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Topological conjugacy of PM functions with height equaling ∞

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Abstract

It is known that topological conjugacy is a basic equivalence relation in dynamical systems. In this paper we study a class of piecewise monotone and continuous functions with infinite height. Those functions are topologically conjugate with each other if and only if they have same sequences describing itineraries of all forts, endpoints, and fixed points. We construct the topological conjugacy by extension, which partly generalizes previous results.

Keywords: Topological conjugacy, homeomorphism, piecewise monotone, fort.
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1. Introduction

Let X, Y be topological spaces, and $f : X \to X$ and $g : Y \to Y$ are continuous maps. We say that f is topologically conjugate to g (denoted by $f \sim g$) if there exists a homeomorphism $\varphi : X \to Y$ satisfying the equation

$$\varphi \circ f = g \circ \varphi, \tag{1.1}$$

φ is called topological conjugacy. Topological conjugacy is an equivalent relation, which means the iterative orbits of f and g have the same dynamical properties. As one of important tools in studying dynamical systems and functional equations, topological conjugacy often appears in linearization [17, 19], normal form [2, 7], and iterative functional equations [9, 10, 21].

Let $I := [a, b]$ be a closed interval and $f : I \to I$ be a continuous function. An interior point c of I is called a fort if f is strictly monotone in no neighborhood of c. A continuous function $f : I \to I$ is called *piecewise monotonic* (abbreviated as PM function) if the number $N(f)$ of forts is finite. Let the least $k \in \mathbb{N} \cup \{0\}$ satisfying $N(f^k) = N(f^{k+1})$, if such k exists and ∞ otherwise, be the *non-monotonicity height* $H(f)$ of f (cf. [20]), abbreviated as height. In general, the larger the number $H(f)$ the more complexity of the dynamics of f.

It seems that it was Parry [15] who first concerned with the topological conjugacy of PM functions, he presented sufficient conditions for a PM function to be conjugate to a uniformly piecewise linear function. In 1990, Baldwin [1] gave a sufficient and necessary condition for the existence of topological conjugacy,

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i.e., PM functions $f \sim g$ if and only if they have same invariants. Later, Brin and Stuck ([5, Corollary 7.4.8]) considered special PM functions having no wandering intervals, no attracting periodic points and no intervals of periodic points, and proved that $f \sim g$ if f and g have the same kneading invariants and endpoint itineraries. In 2013, Shi et al. [18] considered a class of PM functions with $H(f) = 1$, they constructed the topological conjugacy φ by extension when there exists a topological conjugacy φ_0 on initial subinterval. As a continuation of [18], Ref. [11] investigated the more generally cases $H(f) \geqslant 1$, i.e., $f([a, t_1]) \subset [a, t_1]$ and $f(t) < t$ for all $t \in (t_1, b)$ (here, t_1 is the smallest fort). Under the assumptions that f is topologically conjugate to g by φ_0 on subinterval (a, t_1) and has the same sequences of endpoints and forts as that of g, they proved that $f \sim g$ by φ, an extension of φ_0. Recently, under the condition $H(f) = H(g) < \infty$, Li [12, Lemma 2.1] presented a sufficient and necessary condition for $f \sim g$ by using "characteristic interval" and extension method. Besides this, conjugacy or semi-conjugacy is also investigated for special PM functions, such as linear Markov [4], Collet-Eckmann's map [14], piecewise expansive mappings [8], topological semi-conjugacy [6, 13], as well as weakly multimodal mappings [16], a class of generalized PM functions.

In the present paper, we remain interested in the topological conjugacy of PM functions but we restrict the study to a class of PM functions with $H(f) = \infty$, not involved in most of the mentioned references. What is claimed to be new is a sufficient and necessary condition for the existence of topological conjugacy φ and the construction for φ by extension. It is worth mentioning that we need not the assumption on the existence of the topological conjugacy on a subinterval, which is different from Refs. [11, 18] and Ref. [12]. Section 2 gives definitions and lemmas. In Section 3, we describe an equivalent condition for the existence of φ and give its construction. Two examples are presented to illustrate our results at the end of the paper.

2. Preliminaries

A continuous map $f : I \to I$ is said to be an r-modal map if there exists r forts

$$a := c_0 < c_1 < \cdots < c_r < c_{r+1} := b.$$

Clearly, $f_{i+1} := f|_{[c_i, c_{i+1}]}$ is alternately strictly increasing and strictly decreasing for all $i = 0, 1, \ldots, r$. Let $\mathscr{A}_f := \{c_i, f(c_i), f^2(c_i), i = 0, 1, \ldots, r+1\}$ and $PM_r(I, I)$ be the set of r-modal maps. Suppose that f has fixed points $F(f) := \{s_1, s_2, \ldots, s_k\}$, it is clear that $F(f)$ is nonempty since f is a self-mapping on compact interval. As we all know, an isolated fixed point c is the simplest non-trivial invariant set. In a neighborhood $U(c)$ of the point c, there are those patterns of dynamical behavior: an attractor, a repeller, an overflow from left to right, and an overflow from right to left ([3, pp 23]). In addition, the point c maybe be a 2-period point. We say that $f \in PM_r(I, I)$ and $g \in PM_r(J, J)$ are of the same type if f and g have the same type of fixed points and the same monotonicity in subintervals in ascending order.

Let

$$\mathscr{M}(f) := \mathscr{A}_f \cup F(f) \quad \text{and} \quad \#\mathscr{M}(f) := p,$$

in which $\#A$ denotes the cardinality of the set A. Rewritten those p different points as $\mu_{f,i} (i = 0, 1, \ldots, p-1)$, then

$$a := \mu_{f,0} < \mu_{f,1} < \cdots < \mu_{f,p-2} < \mu_{f,p-1} := b$$

gives a partition T_p of I. Now we define a sequence which plays an important role in our result. The itinerary of $t \in I$ with respect to $f \in PM_r(I, I)$ is a sequence of nonnegative integers $I_f(t) = (i_n(t))_{n \in \mathbb{N}}$ defined by

$$i_n(t) := \begin{cases} k, & \text{if } f^n(t) = \mu_{f,k}, \quad\quad k \in \{0, 1, \ldots, p-1\}, \\ \tilde{h}, & \text{if } f^n(t) \in (\mu_{f,h}, \mu_{f,h+1}), \quad h \in \{0, 1, \ldots, p-2\}. \end{cases}$$

We remark that, the sequence $(i_n(t))_{n \in \mathbb{N}}$ is different from $(i_k(x))_{k \in \mathbb{N}_0}$ in Ref. [11] since the set $\mathscr{A}_f$ is not equal to $S(f)$ in there. In fact, the sequence $(i_n(t))_{n \in \mathbb{N}}$ is not necessarily monotone, now we present two examples to describe the monotone and non-monotone cases, respectively.

Example 2.1. Consider the map $f:[0,1]\to[0,1]$ defined by

$$f(t):=\begin{cases}\frac{5}{3}t+\frac{1}{10}, & t\in[0,\frac{3}{10}),\\ -t+\frac{9}{10}, & t\in[\frac{3}{10},\frac{2}{5}),\\ \frac{1}{2}t+\frac{3}{10}, & t\in[\frac{2}{5},\frac{4}{5}),\\ -\frac{1}{3}t+\frac{29}{30}, & t\in[\frac{4}{5},\frac{19}{20}),\\ 6t-\frac{101}{20}, & t\in[\frac{19}{20},1].\end{cases}$$

We see that

$$\mathscr{A}_f:=\{c,f(c),f^2(c)\},\quad F(f)=\{\frac{3}{5}\}$$

for all $c\in\{\frac{3}{10},\frac{2}{5},\frac{4}{5},\frac{19}{20},1\}$. By simple calculation

$$f(t)\leqslant f^2(t)\leqslant\cdots\leqslant f^n(t)\leqslant\cdots\leqslant\frac{3}{5}\quad\text{for}\quad t\in\{0,\frac{3}{10},\frac{2}{5}\},$$

and

$$f(t)\geqslant f^2(t)\geqslant\cdots\geqslant f^n(t)\geqslant\cdots>\frac{3}{5}\quad\text{for}\quad t\in\{\frac{4}{5},\frac{19}{20},1\}.$$

Thus, the sequence $(i_n(t))_{n\in\mathbb{N}}$ is increasing for all $t\in[0,\frac{3}{5})$ and decreasing for all $t\in(\frac{3}{5},1]$.

Example 2.2. Consider the map $g:[0,1]\to[0,1]$ defined by

$$g(t):=\begin{cases}2t+\frac{1}{5}, & t\in[0,\frac{3}{10}),\\ -\frac{11}{6}t+\frac{27}{20}, & t\in[\frac{3}{10},\frac{3}{5}),\\ t-\frac{7}{20}, & t\in[\frac{3}{5},\frac{7}{10}),\\ -\frac{1}{6}t+\frac{7}{15}, & t\in[\frac{7}{10},1].\end{cases}$$

Note that

$$\mathscr{A}_g:=\{d,f(d),f^2(d)\},\quad F(g)=\{\frac{81}{170}\}$$

for all $d\in\{0,\frac{3}{10},\frac{3}{5},\frac{7}{10},1\}$. By simple calculation

$$g(0)=\frac{1}{5},\quad g^2(0)=\frac{3}{5},\quad g^3(0)=\frac{1}{4}.$$

The inequality $g(0)<g^3(0)<g^2(0)$ shows that the sequence $(i_n(0))_{n\in\mathbb{N}}$ is non-monotone. By this manner we can prove the sequence $(i_n(d))_{n\in\mathbb{N}}$ is non-monotone for all $d\in\{0,\frac{3}{10},\frac{3}{5},\frac{7}{10},1\}$. However, the sequence $(i_n(\frac{81}{170}))_{n\in\mathbb{N}}$ is monotone as $g^n(\frac{81}{170})=\frac{81}{170}$ for all integers n.

Suppose that $f\in PM_{r_f}(I,I)$ and $g\in PM_{r_g}(J,J)$ together with $\mathscr{M}(f)$ and $\mathscr{M}(g)$, respectively, in which

$$\mathscr{A}_f:=\{c_i,f(c_i),f^2(c_i),i=0,1,\dots,r_f+1\},\qquad F(f):=\{s_1,s_2,\dots,s_{k_f}\},$$
$$\mathscr{A}_g:=\{d_i,g(d_i),g^2(d_i),i=0,1,\dots,r_g+1\},\qquad F(g):=\{\bar{s}_1,\bar{s}_2,\dots,\bar{s}_{k_g}\}.$$

In the sequel we give two lemmas.

Lemma 2.3. *Suppose that $f\in PM_{r_f}(I,I)$ and $g\in PM_{r_g}(J,J)$. If f is topologically conjugate to g via a homeomorphism $\varphi:I\to J$, then $r_f=r_g$ and $\#F(f)=\#F(g)$.*

Proof. The proof of equality $r_f=r_g$ comes from Lemma 3.1 in [11], we briefly repeat it here. The topological conjugacy φ is strictly monotone since it is a homeomorphism. For arbitrary subinterval $(\alpha,\beta)\subset[c_i,c_{i+1}]$ for all $i\in\{0,1,\dots,r_f\}$, the function

$$g|_{\varphi(\alpha,\beta)}=\varphi|_{f(\alpha,\beta)}\circ f|_{(\alpha,\beta)}\circ\varphi^{-1}|_{\varphi(\alpha,\beta)}$$

is strictly monotone from the strict monotonicity of φ and f. Thus $\varphi(\alpha,\beta)$ contains no forts of g. Conversely, given any $(\gamma,\eta)\subset[d_j,d_{j+1}]$ for all $j\in\{0,1,\dots,r_g\}$, then $\varphi^{-1}(\gamma,\eta)$ has no forts of f as the same reason. Hence, φ maps $[c_i,c_{i+1}]$ into $[d_j,d_{j+1}]$ and maps $\mathscr{A}_f$ onto $\mathscr{A}_g$, implying $r_f=r_g$.

For any fixed point $s\in F(f)$, from the conjugacy equation (1.1) we have

$$g(\varphi(s))=\varphi(s),$$

which shows that $\varphi(s)$ is a fixed point of g, thus $\#F(f)\leqslant\#F(g)$. By the same argument we see that $\#F(g)$ is no more than $\#F(f)$, thus

$$\#F(f)=\#F(g).$$

This completes the proof. □

Lemma 2.4. *Suppose that* $f\in PM_r(I,I)$, $g\in PM_r(J,J)$. *If* f *is topologically conjugate to* g *via an orientation-preserving homeomorphism* $\varphi:I\to J$, *then*

(i) f *and* g *have the same type;*

(ii) $I_f(\mu_{f,i})=I_g(\mu_{g,i})$ *for all* $\mu_{f,i}\in\mathscr{M}(f)$, $\mu_{g,i}\in\mathscr{M}(g)$.

Proof.

(i). Without loss of generality, assume that $s\in[c_i,c_{i+1}]$ is an attractive fixed point of f. From conjugacy equation (1.1) we have

$$\varphi\circ f^n=g^n\circ\varphi, \tag{2.1}$$

then for a small enough neighborhood $U(s)$ of s and any $t\in U(s)$, we get

$$\lim_{n\to\infty}\varphi\circ f^n(t)=\varphi\circ\lim_{n\to\infty}f^n(t)=\varphi(s). \tag{2.2}$$

We see that (2.1) and (2.2) yield

$$\lim_{n\to\infty}g^n\circ\varphi(t)=\varphi(s),$$

which implies that g has an attractive fixed point $\varphi(s)$. That is to say, φ maps an attractive fixed point s of f to an attractive fixed point $\varphi(s)$ of g. By this manner, we can prove φ maps the remainder kinds of fixed points to the corresponding ones. Note that $\varphi:I\to J$ is an orientation-preserving homeomorphism, then f and g are of the same type of fixed points.

Lemma 2.3 shows that the homeomorphism φ maps $[c_i,c_{i+1}]$ to $[d_j,d_{j+1}]$, where $i,j=0,1,\dots,r$. Thus we have

$$\varphi(c_i)=d_i$$

as $\varphi:I\to J$ is orientation-preserving. In view of $\varphi\circ f|_{[c_i,c_{i+1}]}$ and $g\circ\varphi|_{[c_i,c_{i+1}]}$ having the same monotonicity from (1.1), consequently, $f|_{[c_i,c_{i+1}]}$ and $g|_{[d_i,d_{i+1}]}$ are increasing or decreasing simultaneously. Thus, we prove that f and g have the same type.

(ii). In view of (i), it is reasonable to assume that

$$\mathscr{A}_f:=\{c_i,f(c_i),f^2(c_i),\ i=0,1,\dots,r+1\},\qquad F(f):=\{s_1,s_2,\dots,s_k\},$$
$$\mathscr{A}_g:=\{d_i,g(d_i),g^2(d_i),\ i=0,1,\dots,r+1\},\qquad F(g):=\{\bar{s}_1,\bar{s}_2,\dots,\bar{s}_k\}.$$

Using (2.1), we have

$$\varphi\circ f^n(c_i)=g^n\circ\varphi(c_i)=g^n(d_i),\ \ i=0,1,\dots,r+1, \tag{2.3}$$

and

$$\varphi\circ f^n(s_j)=g^n\circ\varphi(s_j)=g^n(\bar{s}_j),\ \ j=1,2,\dots,k. \tag{2.4}$$

Since φ is an orientation-preserving homeomorphism, (2.3) and (2.4) lead to

$$I_f(\mu_{f,i})=I_g(\mu_{g,i})$$

for all $\mu_{f,i}\in\mathscr{M}(f)$ and $\mu_{g,i}\in\mathscr{M}(g)$. This completes the proof. □

In this paper we consider the following r-modal maps

$$\Lambda_r(I) := \{f \in PM_r(I, I) : f(c_0) = c_0, f(c_1) > c_1, f(c_2) < c_2, f([c_1, c_{r+1}]) \subseteq (c_1, c_2)\},$$

where $[c_1, c_2]$ belongs to the basin of attraction of the fixed point $s_2 \in (c_1, c_2)$.

Suppose that $f \in \Lambda_r(I)$. According to $\Lambda_r(I)$, we say that $\{f^{-n}(c_1)\}_{n\in\mathbb{N}}$ is an infinite sequence. Thus, f^{n+1} has a new fort $f^{-n}(c_1)$ than that of f^n at least. Therefore, we get $H(f) = \infty$ for each $f \in \Lambda_r(I)$.

3. Main result

Theorem 3.1. *Let $f \in \Lambda_r(I)$ and $g \in \Lambda_r(J)$. Then f is topologically conjugate to g via an orientation-preserving homeomorphism $\varphi : I \to J$ if and only if $I_f(\mu_{f,i}) = I_g(\mu_{g,i})$ for all $\mu_{f,i} \in \mathscr{M}(f), \mu_{g,i} \in \mathscr{M}(g)$.*

Proof. The necessity directly follows from (ii) of Lemma 2.4.

Sufficiency. It suffices to construct an orientation-preserving homeomorphism $\varphi : I \to J$ satisfying (1.1). In order to investigate the construction method of φ and avoid complicated statement, we only consider $f \in \Lambda_2(I)$ and $g \in \Lambda_2(J)$ in detail. Our method can be generalized to the cases $2 < r < \infty$ without essential difficulty. The construction of φ goes by three steps: we first define an orientation-preserving homeomorphism $\varphi_2 : [c_1, c_2] \to [d_1, d_2]$, that is a critical step for φ, then give $\varphi_1 : [c_0, c_1] \to [d_0, d_1]$ and $\varphi_3 : [c_2, c_3] \to [d_2, d_3]$ by extension from φ_2, respectively. Finally, we link $\varphi_i (i = 1, 2, 3)$ as orientation-preserving homeomorphism $\varphi : I \to J$ and show that φ is a topological conjugacy from f to g. Let

$$\mathscr{A}_f := \{c_i, f(c_i), f^2(c_i), i = 0, 1, 2, 3\}, \qquad F(f) := \{s_f\},$$
$$\mathscr{A}_g := \{d_i, g(d_i), g^2(d_i), i = 0, 1, 2, 3\}, \qquad F(g) := \{s_g\}.$$

Then, there are four cases of $f(c_3)$ that are possible, in the following we construct φ for them, respectively:

(i) $f(c_3) = f(c_1)$, (ii) $f(c_3) = s_f$,
(iii) $f(c_3) \in (f(c_2), s_f)$, (iv) $f(c_3) \in (s_f, f(c_1))$.

Case (i). Note that

$$c_1 < f(c_2) < f^2(c_1) < s_f < f(c_1) < c_2, \quad d_1 < g(d_2) < g^2(d_1) < s_g < g(d_1) < d_2.$$

Let $\varphi_{2,2,1} : [c_1, f^2(c_1)] \to [d_1, g^2(d_1)]$ be an arbitrary orientation-preserving homeomorphism satisfying

$$\varphi_{2,2,1}(f(c_2)) = g(d_2),$$

then we define orientation-preserving homeomorphism $\varphi_2 : [c_1, c_2] \to [d_1, d_2]$ as follows

$$\varphi_2(t) := \begin{cases} s_g, & t = s_f, \\ \varphi_{2,2,1}(t), & c_1 \leqslant t \leqslant f^2(c_1), \\ \varphi_{2,2,2i}(t) := g_2 \circ \varphi_{2,2,2i-1} \circ f_2^{-1}(t), & f^{2i+1}(c_1) \leqslant t \leqslant f^{2i-1}(c_1), \\ \varphi_{2,2,2i+1}(t) := g_2 \circ \varphi_{2,2,2i} \circ f_2^{-1}(t), & f^{2i}(c_1) \leqslant t \leqslant f^{2i+2}(c_1), \\ \varphi_{2,3}(t) := g_2^{-1} \circ \varphi_{2,2,1} \circ f_2(t), & f(c_1) \leqslant t \leqslant c_2, \end{cases} \tag{3.1}$$

where $i = 1, 2, \ldots$. Next, define $\varphi_1 : [c_0, c_1] \to [d_0, d_1]$ such as

$$\varphi_1(t) := \begin{cases} d_0, & t = c_0, \\ \varphi_{1,1}(t) := g_1^{-1} \circ \varphi_2 \circ f_1(t), & f_1^{-1}(s_f) \leqslant t \leqslant c_1, \\ \varphi_{1,i+1}(t) := g_1^{-1} \circ \varphi_{1,i} \circ f_1(t), & f^{-i-1}(s_f) \leqslant t \leqslant f^{-i}(s_f) \end{cases} \tag{3.2}$$

for $i = 1, 2, \dots$ and $\varphi_3 : [c_2, c_3] \to [d_2, d_3]$ as

$$\varphi_3(t) := g_3^{-1} \circ \varphi_2 \circ f_3(t). \tag{3.3}$$

Joining (3.1) and (3.2) with (3.3) we get $\varphi : I \to J$. We say that φ is orientation-preserving from its definition. It remains to prove that φ is a topological conjugacy from f to g. For subinterval $[c_1, c_2]$, from (3.1) we have

$$\begin{aligned}
\varphi \circ f(s_f) &= g \circ \varphi(s_f), \\
\varphi \circ f(t) &= \varphi_{2,2,2i+1} \circ f_2(t) \\
&= g_2 \circ \varphi_{2,2,2i} \circ f_2^{-1} \circ f_2(t) \\
&= g_2 \circ \varphi_{2,2,2i}(t) = g \circ \varphi(t), \qquad \forall t \in [f^{2i+1}(c_1), f^{2i-1}(c_1)], \\
\varphi \circ f(t) &= \varphi_{2,2,2i+2} \circ f_2(t) \\
&= g_2 \circ \varphi_{2,2,2i+1} \circ f_2^{-1} \circ f_2(t) \\
&= g_2 \circ \varphi_{2,2,2i+1}(t) = g \circ \varphi(t), \qquad \forall t \in [f^{2i-2}(c_1), f^{2i}(c_1)], \ \ i = 1, 2, 3 \dots,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
g \circ \varphi(t) &= g_2 \circ \varphi_{2,3}(t) \\
&= g_2 \circ g_2^{-1} \circ \varphi_{2,2,1} \circ f_2(t) = \varphi_{2,2,1} \circ f_2(t) = \varphi \circ f(t), \qquad \forall t \in [f(c_1), c_2]],
\end{aligned} \tag{3.5}$$

which imply that

$$\varphi \circ f(t) = g \circ \varphi(t), \qquad \forall t \in [c_1, c_2]. \tag{3.6}$$

For $[c_0, c_1]$, in view of (3.2) we have

$$g \circ \varphi(c_0) = \varphi \circ f(c_0),$$

and

$$\begin{aligned}
g \circ \varphi(t) &= g_1 \circ \varphi_{1,1}(t) = g_1 \circ g_1^{-1} \circ \varphi_2 \circ f_1(t) = \varphi_2 \circ f_1(t) = \varphi \circ f(t), \qquad \forall t \in [f_1^{-1}(s_f), c_1], \\
g \circ \varphi(t) &= g_1 \circ \varphi_{1,i+1}(t) \\
&= g_1 \circ g_1^{-1} \circ \varphi_{1,i} \circ f_1(t) = \varphi_{1,i} \circ f_1(t) = \varphi \circ f(t), \ \ \forall t \in [f_1^{-i-1}(s_f), f_1^{-i}(s_f)], \ i = 1, 2, \dots,
\end{aligned}$$

implying

$$\varphi \circ f(t) = g \circ \varphi(t), \qquad \forall t \in [c_0, c_1]. \tag{3.7}$$

For subinterval $[c_2, c_3]$, we have

$$g \circ \varphi(t) = g_3 \circ \varphi_3(t) = g_3 \circ g_3^{-1} \circ \varphi_2 \circ f_3(t) = \varphi_2 \circ f_3(t) = \varphi \circ f(t), \ \ \forall t \in [c_2, c_3]. \tag{3.8}$$

Thus, we show that φ is a topological conjugacy from f to g by (3.6), (3.7), and (3.8).

Case (ii). The construction of φ is same as that of Case (i).

Case (iii). Without loss of generality, assume that $f^2(c_1) < f(c_3) < s_f$ (the remainder cases are discussed similarly). That is

$$c_1 < f(c_2) < f^2(c_1) < f(c_3) < s_f < f(c_1) < c_2, \quad d_1 < g(d_2) < g^2(d_1) < g(d_3) < s_g < g(d_1) < d_2.$$

Denote $\varphi_{2,2,1} : [c_1, f(c_3)] \to [d_1, g(d_3)]$ an arbitrary orientation-preserving homeomorphism satisfying

$$\begin{cases} \varphi_{2,2,1}(f(c_2)) = g(d_2), \\ \varphi_{2,2,1}(f^2(c_1)) = g^2(d_1). \end{cases}$$

On the basis of $\varphi_{2,2,1}$, we define orientation-preserving homeomorphism $\varphi_2 : [c_1, c_2] \to [d_1, d_2]$ as

$$\varphi_2(t) := \begin{cases} s_g, & t = s_f, \\ \varphi_{2,2,1}(t), & c_1 \leqslant t \leqslant f(c_3), \\ \varphi_{2,2,2i}(t) := g_2 \circ \varphi_{2,2,2i-1} \circ f_2^{-1}(t), & f^{2i}(c_3) \leqslant t \leqslant f^{2i-1}(c_1), \\ \varphi_{2,2,2i+1}(t) := g_2 \circ \varphi_{2,2,2i} \circ f_2^{-1}(t), & f^{2i}(c_1) \leqslant t \leqslant f^{2i+1}(c_3), \\ \varphi_{2,3}(t) := g_2^{-1} \circ \varphi_{2,2,1} \circ f_2(t), & f(c_1) \leqslant t \leqslant c_2, \end{cases} \tag{3.9}$$

where $i = 1, 2, \ldots$. Consequently, define homeomorphism $\varphi_1 : [c_0, c_1] \to [d_0, d_1]$ as (3.2) and $\varphi_3 : [c_2, c_3] \to [d_2, d_3]$ as (3.3).

Connecting (3.9) and (3.2) with (3.3) as $\varphi : I \to J$, then φ is orientation-preserving from the construction process. We only need to prove φ is a topological conjugacy from f to g. For subinterval $[c_1, c_2]$, from (3.9) we have

$$\begin{aligned} \varphi \circ f(t) &= \varphi_{2,2,2i+1} \circ f_2(t) \\ &= g_2 \circ \varphi_{2,2,2i} \circ f_2^{-1} \circ f_2(t) = g_2 \circ \varphi_{2,2,2i}(t) = g \circ \varphi(t), \qquad \forall t \in [f^{2i}(c_3), f^{2i-1}(c_1)], \\ \varphi \circ f(t) &= \varphi_{2,2,2i+2} \circ f_2(t) \\ &= g_2 \circ \varphi_{2,2,2i+1} \circ f_2^{-1} \circ f_2(t) \\ &= g_2 \circ \varphi_{2,2,2i+1}(t) = g \circ \varphi(t), \qquad \forall t \in [f^{2i-2}(c_1), f^{2i-1}(c_3)], \quad i = 1, 2, 3 \ldots, \end{aligned}$$

and (3.4), (3.5), which imply that

$$\varphi \circ f(t) = g \circ \varphi(t), \qquad \forall t \in [c_1, c_2]. \tag{3.10}$$

Thus, we prove that φ is an orientation-preserving topological conjugacy from f to g by (3.10), (3.7), and (3.8).

Case (iv). Without loss of generality, assume that $f^2(c_1) < f^2(c_3) < s_f$ (the remainder cases can be discussed similarly). That is

$$\begin{aligned} &c_1 < f(c_2) < f^2(c_1) < f^2(c_3) < s_f < f(c_3) < f(c_1) < c_2, \\ &d_1 < g(d_2) < g^2(d_1) < g^2(d_3) < s_g < g(d_3) < g(d_1) < d_2. \end{aligned}$$

Let $\varphi_{2,2,1} : [c_1, f^2(c_3)] \to [d_1, g^2(d_3)]$ be an arbitrary orientation-preserving homeomorphism satisfying

$$\begin{cases} \varphi_{2,2,1}(f(c_2)) = g(d_2), \\ \varphi_{2,2,1}(f^2(c_1)) = g^2(d_1). \end{cases}$$

On the basis of $\varphi_{2,2,1}$, we define orientation-preserving homeomorphism $\varphi_2 : [c_1, c_2] \to [d_1, d_2]$ as

$$\varphi_2(t) := \begin{cases} s_g, & t = s_f, \\ \varphi_{2,2,1}(t), & c_1 \leqslant t \leqslant f^2(c_3), \\ \varphi_{2,2,2i}(t) := g_2 \circ \varphi_{2,2,2i-1} \circ f_2^{-1}(t), & f^{2i+1}(c_3) \leqslant t \leqslant f^{2i-1}(c_1), \\ \varphi_{2,2,2i+1}(t) := g_2 \circ \varphi_{2,2,2i} \circ f_2^{-1}(t), & f^{2i}(c_1) \leqslant t \leqslant f^{2i+2}(c_3), \\ \varphi_{2,3}(t) := g_2^{-1} \circ \varphi_{2,2,1} \circ f_2(t), & f(c_1) \leqslant t \leqslant c_2, \end{cases} \tag{3.11}$$

where $i = 1, 2, \ldots$. Consequently, define $\varphi_1 : [c_0, c_1] \to [d_0, d_1]$ as (3.2) and $\varphi_3 : [c_2, c_3] \to [d_2, d_3]$ as (3.3).

Connecting (3.11) and (3.2) with (3.3) as orientation-preserving homeomorphism $\varphi : I \to J$, we again prove φ is a topological conjugacy from f to g. For subinterval $[c_1, c_2]$, in view of (3.9) we have

$$\varphi \circ f(t) = \varphi_{2,2,2} \circ f_2(t) = g_2 \circ \varphi_{2,2,1} \circ f_2^{-1} \circ f_2(t) = g_2 \circ \varphi_{2,2,1}(t) = g \circ \varphi(t), \qquad \forall t \in [c_1, f^2(c_3)],$$

$$\begin{aligned}\varphi\circ f(t) &= \varphi_{2,2,2i+1}\circ f_2(t)\\ &= g_2\circ\varphi_{2,2,2i}\circ f_2^{-1}\circ f_2(t) = g_2\circ\varphi_{2,2,2i}(t) = g\circ\varphi(t), \qquad \forall t\in[f^{2i+1}(c_3), f^{2i-1}(c_1)],\\ \varphi\circ f(t) &= \varphi_{2,2,2i+2}\circ f_2(t)\\ &= g_2\circ\varphi_{2,2,2i+1}\circ f_2^{-1}\circ f_2(t)\\ &= g_2\circ\varphi_{2,2,2i+1}(t) = g\circ\varphi(t),\ \forall t\in[f^{2i}(c_1), f^{2i+2}(c_3)],\ \ i=1,2,3\ldots,\end{aligned}$$

and (3.4), (3.5), which imply that

$$\varphi\circ f(t) = g\circ\varphi(t), \qquad \forall t\in[c_1,c_2]. \tag{3.12}$$

Thus, we show that φ is an orientation-preserving topological conjugacy from f to g by (3.12), (3.7), and (3.8). □

In the sequel we present some examples to illustrate our result.

Example 3.2. Consider $f_1\in\Lambda_2([0,1])$ and $g_1\in\Lambda_2([0,1])$ defined by

$$f_1(t) = \begin{cases}\frac{7}{2}t, & t\in[0,\frac{1}{4}),\\ -\frac{1}{5}t+\frac{37}{40}, & t\in[\frac{1}{4},\frac{7}{8}),\\ t-\frac{1}{8}, & t\in[\frac{7}{8},1],\end{cases} \quad\text{and}\quad g_1(t) = \begin{cases}\frac{3}{2}t, & t\in[0,\frac{1}{2}),\\ -\frac{1}{3}t+\frac{11}{12}, & t\in[\frac{1}{2},\frac{7}{8}),\\ t-\frac{1}{4}, & t\in[\frac{7}{8},1],\end{cases}$$

respectively. By simple calculation, we know that

$$F(f_1) = \{\frac{37}{72}\},\ \ F(g_1) = \{\frac{11}{16}\}.$$

Since

$$\begin{aligned}I_{f_1}(0) &= I_{g_1}(0) = \{0,0,0,0,0,0,\ldots\},\\ I_{f_1}(\frac{1}{4}) &= I_{g_1}(\frac{1}{2}) = \{4,2,\tilde{3},\tilde{2},\tilde{3},\tilde{2},\ldots\},\\ I_{f_1}(\frac{37}{72}) &= I_{g_1}(\frac{11}{16}) = \{3,3,3,3,3,3,\ldots\},\\ I_{f_1}(\frac{3}{4}) &= I_{g_1}(\frac{3}{4}) = \{2,\tilde{3},\tilde{2},\tilde{3},\tilde{2},\tilde{3},\tilde{2},\ldots\},\\ I_{f_1}(1) &= I_{g_1}(1) = \{4,2,\tilde{3},\tilde{2},\tilde{3},\tilde{2},\ldots\},\end{aligned}$$

we have $f_1\sim g_1$ from Theorem 3.1.

Example 3.3. Consider $f_2\in\Lambda_2([0,1])$ and $g_2\in\Lambda_2([0,1])$ defined by

$$f_2(t) = \begin{cases}3t, & t\in[0,\frac{1}{4}),\\ -\frac{1}{2}t+\frac{7}{8}, & t\in[\frac{1}{4},\frac{3}{4}),\\ \frac{1}{3}t+\frac{1}{4}, & t\in[\frac{3}{4},1],\end{cases} \quad\text{and}\quad g_2(t) = \begin{cases}\frac{5}{3}t, & t\in[0,\frac{3}{8}),\\ -\frac{1}{2}t+\frac{13}{16}, & t\in[\frac{3}{8},\frac{5}{8}),\\ \frac{1}{3}t+\frac{7}{24}, & t\in[\frac{5}{8},1],\end{cases}$$

respectively. Note that

$$F(f_2) = \{\frac{37}{72}\},\ \ F(g_2) = \{\frac{11}{16}\}$$

and

$$I_{f_2}(0) = I_{g_2}(0) = \{0,0,0,0,0,0,\ldots\},$$

$$
\begin{aligned}
I_{f_2}(\frac{1}{4}) = I_{g_2}(\frac{3}{8}) &= \{4, 2, \tilde{3}, \tilde{2}, \tilde{3}, \tilde{2}, \ldots\},\\
I_{f_2}(\frac{7}{12}) = I_{g_2}(\frac{13}{24}) &= \{3, 3, 3, 3, 3, 3, \ldots\},\\
I_{f_2}(\frac{3}{4}) = I_{g_2}(\frac{5}{8}) &= \{2, \tilde{3}, \tilde{2}, \tilde{3}, \tilde{2}, \tilde{3}, \tilde{2}, \ldots\},\\
I_{f_2}(1) = I_{g_2}(1) &= \{3, 3, 3, 3, 3, 3, \ldots\},
\end{aligned}
$$

we get $f_2 \sim g_2$ by Theorem 3.1.

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