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Dynamics of the fuzzy difference equation $z_n = \max\{\frac{1}{z_{n-m}}, \frac{\alpha_n}{z_{n-r}}\}$

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Abstract

In this paper, we study the eventual periodicity of the following fuzzy max-type difference equation

$$z_n = \max\{\frac{1}{z_{n-m}}, \frac{\alpha_n}{z_{n-r}}\}, \quad n = 0, 1, \ldots,$$

where $\{\alpha_n\}_{n\geqslant 0}$ is a periodic sequence of positive fuzzy numbers and the initial values $z_{-d}, z_{-d+1}, \ldots, z_{-1}$ are positive fuzzy numbers with $d = \max\{m, r\}$. We show that if $\max(\text{supp } \alpha_n) < 1$, then every positive solution of this equation is eventually periodic with period $2m$.

Keywords: Fuzzy max-type difference equation, positive solution, eventual periodicity.

2010 MSC: 39A10, 39A11.

1. Introduction

In this paper, our goal is to investigate the eventual periodicity of the following fuzzy max-difference equation

$$z_n = \max\{\frac{1}{z_{n-m}}, \frac{\alpha_n}{z_{n-r}}\}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $m, r \in \mathbf{N} \equiv \{1, 2, 3, \ldots\}$, α_n is a periodic sequence of positive fuzzy numbers, and $z_{-d}, z_{-d+1}, \ldots, z_{-1}$ are positive fuzzy numbers with $d = \max\{m, r\}$.

Recently, there has been an increase in interest in the study of fuzzy difference equations because many models in automatic control theory and finance are represented by these equations naturally (see, e.g., [1–5, 7, 9–15, 17–21]).

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In 2014, Zhang et al. [21] studied the existence, the boundedness, and the asymptotic behavior of the positive solutions to a first order fuzzy Ricatti difference equation

$$z_{n+1} = \frac{\alpha + z_n}{\beta + z_n}, \quad n = 0, 1, \dots,$$

where α, β and the initial condition z_0 are positive fuzzy numbers.

In 2005, Stefanidou and Papaschinopoulos [13] studied the periodicity of the following fuzzy max-difference equations

$$z_{n+1} = \max\{\frac{\alpha}{z_n}, \frac{\alpha}{z_{n-1}}, \dots, \frac{\alpha}{z_{n-k}}\}, \quad n = 0, 1, \dots$$

and

$$z_{n+1} = \max\{\frac{\alpha}{z_n}, \frac{\beta}{z_{n-1}}\}, \quad n = 0, 1, \dots,$$

where $k \in \mathbf{N}$, α, β and the initial conditions $z_{-k}, z_{-k+1}, \dots, z_0$ are positive fuzzy numbers.

In 2006, Stefanidou and Papaschinopoulos [14] studied the periodicity of the following fuzzy max-difference equation

$$z_n = \max\{\frac{\alpha}{z_{n-k}}, \frac{\beta}{z_{n-m}}\}, \quad n = 0, 1, \dots,$$

where α, β and the initial conditions $z_{-d}, z_{-d+1}, \dots, z_{-1}$ with $d = \max\{k, m\}$ are positive fuzzy numbers.

In 2014, He et al. [3] investigated the periodicity of the positive solutions of the fuzzy max-difference equation

$$x_{n+1} = \max\{\frac{A_n}{x_{n-m}}, x_{n-k}\}, \quad n = 0, 1, \dots, \tag{1.2}$$

where $k, m \in \mathbf{N}$, A_n is a periodic sequence of fuzzy numbers and $x_{-d}, x_{-d+1}, \dots, x_0$ are positive fuzzy numbers with $d = \max\{m, k\}$, and showed that every positive solution of (1.2) is eventually periodic with period $k + 1$.

The rest of this paper is organized as follows. We give the some definitions and notations in Section 2. We give the main result and its proof of this paper in Section 3.

2. Preliminaries

In this section, we give the following definitions and notations. A function $P : \mathbf{R}(\equiv (-\infty, +\infty)) \to [0, 1]$ is said to be a fuzzy number if the following statements hold (see [9]):

(1) P is normal (i.e., $P(t) = 1$ for some $t \in \mathbf{R}$).
(2) P is a convex fuzzy set (i.e., $P(\lambda t_1 + (1-\lambda)t_2) \geqslant \min\{P(t_1), P(t_2)\}$ for any $\lambda \in [0, 1]$ and any $t_1, t_2 \in \mathbf{R}$).
(3) P is upper semicontinuous.
(4) $\text{supp } P = \overline{\bigcup_{c\in(0,1]}[P]_c} = \overline{\{t : P(t) > 0\}}$ is compact.

Where $[P]_c = \{t \in \mathbf{R} : P(t) \geqslant c\}$ (for any $c \in (0, 1]$) (which are called the c-cuts of the fuzzy number P) and $\overline{W}$ is the closure of set W. We see from Theorems 3.1.5 and 3.1.8 of [8] that every c-cut of the fuzzy number P is a closed interval.

A fuzzy number P is said to be positive if $\min(\text{supp } P) > 0$. If $P \in (0, +\infty)$, then P is a positive fuzzy number (it is called a trivial fuzzy number also) with $[P]_c = [P, P]$ for any $c \in (0, 1]$.

For some $k \in \mathbf{N}$, let $P_1, P_2, \dots, P_k$ be fuzzy numbers and $c \in (0, 1]$ with

$$[P_i]_c = [P_{i,l,c}, P_{i,r,c}] \text{ for } 0 \leqslant i \leqslant k.$$

Write

$$Q_{l,c} = \max\{P_{i,l,c} : 0 \leqslant i \leqslant k\} \text{ and } Q_{r,c} = \max\{P_{i,r,c} : 0 \leqslant i \leqslant k\}.$$

Then we know from Theorem 2.1 of [16] that there exists a fuzzy number Q such that

$$[Q]_c = [Q_{l,c}, Q_{r,c}] \text{ for any } c \in (0,1].$$

By [6] and Lemma 2.3 of [9] one can define

$$Q = \max\{P_i : 0 \leqslant i \leqslant k\}.$$

A sequence of positive fuzzy numbers $\{z_n\}_{n=-d}^{\infty}$ is called a solution of the equation (1.1) if it satisfies (1.1). $\{z_n\}_{n=-d}^{\infty}$ is said to be eventually periodic with period T if there exists $M \in \mathbf{N}$ such that $z_{n+T} = z_n$ holds for all $n \geqslant M$.

Proposition 2.1. *Suppose that $z_{-d}, z_{-d+1}, \ldots, z_{-1}$ are positive fuzzy numbers. Then there exists a unique positive solution $\{z_n\}_{n=-d}^{\infty}$ of (1.1) with initial values $z_{-d}, z_{-d+1}, \ldots, z_{-1}$.*

Proof. Suppose that $[\alpha_n]_c = [\alpha_{n,l,c}, \alpha_{n,r,c}]$ for any $c \in (0,1]$ and any $n \geqslant 0$, and

$$[z_i]_c = [p_{i,c}, q_{i,c}] \text{ for } -d \leqslant i \leqslant 0 \text{ and any } c \in (0,1], \tag{2.1}$$

and $\{(p_{n,c}, q_{n,c})\}_{n=-d}^{\infty}(c \in (0,1])$ is the unique positive solution of the following system of difference equations

$$p_{n,c} = \max\{\frac{1}{q_{n-m,c}}, \frac{\alpha_{n,l,c}}{q_{n-r,c}}\}, \quad q_{n,c} = \max\{\frac{1}{p_{n-m,c}}, \frac{\alpha_{n,r,c}}{p_{n-r,c}}\} \tag{2.2}$$

with initial values $(p_{i,c}, q_{i,c})(-d \leqslant i \leqslant 0)$. Arguing as in Proposition 3.1 of [13] we can show that $\{(p_{n,c}, q_{n,c})\}_{n=-d}^{\infty}(c \in (0,1])$ determines a sequence of positive fuzzy numbers $\{z_n\}_{n=-d}^{\infty}$ such that

$$[z_n]_c = [p_{n,c}, q_{n,c}],\ n \geqslant -d,\ c \in (0,1], \tag{2.3}$$

and that $\{z_n\}_{n=-d}^{\infty}$ is the unique positive solution of (1.1) with initial values $z_{-d}, z_{-d+1}, \ldots, z_{-1}$. This completes the proof of the Proposition. □

3. Main result and proof

In the sequel, we consider the system of difference equations

$$y_n = \max\{\frac{1}{z_{n-m}}, \frac{\alpha_n}{z_{n-r}}\}, \quad z_n = \max\{\frac{1}{y_{n-m}}, \frac{\beta_n}{y_{n-r}}\}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $\alpha_n, \beta_n \in (0,1)$ are two periodic sequences. Let $\{(y_n, z_n)\}_{n \geqslant -d}$ be a solution of (3.1) with the initial values $y_{-d}, z_{-d}, y_{-d+1}, z_{-d+1}, \ldots, y_{-1}, z_{-1} \in (0, +\infty)$. The main result of this paper is established through the following lemmas.

Lemma 3.1. *The following statements hold:*

(1) $y_n z_{n-m} \geqslant 1$ *and* $z_n y_{n-m} \geqslant 1$ *for all* $n \geqslant 0$;
(2) $y_n \leqslant \max\{y_{n-2m}, \alpha_n y_{n-m-r}\}$ *and* $z_n \leqslant \max\{z_{n-2m}, \beta_n z_{n-m-r}\}$ *for all* $n \geqslant d$;
(3) *if* $y_n = 1/z_{n-m}$ *(Resp.* $z_n = 1/y_{n-m}$*) for some* $n \geqslant m$*, then* $y_n \leqslant y_{n-2m}$ *(Resp.* $z_n \leqslant z_{n-2m}$*). If* $y_n = \alpha_n/z_{n-r} > 1/z_{n-m}$ *(Resp.* $z_n = \beta_n/y_{n-r} > 1/y_{n-m}$*) for some* $n \geqslant d$*, then* $y_n > y_{n-2m}$ *(Resp.* $z_n > z_{n-2m}$*).*

Proof.

(1) It is obvious since $y_n \geqslant 1/z_{n-m}$ (Resp. $z_n \geqslant 1/y_{n-m}$) for all $n \geqslant 0$.

(2) It follows from (1) that for any $n \geqslant d$,

$$y_n = \max\left\{\frac{y_{n-2m}}{z_{n-m}y_{n-2m}}, \frac{\alpha_n y_{n-m-r}}{y_{n-m-r}z_{n-r}}\right\} \leqslant \max\{y_{n-2m}, \alpha_n y_{n-m-r}\}.$$

For the another case, the proof is similar.

(3) If $y_n = 1/z_{n-m}$ for some $n \geqslant m$, then we see from (1) that

$$y_n = \frac{y_{n-2m}}{z_{n-m}y_{n-2m}} \leqslant y_{n-2m}.$$

If $y_n = \alpha_n/z_{n-r} > 1/z_{n-m}$ for some $n \geqslant d$, then we have from (1) that

$$1 < y_n z_{n-m} = \max\{\frac{y_n}{y_{n-2m}}, \frac{y_n z_{n-r}\beta_{n-m}}{y_{n-r-m}z_{n-r}}\} \leqslant \max\{\frac{y_n}{y_{n-2m}}, \alpha_n\beta_{n-m}\} = \frac{y_n}{y_{n-2m}}.$$

This implies $y_n > y_{n-2m}$. For the another case, the proof is similar. This completes the proof of the Lemma. □

Lemma 3.2. *Define*

$$\Phi_n = \max\{y_{n-1}, y_{n-2}, \ldots, y_{n-m-d}\} \quad (n \geqslant d)$$

and

$$\Upsilon_n = \max\{z_{n-1}, z_{n-2}, \ldots, z_{n-m-d}\} \quad (n \geqslant d).$$

Then the following statements hold:

(1) $y_n \leqslant \Phi_n$ *and* $z_n \leqslant \Upsilon_n$ *for any* $n \geqslant d$ *and* $\{\Phi_n\}_{n\geqslant d}$ *and* $\{\Upsilon_n\}_{n\geqslant d}$ *are decreasing;*

(2) *there exist constants* M *and* M' *with* $M \geqslant M' > 0$ *such that* $y_n, z_n \in [M', M]$ *for any* $n \geqslant -d$.

Proof.

(1) Noting Lemma 3.1 (2), we obtain that $y_n \leqslant \Phi_n$ (Resp. $z_n \leqslant \Upsilon_n$) $(n \geqslant d)$, thus

$$\Phi_{n+1} = \max\{y_n, y_{n-1}, \ldots, y_{n-m-d+1}\} \leqslant \Phi_n$$

and

$$\Upsilon_{n+1} = \max\{z_n, z_{n-1}, \ldots, z_{n-m-d+1}\} \leqslant \Upsilon_n.$$

(2) Choose $M = \max\{\Phi_d, \Upsilon_d, y_{d-1}, z_{d-1}, \ldots, y_{-d}, y_{-d}\}$ $(\geqslant 1)$ and $M' = \min\{1/M, y_{-1}, z_{-1}, \ldots, y_{-d}, z_{-d}\}$ (> 0). Then $y_n, z_n \in [M', M]$ for any $n \geqslant -d$. This completes the proof of the Lemma. □

In the following, suppose that $\lim_{n\to\infty}\Phi_n = \Phi$ and $\lim_{n\to\infty}\Upsilon_n = \Upsilon$. Let $\liminf_{n\to\infty} y_n = \phi$ and $\liminf_{n\to\infty} z_n = \gamma$. Then we have the following Corollary 3.3.

Corollary 3.3. *There exists a sequence* $1 < n_1 < n_2 < \cdots < n_k < \cdots$ *satisfying* $n_{k+1} - n_k \leqslant m + d$ *such that* $y_{n_k} \geqslant \Phi$ *(Resp.* $z_{n_k} \geqslant \Upsilon$*).*

Proof. We know from Lemma 3.2 (1) that for any $n \geqslant d$, $\Phi_n = \max\{y_{n-1}, y_{n-2}, \ldots, y_{n-m-d}\} \geqslant \Phi$ (Resp. $\Upsilon_n = \max\{z_{n-1}, z_{n-2}, \ldots, z_{n-m-d}\} \geqslant \Upsilon$), which implies that there exists $j \in \{n-1, \ldots, n-m-d\}$ such that $y_j \geqslant \Phi$ (Resp. $z_j \geqslant \Upsilon$).

Again using Lemma 3.2 (1), we have $\Phi_{j+m+d+1} = \max\{y_{j+m+d}, y_{j+m+d-1}, \ldots, y_{j+1}\} \geqslant \Phi$ (Resp. $\Upsilon_{j+m+d+1} = \max\{z_{j+m+d}, z_{j+m+d-1}, \ldots, z_{j+1}\} \geqslant \Upsilon$), which implies that there exists $i \in \{j+m+d, \ldots, j+1\}$ such that $y_i \geqslant \Phi$ (Resp. $z_i \geqslant \Upsilon$) with $1 \leqslant i - j \leqslant m + d$. Therefore, there exists a sequence $1 < n_1 < n_2 < \cdots < n_k < \cdots$ satisfying $n_{k+1} - n_k \leqslant m + d$ such that $y_{n_k} \geqslant \Phi$ (Resp. $z_{n_k} \geqslant \Upsilon$). This completes the proof of the Corollary. □

Lemma 3.4. *The following statements hold:*

(1) $\Phi = \limsup_{n\to\infty} y_n$ *and* $\Upsilon = \limsup_{n\to\infty} z_n$*;*
(2) $\mathrm{Card}(\{n : \Phi \leqslant y_n = \alpha_n/z_{n-r}\}) < \infty$ *(Resp.* $\mathrm{Card}(\{n : \Upsilon \leqslant z_n = \beta_n/z_{n-r}\}) < \infty$*), where* $\mathrm{Card}(A)$ *denotes the cardinality of the set* A*;*
(3) *there exists* $N \in \mathbf{N}$ *such that:* (i) $y_{N+2m\lambda} \geqslant \Phi$ *and* $y_{N+2m\lambda} = 1/z_{N+2m\lambda-m}$ *for any* $\lambda \geqslant 0$*, and* $y_{N+2m\lambda}$ *is decreasing;* (ii) $\lim_{\lambda\to\infty} z_{N+2m\lambda-m} = \gamma = 1/\Phi$ *(Resp.* (i) $z_{N+2m\lambda} \geqslant \Upsilon$ *and* $z_{N+2m\lambda} = 1/y_{N+2m\lambda-m}$ *for any* $\lambda \geqslant 0$*, and* $z_{N+2m\lambda}$ *is decreasing;* (iii) $\lim_{\lambda\to\infty} y_{N+2m\lambda-m} = \phi = 1/\Upsilon$*).*

Proof.

(1) Since $y_n \leqslant \Phi_n$ and $z_n \leqslant \Upsilon_n$ for all $n \geqslant d$, by Corollary 3.3, we have

$$\Phi \leqslant \limsup_{n\to\infty} y_n \leqslant \limsup_{n\to\infty} \Phi_n = \Phi$$

and

$$\Upsilon \leqslant \limsup_{n\to\infty} z_n \leqslant \limsup_{n\to\infty} \Upsilon_n = \Upsilon.$$

(2) Assume on the contrary that there exists a sequence $r < n_1 < n_2 < \cdots < n_\mu < \cdots$ such that

$$\Phi \leqslant y_{n_\mu} = \frac{\alpha_{n_\mu}}{z_{n_\mu - r}} \leqslant \alpha_{n_\mu} y_{n_\mu - r - m}.$$

By taking subsequence, we suppose that $\lim_{\mu\to\infty} y_{n_\mu - r - m} = L$. This implies $\Phi = \lim_{\mu\to\infty} y_{n_\mu} \leqslant \lim_{\mu\to\infty} y_{n_\mu - r - m} \max\{\alpha_n : n \in \mathbf{N}\} = \max\{\alpha_n : n \in \mathbf{N}\} L < \Phi$, which is a contradiction. For the another case, the proof is similar.

(3) It follows from (2) that there exists $N_1 \in \mathbf{N}$ such that if $n \geqslant N_1$ and $y_n \geqslant \Phi$, then $y_n = 1/z_{n-m}$. By taking subsequence, we may choose a sequence $N_1 \leqslant n_1 < n_2 < \cdots < n_k < \cdots$ satisfying $n_{k+1} - n_k \equiv 0 \pmod{2m}$ such that $y_{n_k} \geqslant \Phi$ (for all $k \geqslant 1$) and $\lim_{k\to\infty} y_{n_k} = \Phi$. Then $y_{n_k} = 1/z_{n_k - m}$. Write $N = n_1$. According to Lemma 3.1 (3), we know that $y_{N+2m\lambda} \geqslant \Phi$ and $1/z_{N+2m\lambda-m} = y_{N+2m\lambda} \geqslant y_{N+2m(\lambda+1)} = 1/z_{N+2m(\lambda+1)-m}$ for any $\lambda \geqslant 0$.

Let $s_k \to +\infty$ such that $z_{s_k} \to \gamma$ and $y_{s_k - m} \to U$. Then

$$\frac{1}{\Phi} = \lim_{\lambda\to\infty} \frac{1}{y_{N+2m\lambda}} = \lim_{\lambda\to\infty} z_{N+2m\lambda-m} \geqslant \gamma = \lim_{k\to\infty} z_{s_k} \geqslant \lim_{k\to\infty} \frac{1}{y_{s_k - m}} = \frac{1}{U} \geqslant \frac{1}{\Phi},$$

which implies $\lim_{\lambda\to\infty} z_{N+2m\lambda-m} = \gamma = 1/\Phi$. For the another case, the proof is similar. This completes the proof of the Lemma. □

Lemma 3.5. *Let* $M, p, q \in \mathbf{N}$ *with* $q \geqslant 2$ *such that*

(1) $\{z_{M+2m\mu}\}_{\mu\geqslant 0}$ *is monotone;*
(2) $y_{M+2m(p+\lambda)+r} = \alpha_{M+2m(p+\lambda)+r}/z_{M+2m(p+\lambda)} > 1/z_{M+2m(p+\lambda)+r-m}$ *for* $\lambda \in \{0, q\}$*;*
(3) $y_{M+2m(p+\lambda)+r} = 1/z_{M+2m(p+\lambda)+r-m}$ *for every* $1 \leqslant \lambda \leqslant q-1$*.*

Then $y_{M+2m(p+\lambda)+r} = y_{M+2m(p+\lambda+1)+r}$ *for every* $0 \leqslant \lambda \leqslant q-2$*.*

Proof. For $1 \leqslant \lambda \leqslant q-1$, if $\{z_{M+2m\mu}\}_{\mu\geqslant 0}$ is decreasing, then from $\alpha_{M+2mp+r}\beta_{M+2m(p+\lambda)+r-m} < 1 \leqslant z_{M+2m(p+\lambda)} y_{M+2m(p+\lambda)-m}$ and Lemma 3.1 (3), it follows that

$$\begin{aligned}\frac{\alpha_{M+2mp+r}}{z_{M+2m(p+\lambda)}} \geqslant \frac{\alpha_{M+2mp+r}}{z_{M+2mp}} &= y_{M+2mp+r} \\ &\geqslant y_{M+2m(p+\lambda)+r} \\ &= \frac{1}{z_{M+2m(p+\lambda)+r-m}}\end{aligned}$$

$$
\begin{aligned}
&= \min\{y_{M+2m(p+\lambda-1)+r}, \frac{y_{M+2m(p+\lambda)-m}}{\beta_{M+2m(p+\lambda)+r-m}}\} \\
&= y_{M+2m(p+\lambda-1)+r} \geqslant y_{M+2m(p+\lambda)+r}.
\end{aligned}
$$

Therefore, $y_{M+2m(p+\lambda-1)+r} = y_{M+2m(p+\lambda)+r}$ for every $1 \leqslant \lambda \leqslant q-1$. If $\{z_{M+2m\mu}\}_{\mu\geqslant 0}$ is increasing, then it follows from $\alpha_{M+2m(p+q)+r}\beta_{M+2m(p+q-1)+r-m} < 1 \leqslant y_{M+2m(p+q-1)}z_{M+2m(p+q-1)-m}$ and Lemma 3.1 (3) that

$$
\begin{aligned}
\frac{\alpha_{M+2m(p+q)+r}}{z_{M+2m(p+q-1)}} \geqslant \frac{\alpha_{M+2m(p+q)+r}}{z_{M+2m(p+q)}} &= y_{M+2m(p+q)+r} \\
&> y_{M+2m(p+q-1)+r} \\
&= \frac{1}{z_{M+2m(p+q-1)+r-m}} \\
&= \min\{y_{M+2m(p+q-2)+r}, \frac{y_{M+2m(p+q-1)-m}}{\beta_{M+2m(p+q-1)+r-m}}\} \\
&= y_{M+2m(p+q-2)+r} \geqslant y_{M+2m(p+q-1)+r}.
\end{aligned}
$$

Therefore, $y_{M+2m(p+q-1)+r} = y_{M+2m(p+q-2)+r}$. In a similar way, we may show that $y_{M+2m(p+q-1)+r} = y_{M+2m(p+\lambda)+r}$ for every $0 \leqslant \lambda \leqslant q-2$. This completes the proof of the Lemma. □

In a similar fashion, we can obtain the following lemma.

Lemma 3.6. *Let $M, p, q \in \mathbf{N}$ with $q \geqslant 2$ such that*

(1) $\{y_{M+2m\mu}\}_{\mu\geqslant 0}$ *is monotone;*
(2) $z_{M+2m(p+\lambda)+r} = \beta_{M+2m(p+\lambda)+r}/y_{M+2m(p+\lambda)} > 1/y_{M+2m(p+\lambda)+r-m}$ *for* $\lambda \in \{0, q\}$;
(3) $z_{M+2m(p+\lambda)+r} = 1/y_{M+2m(p+\lambda)+r-m}$ *for every* $1 \leqslant \lambda \leqslant q-1$.

Then $z_{M+2m(p+\lambda)+r} = z_{M+2m(p+\lambda+1)+r}$ *for every* $0 \leqslant \lambda \leqslant q-2$.

Lemma 3.7. *If there exists $M \in \mathbf{N}$ such that $\{y_{M+2m\mu}\}_{\mu\geqslant 0}$ is monotone, then $\{z_{M+2m\mu+r}\}_{\mu\geqslant 0}$ is eventually monotone.*

Proof. If there exists $K \in \mathbf{N}$ such that $z_{M+2m\mu+r} = 1/y_{M+2m\mu+r-m}$ for all $\mu \geqslant K$ (or $z_{M+2m\mu+r} = \beta_{M+2m\mu+r}/y_{M+2m\mu} > 1/y_{M+2m\mu+r-m}$ for all $\mu \geqslant K$), then by Lemma 3.1 (3) we know that $z_{M+2m\mu+r} \leqslant z_{M+2m(\mu-1)+r}$ for all $\mu \geqslant K$ (or $z_{M+2m\mu+r} > z_{M+2m(\mu-1)+r}$ for all $\mu \geqslant K$). Thus $\{z_{M+2m\mu+r}\}_{\mu\geqslant K}$ is monotone.

If there exists a sequence $1 < s_1 < t_1 < s_2 < t_2 < \cdots < s_\mu < t_\mu < \cdots$ such that

$$
z_{M+2mk+r} = \frac{\beta_{M+2mk+r}}{y_{M+2mk}} > \frac{1}{y_{M+2mk+r-m}} \quad \text{for every} \quad s_i \leqslant k < t_i
$$

and

$$
z_{M+2mk+r} = \frac{1}{z_{M+2mk+r-m}} \quad \text{for every} \quad t_i \leqslant k < s_{i+1},
$$

then by Lemma 3.1 (3) and Lemma 3.6 we know that $z_{M+2m(k-1)+r} < z_{M+2mk+r}$ for every $s_i \leqslant k < t_i$ and $z_{M+2m(k-1)+r} = z_{M+2mk+r}$ for every $t_i \leqslant k < s_{i+1}$, which implies that $\{z_{M+2m\mu+r}\}_{\mu\geqslant s_1}$ is increasing. This completes the proof of the Lemma. □

In a similar fashion, we can obtain the following lemma.

Lemma 3.8. *If there exists $M \in \mathbf{N}$ such that $\{z_{M+2m\mu}\}_{\mu\geqslant 0}$ is monotone, then $\{y_{M+2m\mu+r}\}_{\mu\geqslant 0}$ is eventually monotone.*

Lemma 3.9. $\{(y_n, z_n)\}_{n\geqslant -d}$ *is eventually periodic with period* $2m$.

Proof. First we suppose that $\gcd(r, m) = 1$. According to Lemma 3.4 (3), there exists $N \in \mathbf{N}$ such that the following statements hold:

(1) $y_{N+2mn}z_{N+2mn-m} = z_{N+2mn}y_{N+2mn-m} = 1$ for any $n \geqslant 0$;
(2) y_{N+2mn} is decreasing $(n \geqslant 0)$ and $\lim_{n\to\infty} y_{N+2mn} = \Phi$, $z_{N+2mn-m}$ is increasing $(n \geqslant 0)$ and $\lim_{n\to\infty} z_{N+2mn-m} = \gamma = 1/\Phi$. z_{N+2mn} is decreasing $(n \geqslant 0)$ and $\lim_{n\to\infty} z_{N+2mn} = \Upsilon$, $y_{N+2mn-m}$ is increasing $(n \geqslant 0)$ and $\lim_{n\to\infty} y_{N+2mn-m} = \phi = 1/\Upsilon$.

Using Lemma 3.7 and Lemma 3.8 repeatedly, we see that $\{y_{N+2mn+ir}\}_{n\geqslant0}$, $\{y_{N+2mn-m+ir}\}_{n\geqslant0}$, $\{z_{N+2mn+ir}\}_{n\geqslant0}$, and $\{z_{N+2mn-m+ir}\}_{n\geqslant0}$ are eventually monotone for every $1 \leqslant i \leqslant m-1$. Since $\gcd(r, m) = 1$, we know that for every $j \in \{0, 1, 2, \dots, m-1\}$ there exist some $0 \leqslant i_j \leqslant m-1$ and integer λ_j such that $i_jr = \lambda_jm + j$ and $i_jr - m = (\lambda_j - 1)m + j$, which implies that $\{y_{N+2mn+\lambda_jm+j}\}_{n\geqslant0}$, $\{y_{N+2mn+(\lambda_j-1)m+j}\}_{n\geqslant0}$, $\{z_{N+2mn+\lambda_jm+j}\}_{n\geqslant0}$, and $\{z_{N+2mn+(\lambda_j-1)m+j}\}_{n\geqslant0}$ are eventually monotone for every $j \in \{0, 1, 2, \dots, m-1\}$. Thus $\{y_{2mn+k}\}_{n\geqslant0}$ and $\{z_{2mn+k}\}_{n\geqslant0}$ are eventually monotone for every $0 \leqslant k \leqslant 2m-1$.

Claim 1. $y_{N+2mk+r}$ $(k \geqslant 0)$, $z_{N+2mk+r-m}$ $(k \geqslant 0)$, $z_{N+2mk+r}$ $(k \geqslant 0)$, and $y_{N+2mk+r-m}(k \geqslant 0)$ are constant sequences eventually.

Proof of Claim 1. Since $\alpha_{N+2mk+r}/z_{N+2mk} \leqslant \max\{\alpha_n : n \in \mathbf{N}\}y_{N+2mk-m}$ and $\lim_{k\to\infty} \max\{\alpha_n : n \in \mathbf{N}\}y_{N+2mk-m} = \max\{\alpha_n : n \in \mathbf{N}\}\phi < \phi \leqslant \lim_{k\to\infty} y_{N+2mk+r}$, there exists $p \in \mathbf{N}$ such that for any $k \geqslant p$,

$$y_{N+2mk+r} = \max\{\frac{1}{z_{N+2mk+r-m}}, \frac{\alpha_{N+2mk+r}}{z_{N+2mk}}\} = \frac{1}{z_{N+2mk+r-m}}.$$

Then, by Lemma 3.1 (3) we see that $y_{N+2mk+r}$ $(k \geqslant p)$ is decreasing.

If there exists a sequence $p \leqslant s_1 < s_2 < \cdots < s_i < \cdots$ such that

$$z_{N+2ms_i+r-m} = \frac{\beta_{N+2ms_i+r-m}}{y_{N+2ms_i-m}}$$

for every $i \geqslant 1$, then we have

$$y_{N+2ms_i+r} = \frac{y_{N+2ms_i-m}}{\beta_{N+2ms_i+r-m}}. \tag{3.2}$$

By taking a subsequence we may assume that β_{N+2mk_i+r-m} is a constant sequence since β_n is a periodic sequence. From (3.2) it follows that $y_{N+2mk+r}$ $(k \geqslant 0)$ and $z_{N+2mk+r-m}$ $(k \geqslant 0)$ are constant sequences eventually.

If $z_{N+2mk+r-m} = 1/y_{N+2mk+r-2m}$ eventually, then $y_{N+2mk+r} = y_{N+2m(k-1)+r}$ eventually and $z_{N+2mk+r-m} = z_{N+2m(k-1)+r-m}$ eventually.

In a similar fashion, we may show that $z_{N+2mk+r}$ $(k \geqslant 0)$ and $y_{N+2mk+r-m}$ $(k \geqslant 0)$ are constant sequences eventually. This completes the proof of the Claim.

Claim 2. $y_{N+2mk+2r}$ $(k \geqslant 0)$, $z_{N+2mk+2r-m}$ $(k \geqslant 0)$, $z_{N+2mk+2r}$ $(k \geqslant 0)$, and $y_{N+2mk+2r-m}$ $(k \geqslant 0)$ are constant sequences eventually.

Proof of Claim 2. If there exists a sequence $1 \leqslant s_1 < s_2 < \cdots < s_i < \cdots$ such that

$$y_{N+2ms_i+2r} = \frac{\alpha_{N+2ms_i+2r}}{z_{N+2ms_i+r}}$$

for every $i \geqslant 1$, then $y_{N+2mk+2r}$ $(k \geqslant 0)$ is a constant sequence eventually since α_n is a periodic sequence and $z_{N+2mk+r}$ $(k \geqslant 0)$ is a constant sequence eventually.

If $y_{N+2mk+2r} = 1/z_{N+2mk+2r-m}$ eventually and there exists a sequence $1 \leqslant t_1 < t_2 < \cdots < t_i < \cdots$ such that

$$z_{N+2mt_i+2r-m} = \frac{\beta_{N+2mt_i+2r-m}}{y_{N+2mt_i+r-m}}$$

for every $i \geqslant 1$, then $z_{N+2mk+2r-m}$ $(k \geqslant 0)$ is a constant sequence eventually since β_n is a periodic sequence and $y_{N+2mk+r-m}$ $(k \geqslant 0)$ is a constant sequence eventually. Thus $y_{N+2mk+2r}$ $(k \geqslant 0)$ is a constant sequence eventually.

If $y_{N+2mk+2r} = 1/z_{N+2mk+2r-m}$ eventually and $z_{N+2mk+2r-m} = 1/y_{N+2mk+2r-2m}$ eventually, then $y_{N+2mk+2r}(k \geqslant 0)$ is a constant sequence eventually.

Noting $z_{N+2mk+2r-m} = \max\{1/y_{N+2mk+2r-2m}, \beta_{N+2mk+2r-m}/y_{N+2mk+r-m}\}$, we see that that $z_{N+2mk+2r-m}(k \geqslant 0)$ is also a constant sequence eventually.

In a similar fashion, we can show that $z_{N+2mk+2r}$ $(k \geqslant 0)$ and $y_{N+2mk+2r-m}$ $(k \geqslant 0)$ are constant sequences eventually. This completes the proof of the Claim.

By induction, we may show that $y_{N+2mk+jr}$ $(k \geqslant 0)$, $y_{N+2mk+jr-m}$ $(k \geqslant 0)$, $z_{N+2mk+jr}$ $(k \geqslant 0)$, and $z_{N+2mk+jr-m}$ $(k \geqslant 0)$ are constant sequences eventually for every $1 \leqslant j \leqslant m$, from which it follows that $\{y_{2mn+k}\}_{n\geqslant 0}$ and $\{z_{2mn+k}\}_{n\geqslant 0}$ are constant sequences eventually for every $0 \leqslant k \leqslant 2m-1$. Therefore, $\{(y_n, z_n)\}_{n\geqslant -d}$ is eventually periodic with period $2m$.

If $\gcd(r, m) = s > 1$, then we consider the max-type equation

$$y_n = \max\{\frac{1}{z_{n-sm_1}}, \frac{\alpha_n}{z_{n-sr_1}}\}, \quad z_n = \max\{\frac{1}{y_{n-sm_1}}, \frac{\beta_n}{y_{n-sr_1}}\}, \ n = 0, 1, 2, \ldots, \tag{3.3}$$

where $m = sm_1$ and $r = sr_1$ with $\gcd(m_1, r_1) = 1$. Write $y_n^i = y_{ns+i}$ and $z_n^i = z_{ns+i}$ for every $0 \leqslant i \leqslant s-1$ and $n = 0, 1, 2, \ldots$. Then (3.3) reduces to the equations

$$y_n^i = \max\{\frac{1}{z_{n-m_1}^i}, \frac{\alpha_{ns+i}}{z_{n-r_1}^i}\}, \quad z_n^i = \max\{\frac{1}{y_{n-m_1}^i}, \frac{\beta_{ns+i}}{y_{n-r_1}^i}\}, \ 0 \leqslant i \leqslant s-1, \quad n = 0, 1, 2, \ldots. \tag{3.4}$$

By an analogous way as in the above, we know that $\{(y_n^i, z_n^i)\}_{n\geqslant 0}$ is a positive solution of (3.4) for every $0 \leqslant i \leqslant s-1$. Then $\{(y_n^i, z_n^i)\}_{n\geqslant 0}$ is eventually periodic with period $2m_1$. Therefore $\{(y_n, z_n)\}_{n\geqslant 0}$ is eventually periodic with period $2m$. This completes the proof of the Lemma. □

Now we state and show the main result of this paper.

Theorem 3.10. *If* $\max(\text{supp } \alpha_n) < 1$, *then every positive solution of* (1.1) *is eventually periodic with period* $2m$.

Proof. Let $\{x_n\}_{n=-d}^{\infty}$ be a positive solution of (1.1) with initial values $x_{-d}, x_{-d+1}, \ldots, x_{-1}$ satisfying (2.1) and let (2.3) holds. We see from Proposition 2.1 that $\{(p_{n,c}, q_{n,c})\}_{n=-d}^{\infty}(c \in (0, 1])$ satisfies system (2.2). Using Lemma 3.9 we know that $\{(p_{n,c}, q_{n,c})\}_{n=-d}^{\infty}$ is eventually periodic with period $2m$. Therefore, it follows from (2.2) and Lemma 3.9 that $\{x_n\}_{n=-d}^{\infty}$ is eventually periodic of period $2m$. This completes the proof of Theorem 3.10. □

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