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Hitting probabilities for non-convex lattice

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Abstract

In this paper, we consider three lattices with cells represented in Figures 1, 3, and 5 and we determine the probability that a random segment of constant length intersects a side of the lattice considered.

Keywords: Geometric probability, stochastic geometry, random sets, random convex sets and integral geometry.

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1. Introduction

Caristi and Stoka [7] and [8] introduced in the Buffon-Laplace type problems so-called obstacles. They considered two lattices with axial symmetry and in a first moment [7] they study with eight triangular and circular sector obstacles and in the second moment [8] they analyze twelve obstacles. Several other authors considered different lattices with different types of obstacles and studied the probability that a random body test intersect the fundamental cell [2, 5], and [4]. In particular, in [1], the authors studied a Laplace type problem with obstacles for two Delone hexagonal lattices and in [6] for a regular lattice of Dirichlet-Voronoi. In this study, starting from the results obtained by Duma and Stoka [9] for Buffon type problems with a non-convex lattice we consider a Laplace type problem for three lattices with triangular obstacles, circular sector obstacles and triangular and sectors circular together. We study the probability that a random segment of constant length intersects the fundamental cells in Figures 1, 3, and 5.

2. Obstacles triangular

Let $\mathfrak{R}_1(a, b; m)$ be the lattice with the fundamental cell C_1 represented in Figure 1, where $a < b$ and $m < a/2$. From Figure 1 we have

$$\text{area}C_1 = 3ab - \frac{5}{2}m^2. \tag{2.1}$$

We compute the probability that a random segment s of constant length $l < \frac{a}{2} - m$ intersects a side of lattice $\mathfrak{R}_1$, i.e., the probability $P^{(1)}_{int}$ that the segment s intersects a side of fundamental cell C_1.

The position of segment s is determined by its center and by the angle φ that it formed with the side BC (or AF) of the cell C_1.

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To compute P_{int} we consider the limiting positions of segment s, for a fixed angle of φ, in the cell C_1. We obtain the Figure 2

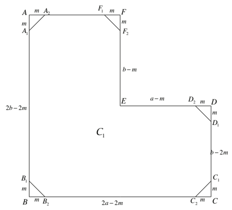

Figure 1

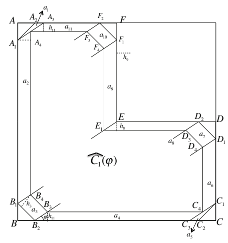

Figure 2

and the formula

$$\text{area}\widehat{C}_1(\varphi) = \text{area}C_1 - \sum_{i=1}^{11} \text{area}a_i(\varphi). \tag{2.2}$$

Theorem 2.1. *We have*

$$P_{int}^{(1)} = \frac{2\left[2(a+b)l - \frac{l^2}{2} - \frac{\pi}{2}m^2\right]}{\pi\left(3ab - \frac{5}{2}m^2\right)}.$$

Proof. By Figure 2 we have

$$\text{area}AA_1A_3 = \frac{ml}{2}\cos\varphi, \qquad \text{area}a_1(\varphi) = \text{area}a_5(\varphi) = \frac{ml}{2}\cos\varphi - \frac{m^2}{2},$$

$$\text{area}a_2(\varphi) = (b-m)\,l\cos\varphi, \qquad \text{area}a_{11}(\varphi) = \frac{al}{2}\sin\varphi - \frac{ml}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi,$$

$$\text{area}a_6(\varphi) = \frac{bl}{2}\cos\varphi - ml\cos\varphi, \qquad \text{area}a_3(\varphi) = \text{area}a_7(\varphi) = \text{area}a_{10}(\varphi) = \frac{ml}{2}(\sin\varphi + \cos\varphi),$$

$$\text{area}a_4(\varphi) = al\sin - \frac{ml}{2}\sin\varphi - \frac{l^2}{4}\sin 2\varphi, \qquad \text{area}a_8(\varphi) = \frac{al}{2}\sin\varphi - \frac{ml}{2}\sin\varphi,$$

$$\text{area}a_9(\varphi) = \frac{bl}{2}\cos\varphi - \frac{ml}{2}\cos\varphi.$$

We can write that

$$A_1(\varphi) = \sum_{i=1}^{11} \text{area}a_i(\varphi) = 2al\sin\varphi + 2bl\cos\varphi - \frac{l^2}{2}\sin 2\varphi - m^2. \tag{2.3}$$

Replacing this relation in formula (2.2) follows

$$\text{area}C_1(\varphi) = \text{area}C_1 - A_1(\varphi). \tag{2.4}$$

We denote with M_1, the set of segments s that they have center in the cell C_1, and with N_1 the set of segments s entirely contained in the cell C_1, so we have [11],

$$P_{int}^{(1)} = 1 - \frac{\mu(N_1)}{\mu(M_1)}, \tag{2.5}$$

where μ is the Lebesgue measure in the euclidean plane.

To compute the measure $\mu(M_1)$ and $\mu(N_1)$ we use the kinematic measure of Poincarè [10]:

$$dk = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinate of center of s and φ the fixed angle.

For $\varphi \in \left[0, \frac{\pi}{2}\right]$ we have

$$\mu(M_1) = \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\epsilon C_1\}} dxdy = \int_0^{\frac{\pi}{2}} (\text{area}C_1)\, d\varphi = \frac{\pi}{2}\text{area}C_1. \tag{2.6}$$

In the same way, considering formula (2.4) we can write

$$\begin{aligned}\mu(N_1) &= \int_0^{\frac{\pi}{2}} d\varphi \iint_{\{(x,y)\epsilon C_{1(\varphi)}\}} dxdy = \int_0^{\frac{\pi}{2}} [\text{area}C_1(\varphi)]\, d\varphi \\ &= \int_0^{\frac{\pi}{2}} [\text{area}C_1 - A_1(\varphi)]\, d\varphi = \frac{\pi}{2}\text{area}C_1 - \int_0^{\frac{\pi}{2}} [A_1(\varphi)]\, d\varphi.\end{aligned} \tag{2.7}$$

Replacing in the (2.5) the relations (2.6) and (2.7) we obtain

$$P_{int}^{(1)} = \frac{2}{\pi \text{area}C_1} \int_0^{\frac{\pi}{2}} [A_1(\varphi)]\, d\varphi. \tag{2.8}$$

Considering formula (2.3) we have

$$\int_0^{\frac{\pi}{2}} [A_1(\varphi)]\, d\varphi = 2(a+b)l - \frac{l^2}{2} - \frac{\pi}{2}m^2. \tag{2.9}$$

The relations (2.1), (2.8) and (2.9) give us the result. □

Remark 2.2. For $m = 0$ the obstacles become points and the probability $P_{int}^{(1)}$ becomes:

$$p^{(1)} = \frac{4(a+b)l - l^2}{3\pi ab}. \tag{2.10}$$

So, we find a result of a previous paper [3].

3. Obstacles circular sectors

We consider the lattice $\mathfrak{R}_2(a, b; m)$ with the fundamental cell C_2 represented in Figure 3.

By this figure we have that the formula (2.2) is valid for the cell C_2. Then we have

$$\text{area}C_2(\varphi) = 3ab - \frac{5\pi}{4}m^2.$$

As in the paragraph 1, we compute the probability $P_{int}^{(2)}$that a segment s intersects a side of fundamental cell C_2.

Considering the limiting positions of segment s, for a fixed angle φ, in the cell C_2. We obtain the Figure 4

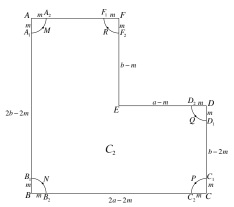

Figure 3

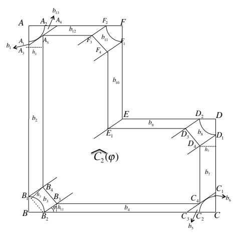

Figure 4

and the formula

$$\text{area}\widehat{C}_2(\varphi) = \text{area}C_2 - \sum_{i=1}^{13} \text{area}b_i(\varphi).$$

Theorem 3.1. *We have*

$$P_{int}^{(2)} = \frac{2\left[2(a+b)l - \frac{l^2}{2} - \frac{\pi m^2(5\pi-6)}{8}\right]}{\pi\left(3ab - \frac{5\pi}{4}m^2\right)}.$$

Remark 3.2. For $m = 0$ the obstacles become points and the probability $P_{int}^{(2)}$ become:

$$p^{(2)} = \frac{4(a+b)l - l^2}{3\pi ab}. \tag{3.1}$$

4. Obstacles triangular and circular sectors

We consider the lattice $\mathfrak{R}_3(a, b; m)$ with the fundamental cell C_3 represented in Figure 5.

From Figure 5 we have

$$\mathrm{area} C_3 = 3ab - m^2 - \frac{3\pi m^2}{4}.$$

As in the previous paragraphs, we compute the probability $P_{int}^{(3)}$ that a segment s intersects a side of fundamental cell C_3.

Considering the limiting positions of segment s, for a fixed angle φ, in the cell C_2. We obtain Figure 6

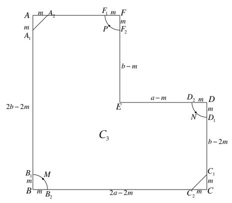

Figure 5

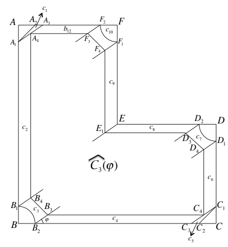

Figure 6

and the formula

$$\mathrm{area}\widehat{C}_3(\varphi) = \mathrm{area} C_3 - \sum_{i=1}^{11} \mathrm{area} c_i(\varphi).$$

Theorem 4.1. *We have*

$$P_{int}^{(3)} = \frac{4(a+b)l - l^2 - \frac{\pi(5\pi-14)m^2}{4}}{\pi\left(3ab - m^2 - \frac{3\pi m^2}{4}\right)}.$$

Remark 4.2. If $m = 0$, the obstacles become points and the probability P_{int} becomes

$$p^{(3)} = \frac{4(a+b)l - l^2}{3\pi ab}, \tag{4.1}$$

Remark 4.3. The relation (2.10), (3.1) and (4.1) give us the evident formula

$$p^{(1)} = p^{(2)} = p^{(3)}.$$

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