrt{(1-\beta)}}{\sqrt{Q_7(\mu,m)-Q_8(\mu,m)}},$$

and

$$|a_{2m+1}| \leqslant \frac{2(1-\beta)}{\{(2m+\mu)+(4m^2-\mu)\}} + \frac{2\{Q_7(\mu,m)+Q_8(\mu,m)\}(1-\beta)^2}{\{(2m+\mu)+(4m^2-\mu)\}\{(m+\mu)+(m^2-\mu)\}^2},$$

where $Q_7(\mu,m)$ *and* $Q_8(\mu,m)$ *are given by* (2.18) *and* (2.19), *respectively.*

For $\lambda = 0$ and $\mu = 1$ in Theorem 2.9, we obtain the following Corollary.

Corollary 2.13 ([27]). *Let* f *given by* (1.2) *be in the class* $\mathcal{H}^{\beta}_{\Sigma,m}$, *then*

$$|a_{m+1}| \leqslant \frac{2\sqrt{(1-\beta)}}{\sqrt{(1+2m)(m+1)}}, \quad \textit{and} \quad |a_{2m+1}| \leqslant \frac{2(1-\beta)}{(2m+1)} + \frac{2(1-\beta)^2}{(m+1)}.$$

For $\lambda = 0$, $m = 1$, and $\mu = 1$, in Theorem 2.9, we obtain the following Corollary.

Corollary 2.14 ([26]). *Let* f *given by* (1.2) *be in the class* $\mathcal{H}^{\beta}_{\Sigma}$, *then*

$$|a_2| \leqslant \sqrt{\frac{2(1-\beta)}{3}}, \quad \textit{and} \quad |a_3| \leqslant \frac{2(1-\beta)}{3} + \frac{(1-\beta)^2}{1}.$$

For $\lambda = 0$ and $\mu = 0$ in Theorem 2.9, we obtain the following Corollary.

Corollary 2.15 ([2]). *Let* f *given by* (1.2) *be in the class* $\mathcal{S}^{\beta}_{\Sigma_m}$, *then*

$$|a_{m+1}| \leqslant \sqrt{\frac{2(1-\beta)}{m}}, \quad \textit{and} \quad |a_{2m+1}| \leqslant \frac{(1-\beta)}{m} + \frac{2(1+m)(1-\beta)^2}{m^2}.$$

For $\lambda = 0$, $m = 1$, and $\mu = 0$, in Theorem 2.9, we obtain the following Corollary.

Corollary 2.16 ([2]). *Let* f *given by* (1.2) *be in the class* $\mathcal{S}^{*}_{\Sigma}(\beta)$, *then*

$$|a_2| \leqslant \sqrt{2(1-\beta)}, \quad \textit{and} \quad |a_3| \leqslant 4(1-\beta)^2 + (1-\beta).$$

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