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Fixed points of weakly compatible mappings satisfying a generalized common limit range property

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Abstract

In this paper, we produce new fixed point theorems for $2n$ self-mappings $\wp_1^a, \wp_2^a, \ldots, \wp_n^a$, $\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b : \mathcal{X} \to \mathcal{X}$ on a metric space $(\mathcal{X}, \rho)$, satisfying a generalized common limit range (CLR) property or $\mathrm{CLR}_{\wp_k^a \gamma_l^b}$ for $k, l = 2, \ldots, n$. Along with the newly introduced property $\mathrm{CLR}_{\wp_k^a \gamma_l^b}$ for $k, l = 2, \ldots, n$ for the $2n$ self-mappings, we also assume that the pairs $(\wp_1^a, \gamma_1^b), (\wp_2^a, \gamma_2^b), \ldots, (\wp_n^a, \gamma_n^b)$ are weakly compatible. From the main result, we produce three more corollaries as its special cases. These results generalize the work of Sarwar et al. [M. Sarwar, M. Bahadur Zada, İ. M. Erhan, Fixed Point Theory Appl., **2015** (2015), 15 pages] and many others in the available literature. Two examples are also presented for the applications of our new FPTs.

Keywords: Weakly compatible mappings, common limit range property, fixed point theorems.
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1. Introduction and preliminaries

Banach's FPT has been utilized in a large number of problems by scientists in different fields like; image processing, selection and matching problems, equilibrium problems, the study of existence and uniqueness of solutions (EUS) for the integral and differential equations and many others. In literature, Banach's FPT has been generalized in different directions for the new FPTs and a lot of applications of the new FPTs were presented [1, 2, 5, 7, 11, 18, 20, 21]. These generalizations were carried out either by the help of the spaces or by the contractions. For example, Bhaskar and Lakshmikantham [11] initiated the concept of coupled FPT which was then followed for the triple and quadruple FPTs. Berinde and

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Borcut [10] worked on the triple FPTs and their applications in partially ordered metric spaces (POMS). Liu [20] studied FPTs for self-quadruple mappings in POMS with the supposition of mixed g-monotone property and illustrated the applications of their results. Aydi et al. [5] discussed FPTs for self-quadruple depending on another function in POMS and some applications were illustrated. Bota et al. [12] studied coupled FPTs and their applications to the EUS of a coupled system of integral equations on the finite interval $[0, \mathcal{T}]$. Mustafa et al. [21] proved FPTs on POMS and generalized some FPTs for the generalized (ϕ, ψ)-contractions in POMS and have worked on the applications of their FPTs. Jleli and Samet [17] provided the idea of generalized metric spaces and have extended some results including Banach's FPT. Shatanawi et al. [24] proved coupled FPTs in POMS for two altering distance functions.

Nadler [22] worked on the FPT for multivalued contraction mappings. Branciari [13] generalized the Banach FPT for a single-valued mapping by the help of integral type of contractions. Stojakovic et al. [25] generalized the concept of the Banach's FPT by the help of integral type contractions by following the work due to Nadler [22]. Sarwar et al. [23] produced an FPT by the help of integral type contractions and provided some applications of their results in dynamic programing. For the applications of the FPTs in fractional differential equations, we refer the readers to [6, 8, 15] and some other related results can be studied in [16, 19].

Inspired from the work [3–5, 14, 19, 19, 22, 23, 25], in this paper, we give the notion of an extended CLR property or $\text{CLR}_{\wp_k^a \gamma_l^b}$ for $k, l = 2, \ldots, m$ for the $2m$ self-mappings $\wp_1^a, \wp_2^a, \ldots, \wp_m^a$, $\gamma_1^b, \gamma_2^b, \ldots, \gamma_m^b$: $\mathcal{X} \to \mathcal{X}$. This new idea of generalization of the CLR property will help us to handle $2m$ self mappings for unique CFPs. Along with the newly introduced property $\text{CLR}_{\wp_k^a \gamma_l^b}$ for $k, l = 2, \ldots, m$ for $2m$ self-mappings, we also assume that the pairs $(\wp_1^a, \wp_2^a, \ldots, \wp_m^a)$ and $(\gamma_1^b, \gamma_2^b, \ldots, \gamma_m^b)$ satisfy the property of weakly compatibility and produce a new FPT as a main result of the paper. Several results are produced from the main result as special cases. These results generalize the work in [23], and many others in the available literature.

Theorem 1.1 ([9]). *If $(\mathcal{X}, d)$ is a complete metric space and $f : \mathcal{X} \to \mathcal{X}$ satisfies that $d(f(x), f(y)) \leqslant \nu\rho(x, y)$, for all $x, y \in \mathcal{X}$ and $\nu \in (0, 1)$, then f has a fixed point in $\mathcal{X}$.*

Definition 1.2. Let $(\mathcal{X}, \rho)$ be a metric space and $\wp_1, \wp_2, \gamma_1, \gamma_2 : \mathcal{X} \to \mathcal{X}$ be quadruple self-mappings. The pairs $(\wp_1^a, \wp_2^a)$ and (γ_1^b, γ_2^b) satisfy the CLR property with respect to mappings $\wp_2^a$ and γ_2^b, denoted by $\text{CLR}_{\wp_2^a \gamma_2^b}$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that

$$\lim_{n\to\infty} \wp_1^a x_n = \lim_{n\to\infty} \wp_2^a x_n = \lim_{n\to\infty} \gamma_1^b y_n = \lim_{n\to\infty} \gamma_2^b y_n = \nu \in \wp_2^a(\mathcal{X}) \cap \gamma_2^b(\mathcal{X})$$

for $x_n, y_n \in \mathcal{X}$ and for all $n \in \mathcal{N} \cup \{0\}$.

Definition 1.3 ([23]). A coincidence point of a pair of self-mapping $\wp_1^a$, $\wp_2^a$,: $\mathcal{X} \to \mathcal{X}$ is a point $z \in \mathcal{X}$ for some $\wp_1^a z = \wp_2^a z$.

A CFP of pair of self-mappings $\wp_1^a$, $\wp_2^a : \mathcal{X} \to \mathcal{X}$ is a point $z \in \mathcal{X}$ for which $\wp_1^a z = \wp_2^a z = z$.

Definition 1.4 ([23]). A pair of self-mappings $\wp_1^a$, $\wp_2^a : \mathcal{X} \to \mathcal{X}$ is weakly compatible if they commute, at their coincidence point that is if there exists a point $z \in \mathcal{X}$ such that $\wp_1^a \wp_2^a z = \wp_2^a \wp_1^a z$.

Definition 1.5 ([14]). Generalized altering distance function is a mapping $\tau : \mathcal{R}^+ \to \mathcal{R}^+$, satisfying that:

(i) τ is a non-decreasing;
(ii) $\tau(t) = 0$ if and only if $t = 0$.

$F = \{\tau : \mathcal{R}^+ \to \mathcal{R}^+ :\ \tau$ satisfying (i) and (ii)$\}$.

$\Phi = \{\phi : \mathcal{R}^+ \to \mathcal{R}^+ : \phi$ is right upper semi-continuous, non-decreasing, and for all $x > 0$, we have $\tau(x) > \phi_1(x)$ and $\tau(x)$ satisfies (i) and (ii)$\}$.

$\Psi_1 = \{\psi_1 : \mathcal{R}_+^6 \to \mathcal{R}_+ :\ \psi_1$ satisfies (A_1)-$(A_3)\}$, where

(A_1) ψ_1 is continuous and non-decreasing in each coordinate;
(A_2) $\psi_1(x,x,x,x,x,x) \leqslant x$ for all $x \geqslant 0$;
(A_3) $\psi_1 = 0$ if and only if all the components of ψ_1 are zero;

$\Psi_2 = \{\psi_2 : \mathcal{R}^{+^4} \to \mathcal{R}^+ :\ \psi_2$ is continuous and $\psi_2 = 0$ if any component of ψ_2 is zero$\}$.

2. Main results

Definition 2.1. Let $\wp_1^a, \wp_2^a, \ldots, \wp_n^a$ and $\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b$ be $2n$ be self-mappings. We say that the pairs $(\wp_1^a, \wp_2^a, \ldots, \wp_n^a)$ and $(\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b)$ satisfy the CLR property with respect to mappings $\wp_2^a, \ldots, \wp_n^a, \gamma_2^b, \ldots, \gamma_n^b$ denoted by $\mathrm{CLR}_{\wp_k^a \gamma_l^b}$ for $l, k = 2, \ldots, n$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that

$$\begin{aligned}\lim_{n\to\infty} \wp_1^a x_n = \lim_{n\to\infty} \wp_2^a x_n = \cdots = \lim_{n\to\infty} \wp_n^a x_n = \lim_{n\to\infty} \gamma_1^b y_n = \lim_{n\to\infty} \gamma_2^b y_n = \cdots = \lim_{n\to\infty} \gamma_n^b y_n \\ = \nu \in \left(\cap_{k=2}^n \wp_k^a(\mathcal{X})\right) \cap \left(\cap_{l=2}^n \gamma_l^b(\mathcal{X})\right)\end{aligned}$$

for $x_n, y_n \in \mathcal{X}$ for $n \in \mathbb{N}$.

Theorem 2.2. *Let $\wp_1^a, \wp_2^a, \ldots, \wp_n^a$ and $\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b$ be $2n$ self-mappings on a metric space $(\mathcal{X}, \rho)$ and satisfying the following conditions:*

(a) *the pairs $\left(\wp_1^a, \wp_2^a, \ldots, \wp_n^a\right)$ and $\left(\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b\right)$ share the property $\mathrm{CLR}_{\wp_k^a \gamma_l^b}$ for $l, k = 2, \ldots, n$;*
(b) *the pairs $(\wp_1^a, \gamma_1^b)$, $(\wp_2^a, \gamma_2^b), \ldots, (\wp_n^a, \gamma_n^b)$ are weakly compatible;*
(c) *the following inequality is satisfied:*

$$\tau\Big(\int_0^{\rho(\wp_1^a x, \gamma_1^b y)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))} \Gamma(t)dt\Big) + \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(x,y))} \Gamma(t)dt\Big), \tag{2.1}$$

where, $\mathcal{L} \geqslant 0$, $\Gamma(t)$ is Lebesgue integrable function such that $\int_0^\delta \Gamma(t)dt > 0$ for any $\delta > 0$ and

$$\begin{aligned}\psi_1(\mathcal{M}(x,y)) = \max\{&\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_n^b y), \rho(\gamma_1^b y, \gamma_n^b y), \rho(\gamma_2^b y, \gamma_n^b y), \\ &\frac{\rho(\wp_1^a x, \gamma_n^b y)}{1 + \sum_{l=2}^n \rho(\wp_1^a x, \gamma_l^b y)}, \frac{\rho(\gamma_1^b x, \gamma_2^b y)}{1 + \sum_{l=2}^n \rho(\wp_1^a x, \gamma_l^b y)}\}, \\ \psi_2(\mathcal{N}(x,y)) = \psi_2\{&\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_n^b y), \rho(\gamma_1^b y, \gamma_n^b y), \frac{\rho(\wp_1^a x, \gamma_n^b y)}{1 + \sum_{l=2}^n \rho(\wp_1^a x, \gamma_l^b y)}\},\end{aligned}$$

then the $2n$ self-mappings $\wp_1^a, \wp_2^a, \ldots, \wp_n^a$ and $\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b$ have a unique CFP in $\left(\mathcal{X}, \rho\right)$.

Proof. By the help of our assumption of the property $\mathrm{CLR}_{\wp_k^a \gamma_l^b}$ of the pairs $(\wp_1^a, \wp_2^a, \ldots, \wp_n^a)$ and $(\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b)$, we may have two sequences $\{x_n\}$ and $\{y_n\}$ in the metric space $(\mathcal{X}, \rho)$, such that

$$\begin{aligned}\lim_{n\to\infty} \wp_1^a x_n = \lim_{n\to\infty} \wp_2^a x_n = \cdots = \lim_{n\to\infty} \wp_n^a x_n = \lim_{n\to\infty} \gamma_1^b y_n = \lim_{n\to\infty} \gamma_2^b y_n = \cdots = \lim_{n\to\infty} \gamma_n^b y_n \\ = \nu \in \left(\cap_{k=2}^n \wp_k^a(\mathcal{X})\right) \cap \left(\cap_{l=2}^n \gamma_l^b(\mathcal{X})\right).\end{aligned} \tag{2.2}$$

From $\nu \in \left(\cap_{k=2}^n \wp_k^a(\mathcal{X})\right) \cap \left(\cap_{l=2}^n \gamma_l^b(\mathcal{X})\right)$, we have two cases; $\nu \in \cap_{k=2}^n \wp_k^a(\mathcal{X})$ and $\nu \in \cap_{l=2}^n \gamma_l^b(\mathcal{X})$.

As $\nu \in \cap_{k=2}^n \wp_k^a(\mathcal{X})$, this implies that $\nu \in \wp_2^a(\mathcal{X})$, which further implies $\nu = \wp_2^a(z)$, for some $z \in \mathcal{X}$, such that

$$\lim_{n\to\infty} \wp_1^a x_n = \lim_{n\to\infty} \wp_2^a x_n = \cdots = \lim_{n\to\infty} \wp_n^a x_n = \lim_{n\to\infty} \gamma_1^b y_n = \lim_{n\to\infty} \gamma_2^b y_n = \cdots = \lim_{n\to\infty} \gamma_n^b y_n = \nu = \wp_2^a(z).$$

Now, we show that $\wp_1^a(z) = \wp_2^a(z)$. For this, we assume the contrary path, i.e., $\wp_1^a(z) \neq \wp_2^a(z)$. By the help of (2.1), we have

$$\tau\Big(\int_0^{\rho(\wp_1^a z, \gamma_1^b y_n)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(z,y_n))} \Gamma(t)dt\Big) + \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(z,y_n))} \Gamma(t)dt\Big) \tag{2.3}$$

for $x = z$ and $y = y_n$ in (2.1), where

$$\begin{aligned}\psi_1(\mathcal{M}(z, y_n)) = \max\{&\rho(\wp_1^a z, \gamma_2^b y_n), \rho(\wp_1^a z, \gamma_n^b y_n), \rho(\gamma_1^b y_n, \gamma_n^b y_n), \rho(\gamma_2^b y_n, \gamma_n^b y_n),\\ &\frac{\rho(\wp_1^a z, \gamma_n^b y_n)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b y_n)}, \frac{\rho(\gamma_1^b x, \gamma_2^b y_n)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b y_n)}\},\end{aligned} \tag{2.4}$$

$$\psi_2(\mathcal{N}(z, y_n)) = \psi_2\{\rho(\wp_1^a z, \gamma_2^b y_n), \rho(\wp_1^a z, \gamma_n^b y_n), \rho(\gamma_1^b y_n, \gamma_n^b y_n), \frac{\rho(\wp_1^a z, \gamma_n^b y_n)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b y_n)}\}. \tag{2.5}$$

Taking $\lim_{n\to\infty}$ in (2.3), (2.4), and (2.5), respectively, we get

$$\begin{aligned}\lim_{n\to\infty} \psi_1(\mathcal{M}(z, y_n)) &= \lim_{n\to\infty} \max\{\rho(\wp_1^a z, \gamma_2^b y_n), \rho(\wp_1^a z, \gamma_n^b y_n), \rho(\gamma_1^b y_n, \gamma_n^b y_n), \rho(\gamma_2^b y_n, \gamma_n^b y_n),\\ &\qquad \frac{\rho(\wp_1^a z, \gamma_n^b y_n)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b y_n)}, \frac{\rho(\gamma_1^b y_n, \gamma_2^b y_n)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b y_n)}\}\\ &= \max\{\rho(\wp_1^a z, \nu), \rho(\wp_1^a z, \nu), \rho(\nu, \nu), \rho(\nu, \nu),\\ &\qquad \frac{\rho(\wp_1^a z, \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \nu)}, \frac{\rho(\wp_1^a z, \nu)}{1 + \sum_{l=2}^n \rho(\nu, \nu)}\}\\ &= \max\{\rho(\wp_1^a z, \nu), \rho(\wp_1^a z, \nu), 0, 0, \frac{\rho(\wp_1^a z, \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \nu)}, 0\}\\ &= \rho(\wp_1^a z, \nu),\\ \psi_2(\mathcal{N}(z, y_n)) &= \psi_2\{\rho(\wp_1^a z, \gamma_2^b y_n), \rho(\wp_1^a z, \gamma_n^b y_n), \rho(\gamma_1^b y_n, \gamma_n^b y_n), \frac{\rho(\wp_1^a z, \gamma_n^b y_n)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b y_n)}\}\\ &= \psi_2\{\rho(\wp_1^a z, \nu), \rho(\wp_1^a z, \nu), \rho(\nu, \nu), \frac{\rho(\wp_1^a z, \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \nu)}\}\\ &= \psi_2\{\rho(\wp_1^a z, \nu), \rho(\wp_1^a z, \nu), 0, \frac{\rho(\wp_1^a z, \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \nu)}\} = 0,\end{aligned}$$

and

$$\lim_{n\to\infty} \tau\Big(\int_0^{\rho(\wp_1^a z, \gamma_1^b y_n)} \Gamma(t)dt\Big) \leqslant \lim_{n\to\infty} \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(z,y_n))} \Gamma(t)dt\Big) + \lim_{n\to\infty} \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(z,y_n))} \Gamma(t)dt\Big).$$

Consequently, we have

$$\tau\Big(\int_0^{\rho(\wp_1^a z, \nu)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\rho(\wp_1^a z, \nu)} \Gamma(t)dt\Big),$$

which is a contradiction and therefore, we have $\wp_1^a z = \nu$. Thus, $\wp_1^a(z) = \wp_2^a(z)$. Next, for $\nu \in \wp_k^a(\mathcal{X})$, we have $z_k \in \mathcal{X}$ for all $k = 2, \ldots, n$, such that $\wp_k^a(z_k) = \nu$. By following the same lines as above, we get

$$\wp_1^a(z_1) = \wp_2^a(z_2) = \cdots = \wp_k^a(z_k) = \cdots = \wp_n^a(z_n) = \nu. \tag{2.6}$$

Next, from (2.2), we also have that $\nu \in \cap_{l=2}^n \gamma_l^b(\mathcal{X})$. Following the same lines as above, we have that there exist $z_2, z_3, \ldots, z_n \in \mathcal{X}$ such that

$$\gamma_1^b(z_1) = \gamma_2^b(z_2) = \cdots = \gamma_k^b(z_k) = \cdots = \gamma_n^b(z_n) = \nu. \tag{2.7}$$

By the help of weakly compatibility of the pairs $(\wp_1^a, \wp_2^a, \ldots, \wp_n^a)$ and $(\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b)$ and (2.6), (2.7), we have

$$\gamma_1^b \nu = \gamma_2^b \nu = \cdots = \gamma_n^b \nu, \quad \wp_1^a \nu = \wp_2^a \nu = \cdots = \wp_n^a \nu.$$

Now, we show that ν is a CFP of the $2n$ self-mappings $\wp_1^a, \wp_2^a, \ldots, \wp_n^a$, $\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b$, for this we assume that $\gamma_1^b \nu \neq \nu$, and putting $x = z$ and $y = \nu$ in (2.1), we have

$$\tau\Big(\int_0^{\rho(\wp_1^a z, \gamma_1^b \nu)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(z,\nu))} \Gamma(t)dt\Big) + \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(z,\nu))} \Gamma(t)dt\Big), \tag{2.8}$$

where

$$\begin{aligned} \psi_1(\mathcal{M}(z,\nu)) &= \max\{\rho(\wp_1^a z, \gamma_2^b \nu), \rho(\wp_1^a z, \gamma_n^b \nu), \rho(\gamma_1^b \nu, \gamma_n^b \nu), \rho(\gamma_2^b \nu, \gamma_n^b \nu), \\ &\qquad \frac{\rho(\wp_1^a z, \gamma_n^b \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b \nu)}, \frac{\rho(\gamma_1^b \nu, \gamma_2^b \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b \nu)}\} \\ &= \max\{\rho(\nu, \gamma_1^b \nu), \rho(\nu, \gamma_1^b \nu), 0, 0, \frac{\rho(\nu, \gamma_1^b \nu)}{1 + \sum_{l=2}^n \rho(\nu, \gamma_l^b \nu)}, \frac{\rho(\nu, \gamma_1^b \nu)}{1 + \sum_{l=2}^n \rho(\nu, \gamma_l^b \nu)}\} \\ &= \rho(\nu, \gamma_1^b \nu), \end{aligned} \tag{2.9}$$

$$\begin{aligned} \psi_2(\mathcal{N}(z,\nu)) &= \psi_2\{\rho(\wp_1^a z, \gamma_2^b \nu), \rho(\wp_1^a z, \gamma_n^b \nu), \rho(\gamma_1^b \nu, \gamma_n^b \nu), \frac{\rho(\wp_1^a z, \gamma_n^b \nu)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b \nu)}\} \\ &= \psi_2\{\rho(\nu, \gamma_1^b \nu), \rho(\nu, \gamma_1^b \nu), \rho(\gamma_1^b \nu, \gamma_1^b \nu), \frac{\rho(\nu, \gamma_1^b \nu)}{1 + \sum_{l=2}^n \rho(\nu, \gamma_l^b \nu)}\} \\ &= \psi_2\{\rho(\nu, \gamma_1^b \nu), \rho(\nu, \gamma_1^b \nu), 0, \frac{\rho(\nu, \gamma_1^b \nu)}{1 + \sum_{l=2}^n \rho(\nu, \gamma_l^b \nu)}\} = 0. \end{aligned} \tag{2.10}$$

By the help of (2.8)-(2.10), we have

$$\tau\Big(\int_0^{\rho(\nu, \gamma_1^b \nu)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\rho(\nu, \gamma_1^b \nu)} \Gamma(t)dt\Big),$$

this is a contradiction of our supposition that $\tau(t) > \phi_1(t)$. This implies that $\nu = \gamma_1^b \nu$, which further implies that $\gamma_1^b \nu = \gamma_2^b \nu = \cdots = \gamma_n^b \nu = \nu = \wp_1^a \nu = \cdots = \wp_n^a \nu$. Thus, ν is a CFP of the $2n$ self-mappings $\wp_1^a, \wp_2^a, \ldots, \wp_n^a$, $\gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b$. Finally, we show that the CFP of the $2n$ self-mappings is unique. For this, we assume once again a contrary path, i.e., the fixed point is not unique and suppose that there are two different points $z, z^* \in \mathcal{X}$, such that

$$\wp_1^a z = \wp_2^a z = \cdots = \wp_n^a z = z, \quad \gamma_1^b z = \gamma_2^b z = \cdots = \gamma_n^b z = z^*.$$

Putting $x = z$ and $y = z$ in (2.1), we have

$$\tau\Big(\int_0^{\rho(\wp_1^a z, \gamma_1^b z)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(z,z))} \Gamma(t)dt\Big) + \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(z,z))} \Gamma(t)dt\Big), \tag{2.11}$$

where

$$\begin{aligned} \psi_1(\mathcal{M}(z,z)) &= \max\{\rho(\wp_1^a z, \gamma_2^b z), \rho(\wp_1^a z, \gamma_n^b z), \rho(\gamma_1^b z, \gamma_n^b z), \rho(\gamma_2^b z, \gamma_n^b z), \\ &\qquad \frac{\rho(\wp_1^a z, \gamma_n^b z)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b z)}, \frac{\rho(\gamma_1^b z, \gamma_2^b z)}{1 + \sum_{l=2}^n \rho(\wp_1^a z, \gamma_l^b z)}\} \\ &= \max\{\rho(z, z^*), \rho(z, z^*), \rho(z^*, z^*), \rho(z^*, z^*), \frac{\rho(z, z^*)}{1 + \sum_{l=2}^n \rho(z, z^*)}, \frac{\rho(z^*, z^*)}{1 + \sum_{l=2}^n \rho(z, z^*)}\} \\ &= \max\{\rho(z, z^*), \rho(z, z^*), 0, 0, \frac{\rho(z, z^*)}{1 + \sum_{l=2}^n \rho(z, z^*)}, 0\} = \rho(z, z^*), \end{aligned} \tag{2.12}$$

$$\begin{aligned}\psi_2(\mathcal{N}(z,z)) &= \psi_2\{\rho(\wp_1^a z,\gamma_2^b z),\rho(\wp_1^a z,\gamma_n^b z),\rho(\gamma_1^b z,\gamma_n^b z),\frac{\rho(\wp_1^a z,\gamma_n^b z)}{1+\sum_{l=2}^{n}\rho(\wp_1^a z,\gamma_l^b z)}\}\\ &= \psi_2\{\rho(z,z^*),\rho(z,z^*),\rho(z^*,z^*),\frac{\rho(z,z^*)}{1+\sum_{l=2}^{n}\rho(z,z^*)}\}\\ &= \psi_2\{\rho(z,z^*),\rho(z,z^*),0,\frac{\rho(z,z^*)}{1+\sum_{l=2}^{n}\rho(z,z^*)}\}=0.\end{aligned} \tag{2.13}$$

By the help of (2.11)-(2.13), we have

$$\tau\Big(\int_0^{\rho(z,z^*)}\Gamma(t)dt\Big)\leqslant\phi_1\Big(\int_0^{\rho(z,z^*)}\Gamma(t)dt\Big),$$

this is a contradiction of our supposition that $\tau(t)>\phi_1(t)$. This implies that $z=z^*$. Thus, the CFP of the $2n$ self-mappings $\wp_1^a,\wp_2^a,\dots,\wp_n^a$, $\gamma_1^b,\gamma_2^b,\dots,\gamma_n^b$ is unique. □

For $n=2$ in Theorem 2.2, we have the following corollary.

Corollary 2.3. *Let $\wp_1^a,\wp_2^a,\dots,\wp_n^a$ and $\gamma_1^b,\gamma_2^b,\dots,\gamma_n^b$ be $2n$ self-mappings on a metric space $(\mathcal{X},\rho)$ and satisfying the following conditions:*

(a) *the pairs $(\wp_1^a,\wp_2^a)$, (γ_1^b,γ_2^b) share the property $CLR_{\wp_2^a\gamma_2^b}$;*
(b) *the pairs $(\wp_1^a,\wp_2^a)$, (γ_1^b,γ_2^b) are weakly compatible;*
(c) *the following inequality is satisfied:*

$$\tau\Big(\int_0^{\rho(\wp_1^a x,\gamma_1^b y)}\Gamma(t)dt\Big)\leqslant\phi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))}\Gamma(t)dt\Big)+\mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(x,y))}\Big),$$

where, $\mathcal{L}\geqslant0$, $\Gamma(t)$ is Lebesgue integrable function such that $\int_0^\delta\Gamma(t)dt>0$ for any $\delta>0$ and

$$\begin{aligned}\psi_1(\mathcal{M}(x,y)) = \max\{&\rho(\wp_1^a x,\gamma_2^b y),\rho(\wp_1^a x,\gamma_2^b y),\rho(\gamma_1^b y,\gamma_2^b y),\rho(\gamma_2^b y,\gamma_2^b y),\\ &\frac{\rho(\wp_1^a x,\gamma_2^b y)}{1+\rho(\wp_1^a x,\gamma_2^b y)},\frac{\rho(\gamma_1^b x,\gamma_2^b y)}{1+\rho(\wp_1^a x,\gamma_2^b y)}\},\end{aligned}$$

$$\psi_2(\mathcal{N}(x,y))=\psi_2\{\rho(\wp_1^a x,\gamma_2^b y),\rho(\wp_1^a x,\gamma_2^b y),\rho(\gamma_1^b y,\gamma_2^b y),\frac{\rho(\wp_1^a x,\gamma_2^b y)}{1+\rho(\wp_1^a x,\gamma_2^b y)}\},$$

then, the self-mappings $\wp_1^a,\wp_2^a,\gamma_1^b,\gamma_2^b$ have a unique CFP in $(\mathcal{X},\rho)$.

For $n=3$ in Theorem 2.2, we have the following corollary.

Corollary 2.4. *Let $\wp_1^a,\wp_2^a$ and γ_1^b,γ_2^b be self-mappings on a metric space $(\mathcal{X},\rho)$ and satisfying the following conditions:*

(a) *the pairs $(\wp_1^a,\wp_2^a,\wp_3^a)$, $(\gamma_1^b,\gamma_2^b,\gamma_3^b)$ share the property $CLR_{\wp_k^a\gamma_l^b}$ for $l,k=2,3$;*
(b) *the pairs $(\wp_1^a,\gamma_1^b)$, $(\wp_2^a,\gamma_2^b)$, $(\wp_3^a,\gamma_3^b)$ are weakly compatible;*
(c) *the following inequality is satisfied:*

$$\tau\Big(\int_0^{\rho(\wp_1^a x,\gamma_1^b y)}\Gamma(t)dt\Big)\leqslant\phi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))}\Gamma(t)dt\Big)+\mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(x,y))}\Big),$$

where, $\mathcal{L}\geqslant0$, $\Gamma(t)$ is Lebesgue integrable function such that $\int_0^\delta\Gamma(t)dt>0$ for any $\delta>0$ and

$$\begin{aligned}\psi_1(\mathcal{M}(x,y)) = \max\{&\rho(\wp_1^a x,\gamma_2^b y),\rho(\wp_1^a x,\gamma_3^b y),\rho(\gamma_1^b y,\gamma_3^b y),\rho(\gamma_2^b y,\gamma_3^b y),\\ &\frac{\rho(\wp_1^a x,\gamma_3^b y)}{1+\sum_{l=2}^{3}\rho(\wp_1^a x,\gamma_l^b y)},\frac{\rho(\gamma_1^b x,\gamma_2^b y)}{1+\sum_{l=2}^{3}\rho(\wp_1^a x,\gamma_l^b y)}\},\end{aligned}$$

$$\psi_2(\mathcal{N}(x,y))=\psi_2\{\rho(\wp_1^a x,\gamma_2^b y),\rho(\wp_1^a x,\gamma_n^b y),\rho(\gamma_1^b y,\gamma_3^b y),\frac{\rho(\wp_1^a x,\gamma_n^b y)}{1+\sum_{l=2}^{3}\rho(\wp_1^a x,\gamma_l^b y)}\},$$

then, the six self-mappings $\wp_1^a, \wp_2^a, \wp_3^a, \gamma_1^b, \gamma_2^b, \gamma_3^b$ have a unique CFP in $(\mathcal{X}, \rho)$.

For $\mathcal{L} = 0$ in Theorem 2.2, we have the following corollary.

Corollary 2.5. *Let $\wp_1^a, \wp_2^a, \dots, \wp_n^a$ and $\gamma_1^b, \gamma_2^b, \dots, \gamma_n^b$ be $2n$ self-mappings on a metric space $(\mathcal{X}, \rho)$ and satisfying the following conditions:*

(a) *the pairs $(\wp_1^a, \wp_2^a, \dots, \wp_n^a)$ and $(\gamma_1^b, \gamma_2^b, \dots, \gamma_n^b)$ share the property $CLR_{\wp_k^a \gamma_l^b}$ for $l, k = 2, \dots, n$;*
(b) *the pairs $(\wp_1^a, \gamma_1^b)$, $(\wp_2^a, \gamma_2^b), \dots, (\wp_n^a, \gamma_n^b)$ are weakly compatible;*
(c) *the following inequality is satisfied:*

$$\tau\Big(\int_0^{\rho(\wp_1^a x, \gamma_1^b y)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))} \Gamma(t)dt\Big),$$

where, $\Gamma(t)$ is Lebesgue integrable function such that $\int_0^\delta \Gamma(t)dt > 0$ for any $\delta > 0$ and

$$\begin{aligned}\psi_1(\mathcal{M}(x,y)) = \max\{&\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_n^b y), \rho(\gamma_1^b y, \gamma_n^b y), \rho(\gamma_2^b y, \gamma_n^b y),\\ &\frac{\rho(\wp_1^a x, \gamma_n^b y)}{1 + \sum_{l=2}^n \rho(\wp_1^a x, \gamma_l^b y)}, \frac{\rho(\gamma_1^b x, \gamma_2^b y)}{1 + \sum_{l=2}^n \rho(\wp_1^a x, \gamma_l^b y)}\},\end{aligned}$$

then, the $2n$ self-mappings $\wp_1^a, \wp_2^a, \dots, \wp_n^a$ and $\gamma_1^b, \gamma_2^b, \dots, \gamma_n^b$ have a unique CFP in $(\mathcal{X}, \rho)$.

3. Applications

In this section, two applications of our main Theorem 2.2 are presented. In the following example, we assume that $n = 2$, and show that the self-mappings $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 2.2.

Example 3.1. Let $(\mathcal{X} = [0, 1], \rho)$ be a metric space with $\rho(x, y) = |x - y|$ for x, $y \in \mathcal{X}$. Define $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b$ as

$$\wp_1^a(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{6} \text{ if } x \in (0.5, 1], \end{cases} \qquad \wp_2^a(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{7} \text{ if } x \in (0.5, 1], \end{cases} \tag{3.1}$$

$$\gamma_1^b(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{8} \text{ if } x \in (0.5, 1], \end{cases} \qquad \gamma_2^b(x) = \begin{cases} \frac{1}{2} \text{ if } x \in [0, 0.5], \\ \frac{1}{9} \text{ if } x \in (0.5, 1]. \end{cases} \tag{3.2}$$

One can easily check that the pair $(\wp_1^a, \gamma_1^b)$ and $(\wp_2^a, \gamma_2^b)$ are weakly compatible. Let us consider the following sequences

$$\{x_n\} = \{\frac{0.1n + 0.21}{n}\}, \qquad \{y_n\} = \{\frac{0.21n + 0.11}{0.71 + n}\}. \tag{3.3}$$

By the help of (3.1), (3.2), and (3.3), we have

$$\begin{aligned} &\lim_{n\to\infty} \wp_1^a(x_n) = \lim_{n\to\infty} \wp_1^a(\frac{0.1n + 0.21}{n}) = \frac{1}{2}, \quad &&\lim_{n\to\infty} \wp_2^a(x_n) = \lim_{n\to\infty} \wp_2^a(\frac{0.1n + 0.21}{n}) = \frac{1}{2},\\ &\lim_{n\to\infty} \gamma_1^b(y_n) = \lim_{n\to\infty} \gamma_1^b(\frac{0.21n + 0.11}{0.71 + n}) = \frac{1}{2}, \quad &&\lim_{n\to\infty} \gamma_2^b(y_n) = \lim_{n\to\infty} \gamma_2^b(\frac{0.21n + 0.11}{0.71 + n}) = \frac{1}{2}. \end{aligned} \tag{3.4}$$

From (3.4), it is proved that the mappings $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b$ share the property $CLR_{\wp_2^a \gamma_2^b}$. Next, we need to determine that the mappings $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b$ satisfy the inequality (2.1). For this, we study two cases, i.e., $x, y \in [0, 0.5]$ and $x, y \in (0.5, 1]$.

Case I. For $x, y \in [0, 0.5]$, we have $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b = \frac{1}{2}$, which implies $\rho(\wp_1^a x, \wp_2^a y) = 0$, $\psi_1(\mathcal{M}) = 0$ and $\psi_2(\mathcal{N}) = 0$. And therefore, the inequality (2.1) is trivially satisfied.

Case II. For $x, y \in (0.5, 1]$, we have $\wp_1^a x = \frac{1}{6}$, $\wp_2^a x = \frac{1}{7}$, $\gamma_1^b y = \frac{1}{8}$, $\gamma_2^b y = \frac{1}{9}$, and

$$\begin{aligned}\psi_1(\mathcal{M}(x,y)) &= \max\{\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_2^b y), \rho(\gamma_1^b y, \gamma_2^b y), \rho(\gamma_2^b y, \gamma_2^b y),\\ &\qquad \frac{\rho(\wp_1^a x, \gamma_2^b y)}{1+\rho(\wp_1^a x, \gamma_2^b y)}, \frac{\rho(\gamma_1^b x, \gamma_2^b y)}{1+\rho(\wp_1^a x, \gamma_2^b y)}\}\\ &= \max\{\rho(\frac{1}{6}, \frac{1}{7}), \rho(\frac{1}{6}, \frac{1}{9}), \rho(\frac{1}{8}, \frac{1}{9}), \rho(\frac{1}{9}, \frac{1}{9}), \frac{\rho(\frac{1}{6}, \frac{1}{9})}{1+\rho(\frac{1}{6}, \frac{1}{9})}, \frac{\rho(\frac{1}{8}, \frac{1}{9})}{1+\rho(\frac{1}{6}, \frac{1}{8})}\}\\ &= 0.05556,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\psi_2(\mathcal{N}(x,y)) &= \psi_2\{\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_2^b y), \rho(\gamma_1^b y, \gamma_2^b y), \frac{\rho(\wp_1^a x, \gamma_2^b y)}{1+\rho(\wp_1^a x, \gamma_2^b y)}\}\\ &= \psi_2\{\rho(\frac{1}{6}, \frac{1}{9}), \rho(\frac{1}{6}, \frac{1}{9}), \rho(\frac{1}{8}, \frac{1}{9}), \frac{\rho(\frac{1}{8}, \frac{1}{9})}{1+\rho(\frac{1}{6}, \frac{1}{8})}\} = 0.013889.\end{aligned} \tag{3.6}$$

For $\tau(t) = 0.9t$, $\phi_1(t) = 0.86t$, $\phi_2(t) = 0.83t$, $\Gamma(t) = 2t$, $\mathcal{L} = 0.21$, (2.1), (3.5), and (3.6), imply

$$0.0015625 = \tau\Big(\int_0^{\rho(\wp_1^a x, \gamma_1^b y)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))} \Gamma(t)dt\Big) + \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(x,y))} \Gamma(t)dt\Big) = 0.00269526.$$

Therefore, the inequality (2.1) is also satisfied. Thus, the self-mappings $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b : \mathcal{X} \to \mathcal{X}$ have a unique CFP 0.5.

In the following example, we assume that $n = 3$, and show that the self-mappings $\wp_1^a, \wp_2^a, \wp_3^a, \gamma_1^b, \gamma_2^b, \gamma_3^b : \mathcal{X} \to \mathcal{X}$ satisfy all the conditions of Theorem 2.2.

Example 3.2. Let $(\mathcal{X} = [0,1], \rho)$ be a metric space with $\rho(x,y) = |x-y|$ for $x, y \in \mathcal{X}$. Define $\wp_1^a, \wp_2^a, \wp_3^a, \gamma_1^b, \gamma_2^b, \gamma_3^b$, as

$$\begin{aligned}\wp_1^a(x) &= \begin{cases} 0.77 - \Big(0.11 + \sin(0.03 + x^2) + 0.1\cos x\Big), & \text{if } x \in (0, 0.35),\\ 0.4, & \text{if } x \in [0.35, 0.5),\\ \frac{2.2 - x^2}{5}, & \text{if } x \in [0.5, 1],\end{cases}\\ \wp_2^a(x) &= \begin{cases} 0.8 - \Big(0.11 + \sin(0.03 + x^2) + 0.1\cos x\Big), & \text{if } x \in (0, 0.35),\\ 0.4, & \text{if } x \in [0.35, 0.5),\\ \frac{2 - x^2}{5}, & \text{if } x \in [0.5, 1],\end{cases}\\ \wp_3^a(x) &= \begin{cases} 0.85 - \Big(0.11 + \sin(0.03 + x^2) + 0.1\cos x\Big), & \text{if } x \in (0, 0.35),\\ 0.4, & \text{if } x \in [0.35, 0.5),\\ \frac{1.95 - x^2}{5}, & \text{if } x \in [0.5, 1],\end{cases}\\ \gamma_1^b(y) &= \begin{cases} 1 - \Big(0.11 + \sin(0.03 + y^2) + 0.1\cos y\Big), & \text{if } y \in (0, 0.35),\\ 0.4, & \text{if } y \in [0.35, 0.5),\\ \frac{2 - y^2}{5}, & \text{if } y \in [0.5, 1],\end{cases}\\ \gamma_2^b(y) &= \begin{cases} 0.9 - \Big(0.11 + \sin(0.03 + y^2) + 0.1\cos y\Big), & \text{if } y \in (0, 0.35),\\ 0.4, & \text{if } y \in [0.35, 0.5),\\ \frac{2 - y^2}{5}, & \text{if } y \in [0.5, 1],\end{cases}\end{aligned} \tag{3.7}$$

$$\gamma_3^b(y)=\begin{cases}0.85-\Big(0.11+\sin(0.03+y^2)+0.1\cos y\Big), & \text{if } y\in(0,0.35),\\ 0.4, & \text{if } y\in[0.35,0.5),\\ \frac{1.95-y^2}{5}, & \text{if } y\in[0.5,1].\end{cases}$$

The pairs $(\wp_1^a,\gamma_1^b)$, $(\wp_2^a,\gamma_2^b)$, $(\wp_3^a,\gamma_3^b)$ are weakly compatible. Let us consider the following sequences

$$\{x_n\}=\{\frac{0.4n+1.4}{n+1}\},\qquad \{y_n\}=\{\frac{0.41n+1.41}{n+1}\}. \tag{3.8}$$

By the help of (3.7), and (3.8), we have

$$\begin{aligned}&\lim_{n\to\infty}\wp_1^a(x_n)=\lim_{n\to\infty}\wp_1^a(\frac{0.4n+1.4}{n+1})=0.4,\quad \lim_{n\to\infty}\wp_2^a(x_n)=\lim_{n\to\infty}\wp_2^a(\frac{0.4n+1.4}{n+1})=0.4,\\ &\lim_{n\to\infty}\wp_3^a(x_n)=\lim_{n\to\infty}\wp_3^a(\frac{0.4n+1.4}{n+1})=0.4,\quad \lim_{n\to\infty}\gamma_1^b(y_n)=\lim_{n\to\infty}\gamma_1^b(\frac{0.41n+1.41}{n+1})=0.4,\\ &\lim_{n\to\infty}\gamma_2^b(y_n)=\lim_{n\to\infty}\gamma_2^b(\frac{0.41n+1.41}{n+1})=0.4,\quad \lim_{n\to\infty}\gamma_3^b(y_n)=\lim_{n\to\infty}\gamma_3^b(\frac{0.41n+1.41}{n+1})=0.4.\end{aligned} \tag{3.9}$$

From (3.9), it is proved that the mappings $\wp_1^a,\wp_2^a,\wp_3^a,\gamma_1^b,\gamma_2^b,\gamma_3^b$ share the property $CLR_{\wp_k^a\gamma_l^b}$ for $k,l=2,3$. Next, we need to determine that the mappings $\wp_1^a,\wp_2^a,\wp_3^a,\gamma_1^b,\gamma_2^b,\gamma_3^b$ satisfy the inequality (2.1). For this, we study three cases, i.e., $x,y\in[0,0.35)$, $x,y\in[0.35,0.5)$, and $x,y\in[0.5,1]$.

Case I. For $x,y\in[0,\,0.35)$, we have

$$\begin{aligned}\wp_1^a x&=0.77-\Big(0.11+\sin(0.03+x^2)+0.1\cos x\Big),\\ \wp_2^a x&=0.8-\Big(0.11+\sin(0.03+x^2)+0.1\cos x\Big),\\ \wp_3^a x&=0.85-\Big(0.11+\sin(0.03+x^2)+0.1\cos x\Big),\\ \gamma_1^b y&=1-\Big(0.11+\sin(0.03+y^2)+0.1\cos y\Big),\\ \gamma_2^b y&=0.9-\Big(0.11+\sin(0.03+y^2)+0.1\cos y\Big),\\ \gamma_3^b y&=0.85-\Big(0.11+\sin(0.03+y^2)+0.1\cos y\Big),\end{aligned}$$

and

$$\begin{aligned}\psi_1(\mathcal{M}(x,y))=\max\{&\rho(\wp_1^a x,\gamma_2^b y),\rho(\wp_1^a x,\gamma_n^b y),\rho(\gamma_1^b y,\gamma_n^b y),\rho(\gamma_2^b y,\gamma_n^b y),\\ &\frac{\rho(\wp_1^a x,\gamma_n^b y)}{1+\sum_{l=2}^n\rho(\wp_1^a x,\gamma_l^b y)},\frac{\rho(\gamma_1^b x,\gamma_2^b y)}{1+\sum_{l=2}^n\rho(\wp_1^a x,\gamma_l^b y)}\}=0.13,\\ \psi_2(\mathcal{N}(x,y))=\psi_2\{&\rho(\wp_1^a x,\gamma_2^b y),\rho(\wp_1^a x,\gamma_n^b y),\rho(\gamma_1^b y,\gamma_n^b y),\frac{\rho(\wp_1^a x,\gamma_n^b y)}{1+\sum_{l=2}^n\rho(\wp_1^a x,\gamma_l^b y)}\}=0.066.\end{aligned}$$

For $\tau(t)=0.9t$, $\varphi_1(t)=0.86t$, $\varphi_2(t)=0.83t$, $\Gamma(t)=2t$, $\mathcal{L}=0.21$, (2.1), (3.5), and (3.6), imply

$$0.04761=\tau\Big(\int_0^{\rho(\wp_1^a x,\gamma_1^b y)}\Gamma(t)dt\Big)\leqslant\varphi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))}\Gamma(t)dt\Big)+\mathcal{L}\varphi_2\Big(\int_0^{\psi_2(\mathcal{N}(x,y))}\Gamma(t)dt\Big)=0.0868436.$$

Case II. For $x,y\in[0.35,\,0.5]$, we have $\wp_1^a x=\wp_2^a x=\wp_3^a x=\gamma_1^b y=\gamma_2^b y=\gamma_3^b y=0.4$, which implies $\rho(\wp_1^a x,\,\wp_2^a y)=0$, $\psi_1(\mathcal{M})=0$ and $\psi_2(\mathcal{N})=0$. And therefore, the inequality (2.1) is trivially satisfied.

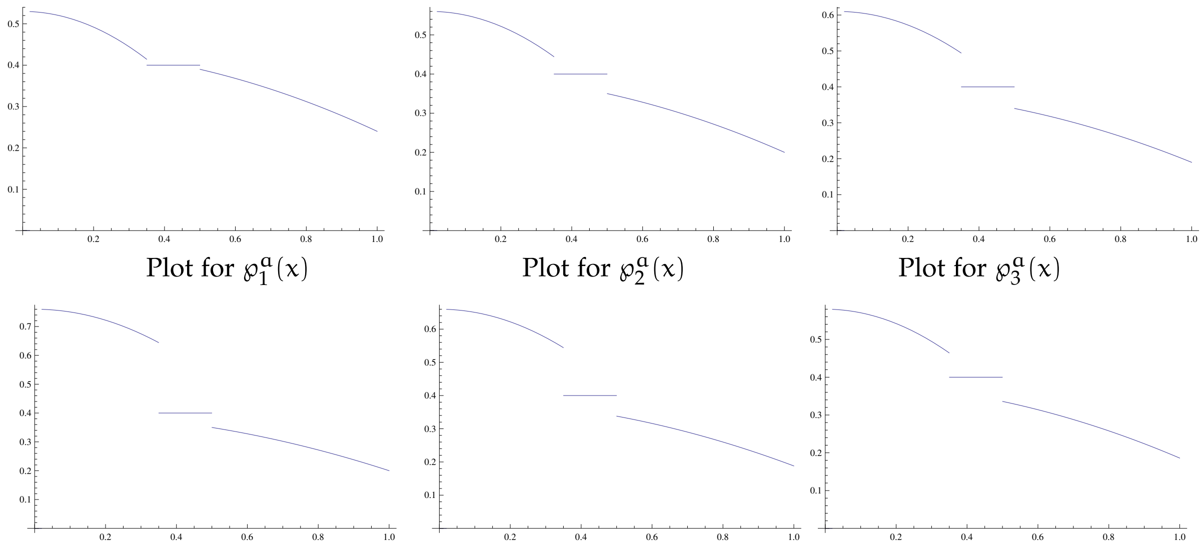
Case III. For $x, y \in (0.5, 1]$, we have $\wp_1^a x = \frac{2.2-x^2}{5}$, $\wp_2^a x = \frac{2-x^2}{5}$, $\wp_3^a x = \frac{1.95-x^2}{5}$, $\gamma_1^b y = \frac{2-x^2}{5}$, $\gamma_2^b y = \frac{2-x^2}{5}$, $\gamma_3^b y = \frac{1.95-x^2}{5}$. Consequently, we have

$$\begin{aligned}\psi_1(\mathcal{M}(x,y)) &= \max\{\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_n^b y), \rho(\gamma_1^b y, \gamma_n^b y), \rho(\gamma_2^b y, \gamma_n^b y), \\ &\qquad \frac{\rho(\wp_1^a x, \gamma_n^b y)}{1+\sum_{l=2}^{n}\rho(\wp_1^a x, \gamma_l^b y)}, \frac{\rho(\gamma_1^b x, \gamma_2^b y)}{1+\sum_{l=2}^{n}\rho(\wp_1^a x, \gamma_l^b y)}\} = 0.1, \\ \psi_2(\mathcal{N}(x,y)) &= \psi_2\{\rho(\wp_1^a x, \gamma_2^b y), \rho(\wp_1^a x, \gamma_n^b y), \rho(\gamma_1^b y, \gamma_n^b y), \frac{\rho(\wp_1^a x, \gamma_n^b y)}{1+\sum_{l=2}^{n}\rho(\wp_1^a x, \gamma_l^b y)}\} = 0.04.\end{aligned}$$

For $\tau(t) = 0.9t$, $\phi_1(t) = 0.86t$, $\phi_2(t) = 0.83t$, $\Gamma(t) = 2t, \mathcal{L} = 0.21$, (2.1), (3.5), and (3.6), imply

$$0.00144 = \tau\Big(\int_0^{\rho(\wp_1^a x, \gamma_1^b y)} \Gamma(t)dt\Big) \leqslant \phi_1\Big(\int_0^{\psi_1(\mathcal{M}(x,y))} \Gamma(t)dt\Big) + \mathcal{L}\phi_2\Big(\int_0^{\psi_2(\mathcal{N}(x,y))} \Gamma(t)dt\Big) = 0.0088656.$$

Therefore, the inequality (2.1) is also satisfied.



Plot for $\wp_1^a(x)$ Plot for $\wp_2^a(x)$ Plot for $\wp_3^a(x)$

Plot for $\gamma_1^b(y)$ Plot for $\gamma_2^b(y)$ Plot for $\gamma_3^b(y)$

Figure 1

Consequently, all the conditions of Theorem 2.2 are satisfied and therefore, the self-mappings $\wp_1^a, \wp_2^a, \wp_3^a, \gamma_1^b, \gamma_2^b, \gamma_3^b$ have a unique CFP. From Fig. 1, we can observe that the unique CFP of the self-mappings $\wp_1^a, \wp_2^a, \wp_3^a, \gamma_1^b, \gamma_2^b, \gamma_3^b$ is 0.4.

4. Conclusion

In this paper, we have given the notion of property $\mathrm{CLR}_{\wp_k^a \gamma_l^b}$ for $k, l = 2, \ldots, n$ for the $2n$ self-mappings $\wp_1^a, \wp_2^a, \ldots, \wp_n^a, \gamma_1^b, \gamma_2^b, \ldots, \gamma_n^b : \mathcal{X} \to \mathcal{X}$. This new idea of generalization of the CLR property will help the scientists to handle $2n$ self-mappings for the unique CFPs and many others. The new definition was utilized for a new FPT along with the assumption that the pairs $(\wp_1^a, \gamma_1^a), (\wp_2^a, \gamma_2^a), \ldots, (\wp_n^a, \gamma_n^a)$ are weakly compatibility in Theorem 2.2. Three corollaries were also produced from Theorem 2.2 as its special cases and two applications were illustrated. Example 3.1 demonstrates the application of our work for $n = 2$ that the four self-mappings $\wp_1^a, \wp_2^a, \gamma_1^b, \gamma_2^b$ have a unique CFP $0.5 \in \mathcal{X} = [0, 1]$. Example 3.2, demonstrates an application of our main Theorem 2.2 for $n = 3$. From the graphical representation of the six self-mappings in the second example, one can easily observe the unique CFP 0.4 in the metric space $(\mathcal{X}, \rho)$.

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