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Inequalities via generalized log $\mathfrak{m}$-convex functions

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Abstract

The main objective of this paper is to introduce and investigate a new class of convex functions, which is called as generalized log $\mathfrak{m}$-convex function. Some new Hermite-Hadamard type of integral inequalities via generalized log $\mathfrak{m}$-convex functions are obtained. Several special cases are also discussed.

Keywords: Generalized convex functions, generalized log $\mathfrak{m}$-convex functions, Hermite-Hadamard type inequalities.
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1. Introduction

Convex analysis has emerged as one of the most interesting and useful field of mathematics in last few decades. In order to study different problems related to pure and applied sciences, the concept of convexity has been extended and generalized in several directions using different new techniques. Consequently various inequalities via convex functions and their variant forms are being developed and studied, see [1–3, 5, 8, 16, 18, 19, 22–25]. Hermite-Hadamard inequality is one of the most important inequalities related to convex function, see [11, 12]. In recent years, much attention has been given to derive the Hermite-Hadamard type inequalities for various types of convex functions, see [9, 13–15, 17, 21]. An important extension of convex functions is an intermediate form between the usual convexity and star shaped property is $\mathfrak{m}$-convex function, defined by the Toader [24]. Gordji et al. [10] also introduced an important class of convex functions, which is called φ-convex function. These φ-convex functions are nonconvex functions. For recent developments, see [6, 7, 10, 19, 20] and the references therein.

Inspired by this ongoing research, we introduce a new class of convex functions, which is called as generalized log $\mathfrak{m}$-convex function. We derive some new Hermite-Hadamard integral inequalities via these non-convex functions. Our results include a wide class of new and known inequalities. This is the main motivation of this paper.

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2. Preliminaries

In this section, we discuss some preliminary concepts and results. Before we move further, let I be an interval in real line $\mathbb{R}$. Let $f : I = [ma, b] \to \mathbb{R}$ be a continuous function and $\eta(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous bifunction. First of all, we recall the following well-known results and concepts.

Definition 2.1. A set $\mathcal{S}$ is said to be m-convex set, if there exists a fixed constant $m \in (0, 1]$ such that

$$(tma + (1-t)b) \in \mathcal{S}, \quad \forall a, b \in \mathcal{S}, t \in [0, 1].$$

Definition 2.2. A function $f : I = [ma, b] \to \mathbb{R}$ is said to be generalized m-convex function with respect to a bifunction $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, if

$$f(tma + (1-t)b) \leqslant f(b) + t\eta(mf(a), f(b)), \quad \forall a, b \in I, t \in [0, 1], m \in (0, 1].$$

If $\eta(mf(a), f(b)) = mf(a) - f(b)$, then Definition 2.2 reduces to the following definition.

Definition 2.3 ([20])**.** A function $f : I = [0, b] \to \mathbb{R}$ is said to be m-convex, where $m \in (0, 1]$, if

$$f(tma + (1-t)b) \leqslant tmf(a) + (1-t)f(b), \quad \forall a, b \in I, t \in [0, 1].$$

It is denoted by $K_m(b)$, the class of all m-convex functions on $[0, b]$.

Definition 2.4. A function $f : I = [ma, b] \to \mathbb{R}^+$ is said to be generalized log m-convex (or log m-convex function) with respect to a bifunction $\eta(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^+$, if

$$f(tma + (1-t)b) \leqslant [mf(a)]^t[mf(a) + \eta(f(b), mf(a))]^{1-t} \tag{2.1}$$

holds for all $a, b \in I, t \in [0, 1]$ and $m \in (0, 1]$.

If $t = \frac{1}{2}$ in (2.1), then

$$f\Big(\frac{ma+b}{2}\Big) \leqslant \sqrt{[mf(a)][mf(a) + \eta(f(b), mf(a))]},$$

which is known as Jensen generalized log m-convex function.

From Definition 2.4, we have

$$f(tma + (1-t)b) \leqslant [mf(a)]^t[mf(a) + \eta(f(b), mf(a)]^{1-t} \leqslant tmf(a) + (1-t)(mf(a) + \eta(f(b), mf(a))).$$

It follows that

$$\log f(tma + (1-t)tb) \leqslant t\log[mf(a)] + (1-t)\log[mf(a) + \eta(f(b), mf(a))].$$

This means that every generalized log m-convex function is a generalized convex function. However the converse is not true, see [4].

Example 2.5 ([6])**.** Consider $f : \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \geqslant 0, \\ e^{x}, & \text{if } x < 0, \end{cases}$$

and define a bifunction $\eta(x, y) = -x - y$ for all $x, y \in \mathbb{R}^- = (-\infty, 0)$. Then f is generalized log m-convex, but the converse is not true.

If $\eta(mf(a), f(b)) = mf(a) - f(b)$, then Definition 2.4 reduces to the following definition.

Definition 2.6. A function $f : I \to [0, \infty)$ is said to be log m-convex, if $\log f$ is convex, or equivalently if for all $a, b \in I$, $m \in (0, 1)$ and $t \in [0, 1]$, one has the inequality

$$f((1-t)ma + tb) \leqslant [mf(a)]^{1-t}[f(b)]^{t}, \quad \forall a, b \in I, t \in [0, 1].$$

We will use the following notations throughout this paper.

1. Arithmetic Mean:
$$A(a, b) = \frac{a+b}{2}, \quad \forall a, b \in \mathbb{R}_{+}.$$
2. Geometric Mean:
$$G(a, b) = \sqrt{ab}, \quad \forall a, b \in \mathbb{R}_{+}.$$
3. Logarithmic mean:
$$L(a, b) = \frac{b-a}{\log b - \log a}, \quad \forall a, b \in \mathbb{R}_{+}, a \neq b.$$
4. Quadratic Mean:
$$K(a, b) = \sqrt{\frac{a^2+b^2}{2}}, \quad \forall a, b \in \mathbb{R}_{+}.$$

3. Main results

In this section, we establish several new integral inequalities of Hermite-Hadamard type via generalized log m-convex functions.

Theorem 3.1. *Let* $f : I = [ma, b] \to (0, \infty)$ *be generalized* log m*-convex function on* I *with* $a < b$*. Then*

$$\begin{aligned}\frac{1}{\sqrt{m}} f\Big(\frac{ma+b}{2}\Big) &- \exp \frac{1}{2(b-ma)} \int_{ma}^{b} \log[mf(x) + \eta(f(ma+b-x), mf(x))]dx \\ &\leqslant \exp \frac{1}{2(b-ma)} \int_{ma}^{b} \log mf(x)dx \\ &\leqslant \sqrt{\Big[mf(a)[mf(a) + \eta(f(b), mf(a))]\Big]}.\end{aligned}$$

Proof. Let f be a generalized log m-convex function on I. Then

$$f(tma + (1-t)b) \leqslant [mf(a)]^{t}[mf(a) + \eta(f(b), mf(a))]^{1-t}.$$

This implies that

$$\log f(tma + (1-t)b) \leqslant t\log[mf(a)] + (1-t)\log[mf(a) + \eta(f(b), mf(a))]. \tag{3.1}$$

Integrating (3.1) with respect to t on $[0, 1]$, we have

$$\begin{aligned}\int_0^1 \log f(tma + (1-t)b)dt &\leqslant \int_0^1 \Big[t\log[mf(a)] + (1-t)\log[mf(a) + \eta(f(b), mf(a))]\Big]dt \\ &= \log \sqrt{[mf(a)][mf(a) + \eta(f(b), mf(a))]}.\end{aligned}$$

Thus

$$\frac{1}{b-ma}\int_{ma}^{b}\log f(x)dx \leqslant \log\sqrt{[mf(a)][mf(a)+\eta(f(b),mf(a))]}. \tag{3.2}$$

Consider

$$\begin{aligned} f\left(\frac{ma+b}{2}\right) &= \frac{f(tma+(1-t)b)+((1-t)ma+tb)}{2} \\ &\leqslant \sqrt{[mf(tma+(1-t)b)][mf(tma+(1-t)b)+\eta(f(1-t)b+tma,mf(tma+(1-t)b))]}\,. \end{aligned}$$

From this it follows that

$$\begin{aligned} \log f\left(\frac{ma+b}{2}\right) \leqslant \frac{1}{2}\Big\{ &\log[mf(tma+(1-t)b)]+\log[mf(tma+(1-t)b) \\ &+\eta(f((1-t)b+tma),mf(tma+(1-t)b)]\Big\}. \end{aligned} \tag{3.3}$$

Integrating (3.3) with respect to t on $[0,1]$, we have

$$\log f\left(\frac{ma+b}{2}\right) \leqslant \frac{1}{2(b-ma)}\int_{ma}^{b}\Big\{\log mf(x)+\log[mf(x)+\eta(f(ma+b-x),mf(x))]\Big\}dx.$$

Thus

$$\begin{aligned} 2\log f\left(\frac{ma+b}{2}\right) &- \frac{1}{(b-ma)}\int_{ma}^{b}\log[mf(x)+\eta(f(ma+b-x),mf(x))]dx \\ &\leqslant \frac{1}{(b-ma)}\int_{ma}^{b}\log mf(x)dx \\ &= \frac{1}{(b-ma)}\int_{ma}^{b}\log m dx + \frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)dx \\ &= \frac{\log m}{(b-ma)}(b-ma) + \frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)dx \\ &= \log m + \frac{1}{(b-ma)}\int_{ma}^{b}\log f(x)dx. \end{aligned} \tag{3.4}$$

Combining (3.2) and (3.4), we have

$$\begin{aligned} 2\log f\left(\frac{ma+b}{2}\right) &- \frac{1}{(b-ma)}\int_{ma}^{b}\log[mf(x)+\eta(f(ma+b-x),mf(x))dx - \log m \\ &\leqslant \frac{1}{(b-ma)}\int_{ma}^{b}\log mf(x)dx \\ &\leqslant \log\sqrt{\Big[mf(a)[mf(a)+\eta(f(b),mf(a))]\Big]}, \end{aligned}$$

which implies that

$$\begin{aligned} \frac{1}{\sqrt{m}}f\left(\frac{ma+b}{2}\right) &- \exp\frac{1}{2(b-ma)}\int_{ma}^{b}\log[mf(x)+\eta(f(ma+b-x),mf(x))]dx \\ &\leqslant \exp\frac{1}{2(b-ma)}\int_{ma}^{b}\log mf(x)dx \\ &\leqslant \sqrt{\Big[mf(a)[mf(a)+\eta(f(b),mf(a))]\Big]}. \end{aligned}$$

This completes the proof. □

Corollary 3.2. *If* $\eta(f(b), mf(a)) = f(b) - mf(a)$, *then under the assumption of Theorem* 3.1, *we have Hermite-Hadamard inequality for* log *m-convex function,*

$$\frac{1}{\sqrt{m}} f\left(\frac{ma+b}{2}\right) \leqslant \exp \frac{1}{(b-ma)} \int_{ma}^{b} \log f(x)dx \leqslant \sqrt{mf(a)f(b)}.$$

Corollary 3.3 ([8])**.** *If* $\eta(f(b), mf(a)) = f(b) - mf(a)$ *and* $m = 1$, *then under the assumption of Theorem* 3.1, *we have Hermite-Hadamard inequality for* log-*convex function is*

$$f\left(\frac{a+b}{2}\right) \leqslant \exp \frac{1}{(b-a)} \int_{a}^{b} \log f(x)dx \leqslant \sqrt{[f(a)f(b)]}.$$

Theorem 3.4. *Let* $f, g : I = [ma, b] \to (0, \infty)$ *be generalized* log m*-convex functions on* I *with* $a < b$. *Then*

$$\begin{aligned}
&\frac{1}{b-ma} \int_{ma}^{b} f(x)g(ma+b-x)dx \\
&\leqslant \frac{mg(a)\{mf(a)+\eta(f(b), mf(a))\} + mf(a)\{mg(a)+\eta(g(b), mg(a))\}}{2} \\
&\leqslant \frac{1}{2}\Big\{A[mf(a)+\eta(f(b), mf(a)), mf(a)]L[mf(a)+\eta(f(b), mf(a)), mf(a)] \\
&\quad + A[mg(a), mg(a)+\eta(g(b), mg(a))]L[mg(a), mg(a)+\eta(g(b), mg(a))]\Big\} \\
&\leqslant \frac{[mf(a)+(mf(a)+\eta(f(b), mf(a)))]^2}{16} + \frac{[mg(a)+(mg(a)+\eta(g(b), mg(a)))]^2}{16} \\
&\quad + \frac{mg(a)\{mf(a)+\eta(f(b), mf(a))\} + \{mf(a)\{mg(a)+\eta(g(b), mg(a))\}}{4},
\end{aligned}$$

where A *and* L *are Arithmetic and Logarithmic means, respectively.*

Proof. Let f, g be generalized log m-convex functions on I. Then

$$\begin{aligned}
f(tma+(1-t)b) &\leqslant [mf(a)]^t[mf(a)+\eta(f(b), mf(a))]^{1-t}, \\
g((1-t)ma+tb) &\leqslant [mg(a)]^{1-t}[mg(a)+\eta(g(b), mg(a))]^t.
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{1}{b-ma} \int_{ma}^{b} f(x)g(ma+b-x)dx \\
&= \int_0^1 f(tma+(1-t)b)g((1-t)ma+tb)dt \\
&\leqslant \int_0^1 \left[\{mf(a)\}^t\{mf(a)+\eta(f(b), mf(a))\}^{1-t}\{mg(a)\}^{1-t}\{mg(a)+\eta(g(b), mg(a))\}^t\right]dt \\
&= [mg(a)(mf(a)+\eta(f(b), mf(a)))] \int_0^1 \left[\frac{mf(a)\{mg(a)+\eta(g(b), mg(a))\}}{mg(a)\{mf(a)+\eta(f(b), mf(a))\}}\right]^t dt \\
&= \frac{\left[mf(a)\{mg(a)+\eta(g(b), mg(a))\} - mg(a)\{mf(a)+\eta(f(b), mf(a))\}\right]}{\log\left[mf(a)\{mg(a)+\eta(g(b), mg(a))\}\right] - \log\left[mg(a)\{mf(a)+\eta(f(b), mf(a))\}\right]} \\
&\leqslant \frac{mg(a)\{mf(a)+\eta(f(b), mf(a))\} + mf(a)\{mg(a)+\eta(g(b), mg(a))\}}{2} \\
&\leqslant \frac{1}{2} \int_0^1 [\{f(tma+(1-t)b)\}^2 + \{g((1-t)ma+tb))\}^2]dt
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\int_0^1 \left\{[\{mf(a)\}^t\{mf(a)+\eta(f(b),mf(a))\}^{1-t}]^2+[\{mg(a)\}^{1-t}\{mg(a)+\eta(g(b),mg(a))\}^t]^2\right\}dt\\
&=\frac{1}{2}\left\{[mf(a)+\eta(f(b),mf(a))]^2\int_0^1\left[\frac{mf(a)}{mf(a)+\eta(f(b),mf(a))}\right]^{2t}dt\right.\\
&\quad\left.+[mg(a)]^2\int_0^1\left[\frac{mg(a)+\eta(g(b),mg(a))}{mg(a)}\right]^{2t}dt\right\}\\
&=\frac{1}{4}\left\{\left[\frac{[mf(a)]^2-[mf(a)+\eta(f(b),mf(a))]^2}{\log[mf(a)]-\log[mf(a)+\eta(f(b),mf(a))]}\right]\right.\\
&\quad\left.+\left[\frac{[mg(a)+\eta(g(b),mg(a))]^2-[mg(a)]^2}{\log[mg(a)+\eta(g(b),mg(a))]-\log[mg(a)]}\right]\right\}\\
&\leqslant\frac{1}{2}\left\{A[mf(a)+\eta(f(b),mf(a)),mf(a)]L[mf(a)+\eta(f(b),mf(a)),mf(a)]\right.\\
&\quad\left.+A[mg(a),mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))]\right\}\\
&\leqslant\frac{1}{4}\int_0^1[f(tma+(1-t)b)+g((1-t)ma+tb)]^2dt\\
&\leqslant\frac{1}{4}\int_0^1\left\{[\{mf(a)\}^t\{mf(a)+\eta(f(b),mf(a))\}^{1-t}]^2+[\{mg(a)\}^{1-t}\{mg(a)+\eta(g(b),mg(a))\}^t]^2\right.\\
&\quad\left.+2[\{mf(a)\}^t\{mf(a)+\eta(f(b),mf(a))\}^{1-t}][\{mg(a)\}^{1-t}\{mg(a)+\eta(g(b),mg(a))\}^t]\right\}dt\\
&=\frac{1}{4}\left\{[mf(a)+\eta(f(b),mf(a))]^2\int_0^1\left[\frac{mf(a)}{mf(a)+\eta(f(b),mf(a))}\right]^{2t}dt\right.\\
&\quad+[mg(a)]^2\int_0^1\left[\frac{mg(a)+\eta(g(b),mg(a))}{mg(a)}\right]^{2t}dt\\
&\quad\left.+2[mg(a)(mf(a)+\eta(f(b),mf(a)))]\int_0^1\left[\frac{mf(a)\{mg(a)+\eta(g(b),mg(a))\}}{mg(a)\{mf(a)+\eta(f(b),mf(a))\}}\right]^t dt\right\}\\
&=\frac{1}{8}\left\{\left[\frac{[mf(a)]^2-[mf(a)+\eta(f(b),mf(a))]^2}{\log[mf(a)]-\log[mf(a)+\eta(f(b),mf(a))]}\right]\right.\\
&\quad\left.+\left[\frac{[mg(a)+\eta(g(b),mg(a))]^2-[mg(a)]^2}{\log[mg(a)+\eta(g(b),mg(a))]-\log mg(a)}\right]\right\}\\
&\quad+\frac{1}{2}\left\{\frac{mf(a)\{mg(a)+\eta(g(b),mg(a))\}-mg(a)\{mf(a)+\eta(f(b),mf(a))\}}{\log[mf(a)\{mg(a)+\eta(g(b),mg(a))\}]-\log\left[mg(a)\{mf(a)+\eta(f(b),mf(a))\}\right]}\right\}\\
&\leqslant\frac{[mf(a)+(mf(a)+\eta(f(b),mf(a)))]^2}{16}+\frac{[mg(a)+(mg(a)+\eta(g(b),mg(a)))]^2}{16}\\
&\quad+\frac{mg(a)\{mf(a)+\eta(f(b),mf(a))\}+\{mf(a)\{mg(a)+\eta(g(b),mg(a))\}}{4}.
\end{aligned}
$$

This completes the proof. □

Corollary 3.5. *If* $\eta(f(b),mf(a))=f(b)-mf(a)$, *then under the assumption of Theorem 3.4, we have*

$$
\begin{aligned}
\frac{1}{b-ma}\int_{ma}^{b}f(x)g(ma+b-x)dx &\leqslant \frac{mg(a)f(b)+mf(a)g(b)}{2}\\
&\leqslant\frac{1}{2}\left\{A[f(b),mf(a)]L[f(b),mf(a)]+A[mg(a),g(b)]L[mg(a),g(b)]\right\}
\end{aligned}
$$

$$\leqslant \frac{[mf(a)+f(b)]^2}{16}+\frac{[g(b)+mg(a)]^2}{16}+\frac{mg(a)f(b)+mf(a)g(b)}{4}.$$

Corollary 3.6 ([18]). *If $\eta(f(b), mf(a)) = f(b) - mf(a)$ and $m = 1$, then under the assumption of Theorem 3.4, we have*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(a+b-x)dx &\leqslant \frac{g(a)f(b)+f(a)g(b)}{2}\\ &\leqslant \frac{1}{2}\Big\{A[f(b),f(a)]L[f(b),f(a)]+A[g(a),g(b)]L[g(a),g(b)]\Big\}\\ &\leqslant \frac{[f(a)+f(b)]^2}{16}+\frac{[g(b)+g(a)]^2}{16}+\frac{g(a)f(b)+f(a)g(b)}{4}.\end{aligned}$$

Theorem 3.7. *Let $f, g : I = [ma, b] \to (0, \infty)$ be generalized $\log$ m-convex functions on I with $a < b$. If $\alpha + \beta = 1$, then*

$$\begin{aligned}&\frac{1}{b-ma}\int_{ma}^b f(x)g(ma+b-x)dx\\ &\leqslant \alpha^2\left[\frac{[mf(a)+\eta(f(b),mf(a))]^{\frac{1}{\alpha}}-[mf(a)]^{\frac{1}{\alpha}}}{[\eta(f(b),mf(a)]}\right]L[mf(a),mf(a)+\eta(f(b),mf(a))]\\ &\quad+\beta^2\left[\frac{[mg(a)]^{\frac{1}{\beta}}-[mg(a)+\eta(g(b),mg(a))]^{\frac{1}{\beta}}}{[\eta(g(b),mg(a))]}\right]L[mg(a)+\eta(g(b),mg(a)),mg(a)],\end{aligned}$$

where L is the Logarithmic mean.

Proof. Let f and g be generalized log m-convex function on I. Then for all $a, b \in I, t \in [0, 1]$, we have

$$\begin{aligned}f(tma+(1-t)b) &\leqslant [mf(a)]^t[mf(a)+\eta(f(b),mf(a))]^{1-t},\\ g((1-t)ma+tb) &\leqslant [mg(a)]^{1-t}[mg(a)+\eta(g(b),m(a))]^t.\end{aligned}$$

Now we use Young's inequality,

$$ab \leqslant \alpha a^{\frac{1}{\alpha}}+\beta b^{\frac{1}{\beta}}, \quad \forall \alpha,\beta>0, \alpha+\beta=1.$$

Consider

$$\begin{aligned}&\frac{1}{b-ma}\int_{ma}^b f(x)g(ma+b-x)dx\\ &=\int_0^1 f(tma+(1-t)b)g((1-t)ma+tb)dt\\ &\leqslant \int_0^1 \left\{\alpha[f(tma+(1-t)b)]^{\frac{1}{\alpha}}+\beta[g((1-t)ma+tb]^{\frac{1}{\beta}}\right\}dt\\ &\leqslant \int_0^1 \Big\{\alpha\{[mf(a)]^t[mf(a)+\eta(f(b),mf(a))]^{1-t}\}^{\frac{1}{\alpha}}\\ &\quad+\beta\{[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^t\}^{\frac{1}{\beta}}\Big\}dt\\ &=\alpha[mf(a)+\eta(f(b),mf(a))]^{\frac{1}{\alpha}}\int_0^1\left[\frac{mf(a)}{mf(a)+\eta(f(b),mf(a))}\right]^{\frac{t}{\alpha}}dt\\ &\quad+\beta[mg(a)]^{\frac{1}{\beta}}\int_0^1\left[\frac{mg(a)+\eta(g(b),mg(a))}{mg(a)}\right]^{\frac{t}{\beta}}dt\end{aligned}$$

$$
\begin{aligned}
&= \alpha^2[mf(a)]^{\frac{1}{\alpha}}\left[\frac{\left(\frac{mf(a)+\eta(f(b),mf(a))}{mf(a)}\right)^u}{\log\frac{mf(a)+\eta(f(b),mf(a))}{mf(a)}}\right]_0^{\frac{1}{\alpha}}\\
&\quad + \beta^2[mg(a)+\eta(g(b),mg(a))]^{\frac{1}{\beta}}\left[\frac{\left(\frac{mg(a)}{mg(a)+\eta(g(b),mg(a))}\right)^u}{\log\frac{mg(a)}{mg(a)+\eta(g(b),mg(a))}}\right]_0^{\frac{1}{\beta}}\\
&= \alpha^2\left[\frac{[mf(a)+\eta(f(b),mf(a))]^{\frac{1}{\alpha}}-[mf(a)]^{\frac{1}{\alpha}}}{[\log[mf(a)+\eta(f(b),mf(a))]-\log mf(a)]}\right]\\
&\quad + \beta^2\left[\frac{[mg(a)]^{\frac{1}{\beta}}-[mg(a)+\eta(g(b),mg(a))]^{\frac{1}{\beta}}}{\log[mg(a)]-\log[mg(a)+\eta(g(b),mg(a))]}\right]\\
&= \alpha^2\left[\frac{[mf(a)+\eta(f(b),mf(a))]^{\frac{1}{\alpha}}-[mf(a)]^{\frac{1}{\alpha}}}{[\eta(f(b),mf(a))]}\right]L[mf(a),mf(a)+\eta(f(b),mf(a))]\\
&\quad + \beta^2\left[\frac{[mg(a)]^{\frac{1}{\beta}}-[mg(a)+\eta(g(b),mg(a))]^{\frac{1}{\beta}}}{[\eta(g(b),mg(a))]}\right]L[mg(a)+\eta(g(b),mg(a)),mg(a)].
\end{aligned}
$$

This completes the proof. □

Corollary 3.8. *If* $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$ *and* $\eta(f(b),mf(a))=f(b)-mf(a)$, *then under the assumption of Theorem 3.7, we have*

$$
\frac{1}{b-ma}\int_{ma}^{b} f(x)g(ma+b-x)dx \leqslant \frac{1}{2}\big\{A[f(b),mf(a)]L[mf(a),f(b)]+A[g(b),mg(a)]L[mg(a),g(b)]\big\}.
$$

Corollary 3.9 ([8]). *If* $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$, $m=1$ *and* $\eta(f(b),mf(a))=f(b)-mf(a)$, *then under the assumption of Theorem 3.7, we have*

$$
\frac{1}{b-a}\int_{a}^{b} f(x)g(a+b-x)dx \leqslant \frac{1}{2}\big\{A[f(b),f(a)]L[f(a),f(b)]+A[g(b),g(a)]L[g(a),g(b)]\big\}.
$$

Corollary 3.10. *If* $\alpha=\frac{1}{4}$, $\beta=\frac{3}{4}$ *and* $\eta(f(b),mf(a))=f(b)-mf(a)$, *then under the assumption of Theorem 3.7, we have*

$$
\begin{aligned}
\frac{1}{b-ma}\int_{ma}^{b} f(x)g(ma+b-x)dx &\leqslant \frac{1}{16}\left[\frac{f^4(b)-mf^4(a)}{f(b)-mf(a)}\right]L[mf(a),f(b)]\\
&\quad+\frac{9}{16}\left[\frac{mg^{\frac{4}{3}}(a)-g^{\frac{4}{3}}(b)}{mg(a)-g(b)}\right]L[mg(a),g(b)].
\end{aligned}
$$

Theorem 3.11. *Let* $f,g: I=[ma,b]\to(0,\infty)$ *be an increasing and generalized* log m*-convex functions on* I *with* $a<b$. *Then*

$$
\begin{aligned}
&8L[mg(a),mg(a)+\eta(g(b),mg(a))]\left\{f\left(\frac{ma+b}{2}\right)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x),f(ma+b-x))dx\right\}\\
&\leqslant \frac{1}{b-ma}\int_{ma}^{b} f^4(x)dx + K^2[mg(a),mg(a)+\eta(g(b),mg(a))]\\
&\quad\times A[mg(a),mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))]+8.
\end{aligned}
$$

Proof. Let f and g be generalized log m-convex functions on I. Then for all $a,b\in I, t\in[0,1]$, we have

$$
\begin{aligned}
f(tma+(1-t)b) &\leqslant [mf(a)]^t[mf(a)+\eta(f(b),mf(a))]^{1-t},\\
g((1-t)ma+tb) &\leqslant [mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^t.
\end{aligned}
$$

Using the inequality,

$$8xy \leqslant x^4 + y^4 + 8. \quad \forall x, y \in \mathbb{R},$$

we have

$$\begin{aligned}&8f(tma+(1-t)b)[mg(a)]^{1-t}t[mg(a)+\eta(g(b),mg(a))]^{t}\\&\quad\leqslant f^4(tma+(1-t)b)+[mg(a)]^{4(1-t)}[mg(a)+\eta(g(b),mg(a))]^{t}+8.\end{aligned}$$

Integrating the above inequality with respect to t on [0, 1], we have

$$\begin{aligned}&8\int_0^1 f(tma+(1-t)b)[mg(a)]^{t}[mg(a)+\eta(g(b),mg(a))]^{1-t}dt\\&\leqslant \int_0^1 f^4(tma+(1-t)b)dt+\int_0^1 [mg(a)]^{4t}[mg(a)+\eta(g(b),mg(a))]^{4-4t}dt+8.\end{aligned}$$

As f and g are increasing functions, then we have

$$\begin{aligned}&8\int_0^1 f(tma+(1-t)b)dt\int_0^1 [mg(a)]^{t}[mg(a)+\eta(g(b),mg(a))]^{1-t}dt\\&\leqslant \int_0^1 f^4(tma+(1-t)b)dt+\int_0^1 [mg(a)]^{4t}[mg(a)+\eta(g(b),mg(a))]^{4-4t}dt+8.\end{aligned}$$

From the above inequality, we observe that

$$\begin{aligned}&\frac{8L[mg(a),mg(a)+\eta(g(b),mg(a))]}{b-ma}\int_{ma}^{b} f(x)dx\\&\leqslant \frac{1}{b-ma}\int_{ma}^{b} f^4(x)dx+K^2[mg(a),mg(a)+\eta(g(b),mg(a))]\\&\quad\times A[mg(a),mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))]+8.\end{aligned} \tag{3.5}$$

Now using the L.H.S of Hermite-Hadamard's inequality in (3.5), we have

$$\begin{aligned}&8L[mg(a),mg(a)+\eta(g(b),mg(a))]\left\{f\left(\frac{ma+b}{2}\right)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x),f(ma+b-x))dx\right\}\\&\leqslant \frac{1}{b-ma}\int_{ma}^{b} f^4(x)dx+K^2[mg(a),mg(a)+\eta(g(b),mg(a))]\\&\quad\times A[mg(a),mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))]+8,\end{aligned}$$

where A, L, and K are Arithmetic, Logarithmic, and Quadratic means, respectively. □

Corollary 3.12. *If* $\eta(f(b), mf(a)) = f(b) - mf(a)$, *then under the assumption of Theorem* 3.11, *we have*

$$8L[mg(a),g(b)]f\left(\frac{ma+b}{2}\right)\leqslant \frac{1}{b-ma}\int_{ma}^{b} f^4(x)dx+K^2[mg(a),g(b)]A[mg(a),g(b)]L[mg(a),g(b)]+8.$$

Corollary 3.13. *If* $f = g$ *and* $m = 1$, *then under the assumption of Theorem* 3.11, *we obtain the result given in* [25].

Theorem 3.14. *Let* $f, g : I = [ma, b] \to (0, \infty)$ *be an incresing and generalized* log m*-convex function on* I *with* $a < b$. *Then*

$$\left\{f\left(\frac{ma+b}{2}\right)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x),f(ma+b-x))dx\right\}L[mg(a),mg(a)+\eta(g(b),mg(a))]$$

$$
\begin{aligned}
&+\left\{g\Big(\frac{ma+b}{2}\Big)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mg(x),g(ma+b-x))dx\right\}L[mf(a),mf(a)+\eta(f(b),mf(a))]\\
&\leqslant \frac{1}{b-ma}\int_{ma}^{b}f(x)g(x)dx+L\Big[f(b)mg(a),[f(b)+\eta(mf(a),f(b))][mg(a)+\eta(g(b),mg(a))]\Big].
\end{aligned}
$$

Proof. Let f and g be generalized log m-convex functions on I. Then for all $a,b\in I, t\in[0,1]$, we have

$$
\begin{aligned}
f(tma+(1-t)b)&\leqslant [mf(a)]^{t}[f(b)+\eta(mf(a),f(b))]^{1-t},\\
g((1-t)ma+tb)&\leqslant [mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}.
\end{aligned}
$$

Using the inequality,

$$
(a-b)(c-d)\geqslant 0,\quad \forall a,b,c,d\in\mathbb{R}, a<b, c<d,
$$

we have

$$
\begin{aligned}
&f(tma+(1-t)b)\big[[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}\big]\\
&\qquad+[g((1-t)ma+tb)]\big[[f(b)]^{1-t}[f(b)+\eta(mf(a),f(b))]^{t}\big]\\
&\quad\leqslant f(tma+(1-t)b)g((1-t)ma+tb)\\
&\qquad+\big[[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}[f(b)]^{1-t}[f(b)+\eta(mf(a),f(b))]^{t}\big].
\end{aligned}
$$

Now integrating the above inequality with respect to t on $[0,1]$, we have

$$
\begin{aligned}
&\int_0^1[f(tma+(1-t)b)]\big[[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}\big]dt\\
&\qquad+\int_0^1[g((1-t)ma+tb)]\big[[f(b)]^{1-t}[f(b)+\eta(mf(a),f(b))]^{t}\big]dt\\
&\quad\leqslant\int_0^1[f(tma+(1-t)b)]g((1-t)ma+tb)dt\\
&\qquad+\int_0^1+\big[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}[f(b)]^{1-t}[f(b)+\eta(mf(a),f(b))]^{t}\big]dt.
\end{aligned}
$$

As f and g are increasing functions, we have

$$
\begin{aligned}
&\int_0^1[f(tma+(1-t)b)]dt\int_0^1\big[[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}\big]dt\\
&\qquad+\int_0^1[g((1-t)ma+tb)]dt\int_0^1\big[[f(b)]^{1-t}[f(b)+\eta(mf(a),f(b))]^{t}\big]dt\\
&\quad\leqslant\int_0^1[f(tma+(1-t)b)]dt\int_0^1[g((1-t)ma+tb)]dt\\
&\qquad+\int_0^1[f(b)mg(a)]^{1-t}\Big[[f(b)+\eta(mf(a),f(b))][mg(a)+\eta(g(b),mg(a))]\Big]^{t}dt.
\end{aligned}
$$

Now after some integration, we have

$$
\begin{aligned}
&\left\{f\Big(\frac{ma+b}{2}\Big)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x),f(ma+b-x))dx\right\}L[mg(a),mg(a)+\eta(g(b),mg(a))]\\
&\quad+\left\{g\Big(\frac{ma+b}{2}\Big)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mg(x),g(ma+b-x))dx\right\}L[mf(a),mf(a)+\eta(f(b),mf(a))]
\end{aligned}
$$

$$\leqslant \frac{1}{b-ma}\int_{ma}^{b} f(x)g(x)dx + L\Big[f(b)mg(a), [f(b)+\eta(mf(a),f(b))][mg(a)+\eta(g(b),mg(a))]\Big].$$

This completes the proof. □

Corollary 3.15. *If* $\eta(f(b), mf(a)) = f(b) - mf(a)$, *then under the assumption of Theorem 3.14, we have*

$$\begin{aligned}&\left\{f\Big(\frac{ma+b}{2}\Big)\right\}L[mg(a),g(b)] + \left\{g\Big(\frac{ma+b}{2}\Big)\right\}L[mf(a),f(b)]\\&\leqslant \frac{1}{b-ma}\int_{ma}^{b} f(x)g(x)dx + L\left[f(b)mg(a), mf(a)g(b)\right].\end{aligned}$$

Corollary 3.16. *If* $\eta(f(b), mf(a)) = f(b) - mf(a)$ *and* $m = 1$, *then under the assumption of Theorem 3.14, we have*

$$f(\frac{a+b}{2})L[g(a),g(b)] + g(\frac{a+b}{2})L[f(a),f(b)] \leqslant \frac{1}{b-a}\int_{a}^{b} f(x)g(x)dx + L\big[f(b)g(a),(f(a)g(b)\big].$$

Theorem 3.17. *Let* $f, g : I = [ma, b] \to (0,\infty)$ *be an increasing and generalized* log m*-convex functions on* I *with* $a < b$. *Then*

$$\begin{aligned}&\frac{1}{b-ma}\int_{ma}^{b} f^2(x)dx + A[f(b), f(b)+\eta(mf(a),f(b))]L[f(b),f(b)+\eta(mf(a),f(b))]\\&\quad + A[mg(a), mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))]\\&\quad\geqslant \left\{f\Big(\frac{ma+b}{2}\Big) - \frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x), f(ma+b-x))dx\right\}L[mf(a)+\eta(f(b),mf(a)),mf(a)]\\&\quad + L[mg(a)(mf(a)+\eta(f(b),mf(a))), mf(a)(mg(a)+\eta(g(b),mg(a)))]\\&\quad + \left\{f\Big(\frac{ma+b}{2}\Big) - \frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x), f(ma+b-x))dx\right\}L[mg(a), mg(a)\\&\quad +\eta(g(b),mg(a))].\end{aligned}$$

Proof. Let f and g be generalized log m-convex functions on I. Then for all $a, b \in I, t \in [0,1]$, we have

$$\begin{aligned}f(tma+(1-t)b) &\leqslant [mf(a)]^t[mf(a)+\eta(f(b),mf(a))]^{1-t},\\ g((1-t)ma+tb) &\leqslant [mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^t.\end{aligned}$$

Using the inequality

$$x^2+y^2+z^2 \geqslant xy+yz+zx, \quad \forall x,y,z \in \mathbb{R},$$

we have

$$\begin{aligned}&f^2(tma+(1-t)b) + [mf(a)]^{2t}[mf(a)+\eta(f(b),mf(a))]^{2(1-t)t}\\&\quad + [mg(a)]^{2(1-t)}[mg(a)+\eta(g(b),mg(a))]^{2t}\\&\quad\geqslant \Big[f(tma+(1-t)b)\big([mf(a)]^t[mf(a)+\eta(f(b),mf(a))]^{1-t}\big)\Big]\\&\quad + \Big[[mf(a)]^t[mf(a)+\eta(f(b),mf(a))]^{1-t}[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^t\Big]\\&\quad + \Big[[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^t f(tma+(1-t)b)\Big].\end{aligned}$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f^2(tma+(1-t)b)dt+\int_0^1 \big[[mf(a)]^{2t}[mf(a)+\eta(f(b),mf(a))]^{2(1-t)}\big]dt\\
&\quad+\int_0^1 \big[[mg(a)]^{2(1-t)}[mg(a)+\eta(g(b),mg(a))]^{2t}\big]dt\\
&\quad\geqslant \int_0^1 \big[f(tma+(1-t)b)[mf(a)]^{t}[mf(a)+\eta(f(b),mf(a))]^{t}\big]dt \qquad (3.6)\\
&\quad+\int_0^1 \big[[mf(a)]^{t}[mf(a)+\eta(f(b),mf(a))]^{1-t}[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}\big]dt\\
&\quad+\int_0^1 \big[[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}f(tma+(1-t)b)\big]dt.
\end{aligned}$$

To solve the integral in (3.6), let

$$\begin{aligned}
\mathbb{A}&=\int_0^1 f^2(tma+(1-t)b)dt+\int_0^1 \big[[mf(a)]^{2t}[mf(a)+\eta(f(b),mf(a))]^{2(1-t)}\big]dt\\
&\quad+\int_0^1 [[mg(a)]^{2(1-t)}[mg(a)+\eta(g(b),mg(a))]^{t}]dt\\
&=\frac{1}{b-ma}\int_{ma}^{b} f^2(x)dx+A[mf(a)+\eta(f(b),mf(a)),mf(a)]\\
&\quad\times L[mf(a)+\eta(f(b),mf(a)),mf(a)]+A[mg(a),mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{B}&=\int_0^1 \big[f(tma+(1-t)b)[mf(a)]^{t}[mf(a)+\eta(f(b),mf(a))]^{1-t}\big]dt\\
&\quad+\int_0^1 \big[[mf(a)]^{t}[mf(a)+\eta(f(b),mf(a))]^{1-t}[mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}\big]dt\\
&\quad+\int_0^1 [mg(a)]^{1-t}[mg(a)+\eta(g(b),mg(a))]^{t}f(tma+(1-t)b)dt\\
&\geqslant f\Big(\frac{ma+b}{2}\Big)L[mf(a)+\eta(f(b),mf(a)),mf(a)]\\
&\quad+L[mg(a)(mf(a)+\eta(f(b),mf(a)),mf(a)(mg(a)+\eta(g(b),mg(a))]\\
&\quad+f\Big(\frac{ma+b}{2}\Big)L[mg(a),mg(a)+\eta(g(b),mg(a))].
\end{aligned}$$

Substituting the values of $\mathbb{A}$ and $\mathbb{B}$ in (3.6), we have

$$\begin{aligned}
&\frac{1}{b-ma}\int_{ma}^{b} f^2(x)dx+A[f(b),f(b)+\eta(mf(a),f(b))]L[f(b),f(b)+\eta(mf(a),f(b))]\\
&\quad+A[mg(a),mg(a)+\eta(g(b),mg(a))]L[mg(a),mg(a)+\eta(g(b),mg(a))]\\
&\quad\geqslant\Big\{f\Big(\frac{ma+b}{2}\Big)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x),f(ma+b-x))dx\Big\}L[mf(a)+\eta(f(b),mf(a)),mf(a)]\\
&\quad+L[mg(a)(mf(a)+\eta(f(b),mf(a))),mf(a)(mg(a)+\eta(g(b),mg(a)))]\\
&\quad+\Big\{f\Big(\frac{ma+b}{2}\Big)-\frac{1}{2(b-ma)}\int_{ma}^{b}\eta(mf(x),f(ma+b-x))dx\Big\}L[mg(a),mg(a)+\eta(g(b),mg(a))].
\end{aligned}$$

This completes the proof. □

Corollary 3.18. *If* $\eta(f(b),mf(a))=f(b)-mf(a)$ *and* $m=1$, *then under the assumption of Theorem 3.17, we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f^2(x)dx + A[f(b),f(a)]L[f(b),f(a)] + A[g(a),g(b)]L[g(a),g(b)]\\&\geqslant f(\frac{a+b}{2})L[f(b),f(a)] + L[f(b)g(a),f(a)g(b)] + f(\frac{a+b}{2})L[g(a),g(b)].\end{aligned}$$

Conclusion

In this paper, we have introduced a new class of convex functions relative to a constant $m \in (0,1]$, which is known as log m-convex function. New integral inequalities are obtained via these nonconvex functions. Some special cases are also discussed which have been obtained from our results. The technique of this paper may motivate new research.

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