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On monotone multivalued transformations

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Abstract

In this work, we discuss the recently introduced monotone τ-Opial condition in Banach spaces which admit a sequence of monotone approximations of the identity. Then we give a fixed point theorem for monotone multivalued nonexpansive mappings in Banach spaces satisfying the monotone τ-Opial condition. This result generalizes those of Markin, Browder and Lami Dozo to monotone mappings.

Keywords: Fixed point, systems of projections, monotone Opial condition, monotone nonexpansive mappings, multivalued mappings.
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1. Introduction

The extension of many fixed point theorems of singlevalued mappings to the multivalued case was never an easy one. Recall that the first attempts was carried by Markin [16] in Hilbert spaces, by Browder [5] in spaces having a weakly continuous duality mapping, and by Lami Dozo [14] in spaces satisfying the weak Opial condition. For a nice survey on multivalued mappings, we recommend the paper by Benavides and Ramírez [4].

The study of fixed points of monotone mappings attracted some attention following the extension of the Banach Contraction Principle [3] by Ran and Reurings [18] in partially ordered metric spaces. For an extensive list of references and historical facts about monotone mappings, we recommend the survey by Bachar and Khamsi [2]. In this work, we extend the main ideas of Lami Dozo [14] to the case of monotone multivalued mappings. In particular, we introduce the concept of monotone approximation of the identity of a Banach space.

For more on the metric fixed point theory, we recommend the book by Khamsi and Kirk [12].

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2. Monotone Opial condition

Throughout, $(X, \|.\|, \preceq)$ stands for a partially ordered Banach space such that $x \preceq y$ implies

$$\alpha\, x + z \preceq \alpha\, y + z$$

for any $x, y, z \in X$ and $\alpha \in [0, +\infty)$. We will assume that order intervals are closed. Recall that an order interval is any of the sets

$$[x, \rightarrow) = \{y \in X;\ x \preceq y\} \text{ and } (\leftarrow, x] = \{y \in X;\ y \preceq x\}$$

for any $x \in X$. Let τ be a topology on X for which order intervals are τ-closed. Let us next recall the definition of τ-Opial condition [6, 17].

Definition 2.1. We will say that X satisfies the τ-Opial condition if for any sequence $\{x_n\}$ in X which τ-converges to x, we have

$$\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|$$

for any $y \in X$ such that $y \neq x$.

Very early on, this property played a major role in the study of the fixed point property of nonexpansive mappings [6]. For example, Opial [17] noted that the classical Banach spaces ℓ^p enjoyed the weak-Opial condition while $L^p([0, 1])$ fails it for $p \neq 2$ and $1 < p < \infty$. In recent study of the fixed point property of monotone nonexpansive mappings, the authors introduced the concept of monotone Opial condition as follows.

Definition 2.2 ([1])**.** We will say that X satisfies the monotone τ-Opial condition if for any monotone increasing (resp. decreasing) sequence $\{x_n\}$ in X which τ-converges to x, we have

$$\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|$$

for any $y \in \bigcap_{n\geqslant 1} [x_n, \rightarrow)$ (resp. $y \in \bigcap_{n\geqslant 1} (\leftarrow, x_n]$) such that $y \neq x$.

Note that since we assumed that order intervals are τ-closed, then we have $y \in \bigcap_{n\geqslant 1} [x_n, \rightarrow)$ if and only if $x \preceq y$ (resp. $y \in \bigcap_{n\geqslant 1} (\leftarrow, x_n]$ if and only if $y \preceq x$). In [1], it is proved that any lattice Banach space with a monotone norm satisfies a large monotone weak-Opial condition. But if the norm is uniformly convex, then we have the monotone weak-Opial condition. This is amazing because it allows one to prove that $L^p([0, 1])$ satisfies the monotone weak-Opial condition, for $p \neq 2$ and $1 < p < \infty$.

Next, we extend the concept of the approximation of the identity as introduced by Lami Dozo [14] to the monotone case.

Definition 2.3. A sequence of bounded linear operators $\{P_n\}_{n\geqslant 1}$ of X is said to be a monotone τ-approximation of the identity, if

(1) $\lim_{n\to\infty} \|x - P_n(x)\| = 0$, for any $x \in X$;
(2) $\lim_{n\to\infty} \|P_i(x_n)\| = 0$, for any $i \geqslant 1$ and any sequence $\{x_n\}$ which τ-converges to 0;
(3) P_n and $Q_n = I - P_n$ are monotone operators, where $I : X \to X$ is the identity mapping.

We will say that $\{P_n\}$ satisfies the property (L) if there exists a continuous function $\delta : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that

$$\|P_n(u) + Q_n(v)\| \geqslant \delta(\|P_n(u)\|, \|Q_n(v)\|)$$

for any $u, v \in X$ such that $u \succeq 0$ and $v \succeq 0$ and $n \geqslant 1$. Moreover the function δ satisfies $\delta(0, r) = \delta(r, 0) = r$, $\delta(r, s) > r$ and $\delta(s, r) > r$ for any $r \geqslant 0$ and $s > 0$.

The origin of the property (L) may be found in [11] where this property was introduced and investigated in relation with the Opial and Kadec-Klee properties. The following example will help shed some interesting properties about this new concept. It is inspired from the original work of Lim [15].

Example 2.4. Consider the Banach space ℓ^p, with $1 < p < \infty$. For any $x = (x_n) \in \ell^p$, define

$$x^+ = (\max(x_n, 0)) \ \text{ and } \ x^- = (-\min(x_n, 0)).$$

Define the new norm in ℓ^p defined by

$$\|x\| = \|x^+\|_p + \|x^-\|_p$$

for any $x \in \ell^p$. It is easy to check that $\|.\|$ is an equivalent norm to $\|.\|_p$. Next consider the sequence of natural projections $\{P_n\}$ associated to the canonical Schauder basis of ℓ^p, i.e.

$$P_n(x) = (x_1, x_2, \cdots, x_n, 0, 0, 0, \cdots), \quad x \in \ell^p.$$

Then $\{Q_n\}$ are also projections defined by

$$Q_n(x) = (0, 0, \cdots, x_{n+1}, x_{n+2}, \cdots), \quad x \in \ell^p.$$

Note that if $u, v \in \ell^p$ are such that $u \succeq 0$ and $v \succeq 0$, then we have $u = u^+$ and $v = v^+$ and

$$\|P_n(u) + Q_n(v)\| = \|P_n(u) + Q_n(v)\|_p = \Big(\|P_n(u)\|_p^p + \|Q_n(v)\|_p^p\Big)^{1/p},$$

which implies

$$\|P_n(u) + Q_n(v)\| = \Big(\|P_n(u)\|^p + \|Q_n(v)\|^p\Big)^{1/p}$$

for any $n \geqslant 1$. If we take $\delta(r, s) = (r^p + s^p)^{1/p}$, the sequence $\{P_n\}$ satisfies the property (L). And if we take τ to be the weak topology, then $\{P_n\}$ is a monotone weak-approximation of the identity. Note that $(\ell^p, \|.\|)$ is not strictly convex since

$$\|e_1\| = \|-e_2\| = \|(e_1 - e_2)/2\| = 1,$$

where $e_1 = (1, 0, 0, \cdots)$ and $e_2 = (0, 1, 0, 0, \cdots)$.

The following result may be seen as an analogue to [14, Theorem 2.1].

Theorem 2.5. *Let $(X, \|.\|, \preceq)$ and τ be as described before. Assume X has a monotone τ-approximation of the identity $\{P_n\}$ which satisfies the property (L). Then X satisfies the monotone τ-Opial condition.*

Proof. Let $\{x_n\}$ be a monotone sequence in X which τ-converges to x. Without loss of generality, we assume $\{x_n\}$ is monotone increasing. Since order intervals are τ-closed, we know that $x_n \preceq x$, for any $n \geqslant 1$. Let $y \neq x$ and $x \preceq y$. We have

$$y - x_n = y - x + (x - x_n), \quad n \geqslant 1.$$

Set $u = y - x$ and $u_n = x - x_n$, for $n \geqslant 1$. Using the properties satisfied by the order $\preceq$ and the topology τ, we conclude that $u \succeq 0$, $u_n \succeq 0$, for any $n \geqslant 1$, $u \neq 0$, and $\{u_n\}$ τ-converges to 0. Since $P_i + Q_i = I$, we get

$$u + u_n = P_i(u) + Q_i(u_n) + Q_i(u) + P_i(u_n)$$

for any $n, i \geqslant 1$. Since $\{P_n\}$ is a monotone τ-approximation of the identity which satisfies the (L) property, we have

$$\lim_{i\to\infty} \|Q_i(u)\| = \lim_{n\to\infty} \|P_i(u_n)\| = 0.$$

Hence

$$\begin{aligned}\|u+u_n\| &\geqslant \|P_i(u)+Q_i(u_n)\| - \|Q_i(u)\| - \|P_i(u_n)\|\\ &\geqslant \delta(\|P_i(u)\|, \|Q_i(u_n)\|) - \|Q_i(u)\| - \|P_i(u_n)\|\end{aligned}$$

for any $n, i \geqslant 1$. If we let $n \to \infty$, we get

$$\limsup_{n\to\infty} \|u+u_n\| \geqslant \delta(\|P_i(u)\|, \limsup_{n\to\infty} \|u_n\|) - \|Q_i(u)\|,$$

since $\limsup_{n\to\infty} \|Q_i(u_n)\| = \limsup_{n\to\infty} \|u_n\|$, for any $i \geqslant 1$. Next, we let $i \to \infty$, we get

$$\limsup_{n\to\infty} \|u+u_n\| \geqslant \delta(\|u\|, \limsup_{n\to\infty} \|u_n\|),$$

since $\limsup_{i\to\infty} \|P_i(u)\| = \|u\|$. Since $\|u\| > 0$ we obtain

$$\limsup_{n\to\infty} \|u+u_n\| > \limsup_{n\to\infty} \|u_n\|,$$

i.e.,

$$\limsup_{n\to\infty} \|x_n - y\| > \limsup_{n\to\infty} \|x_n - x\|.$$

The proof of our claim is complete. □

As a corollary, we have the following result.

Corollary 2.6. *Let $(\ell^p, \|.\|)$, with $1 < p < \infty$, the Banach space described in Example 2.4. Then $(\ell^p, \|.\|)$ satisfies the monotone weak-Opial condition.*

This is truly amazing since $(\ell^p, \|.\|)$ fails to be uniformly convex. In the next section, we give a fixed point result for multivalued monotone nonexpansive mappings in Banach spaces which satisfy the monotone Opial condition.

3. Fixed points of monotone multivalued nonexpansive mappings

Before we give the definition of monotone multivalued nonexpansive mappings, we need the following definition.

Definition 3.1. Let $(X, \|.\|, \preceq)$ be a Banach space endowed with a partial order. Let C be a nonempty subset of X. Let $T : C \to C$ be a map.

(a) T is said to be monotone if

$$x \preceq y \Longrightarrow T(x) \preceq T(y)$$

for any $x, y \in M$.

(b) T is said to be monotone Lipschitzian if T is monotone and there exists $k \geqslant 0$ such that

$$\|T(x) - T(y)\| \leqslant k \, \|x - y\|$$

for any $x, y \in C$ such that $x \preceq y$.

If $k < 1$, then we say that T is a monotone contraction mapping. If $k = 1$, T is called a monotone nonexpansive mapping. A point $x \in C$ is said to be a fixed point of T whenever $T(x) = x$.

Note that classical definitions of nonexpansive multivalued mappings use the Hausdorff distance. This will force the multivalued mappings to have closed and bounded values. Next, we define the concept of monotone nonexpansive multivalued mappings defined on a partially ordered Banach space which, in single-valued case, coincides with the definition of monotone nonexpansive mappings.

Definition 3.2. Let $(X, \|.\|, \preceq)$ be a Banach space endowed with a partial order and C a nonempty subset of X. A multivalued mapping $T : C \to 2^C$ with nonempty values is said to be monotone nonexpansive if for any $x, y \in C$ with $x \preceq y$ and any $u \in T(x)$ there exists $v \in T(y)$ such that

$$u \preceq v \quad \text{and} \quad \|u - v\| \leqslant \|x - y\|.$$

A point x is said to be a fixed point of T if and only if $x \in T(x)$.

As we said before, monotone Lipschitzian mappings are not necessarily continuous. They usually have good topological behaviors on comparable elements. In order to take advantage of this point, one approach to study such mappings is to use iterative methods. Let $(X, \|.\|, \preceq)$ and τ be as described before. Mostly we will consider sequential convergence with respect to τ. A nice discussion about the use of a topology versus a sequential convergence may be found in the work of Dudley [7].

Let C be a convex nonempty subset of a partially ordered Banach space X not reduced to one point. Let $T : C \to 2^C$ be a monotone multivalued nonexpansive mapping with nonempty values. Set

$$C_T = \{x \in C;\ x \preceq y \text{ or } y \preceq x \text{ for some } y \in T(x)\}.$$

Assume that C_T is not empty. Fix $\lambda \in (0, 1)$ and $x_0 \in C_T$. Without loss of generality, assume there exists $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. Then $x_1 = \lambda\, x_0 + (1 - \lambda)\, y_0 \in C$. Since order intervals are convex, we have $x_0 \preceq x_1 \preceq y_0$. Since T is monotone multivalued nonexpansive, then there exists $y_1 \in T(x_1)$ such that $y_0 \preceq y_1$ and $\|y_0 - y_1\| \leqslant \|x_0 - x_1\|$. We have $x_0 \preceq x_1 \preceq y_0 \preceq y_1$. Set $x_2 = \lambda\, x_1 + (1 - \lambda)\, y_1$. Then $x_2 \in C$ and $x_1 \preceq x_2 \preceq y_1$ since order intervals are convex. Since T is monotone multivalued nonexpansive, then there exists $y_2 \in T(x_2)$ such that $y_1 \preceq y_2$ and $\|y_1 - y_2\| \leqslant \|x_1 - x_2\|$. We have $x_1 \preceq x_2 \preceq y_1 \preceq y_2$. By induction, we build two sequences $\{x_n\}$ and $\{y_n\}$ in C defined by what is known as the Krasnoselskii-Ishikawa [10, 13] iteration

$$x_{n+1} = \lambda\, x_n + (1 - \lambda)\, y_n, \quad n \geqslant 1, \tag{3.1}$$

such that $\|y_n - y_{n+1}\| \leqslant \|x_n - x_{n+1}\|$, $y_n \in T(x_n)$ and $x_n \preceq x_{n+1} \preceq y_n \preceq y_{n+1}$, for any $n \in \mathbb{N}$.

In order to proceed further, we will need the following fundamental property satisfied by $\{x_n\}$.

Proposition 3.3 ([8, 9]). *Consider the sequences $\{x_n\}$ and $\{y_n\}$ generated by λ, x_0 and the iteration (3.1). Then the following inequality holds*

$$\text{(GK)} \quad (1 + n\lambda)\ \|y_i - x_i\| \leqslant \|y_{i+n} - x_i\| + (1 - \lambda)^{-n}\Big(\|y_i - x_i\| - \|y_{i+n} - x_{i+n}\|\Big),\ \forall i, n \in \mathbb{N}.$$

If C is assumed to be bounded, then we have $\lim\limits_{n \to +\infty} \|x_n - y_n\| = 0$, i.e., $\{x_n\}$ is an approximate fixed point sequence of T.

The following result will be helpful to prove the main fixed point theorem of this work.

Lemma 3.4. *Assume that C is a convex and bounded nonempty subset of X not reduced to one point. Let $T : C \to 2^C$ be a monotone nonexpansive multivalued mapping with nonempty values. Assume C_T is not empty. Fix $\lambda \in (0, 1)$ and $x_0 \in C_T$. Consider the sequences $\{x_n\}$ and $\{y_n\}$ in C generated by λ, x_0 and the iteration (3.1). Then $\{x_n\}$ has at most one τ-cluster point.*

Proof. Since $x_0 \in C_T$, there exists $y_0 \in T(x_0)$ which is comparable to x_0. Without loss of generality, we assume $x_0 \preceq y_0$. In this case, the sequence $\{x_n\}$ is monotone increasing, i.e., $x_n \preceq x_{n+1}$, for any $n \in \mathbb{N}$. Let z_1 and z_2 be two τ-cluster points of $\{x_n\}$. Fix $k \in \mathbb{N}$. Since $\{x_n\}$ is monotone increasing and the order interval $[x_k, \to)$ is τ-closed, we conclude that $z_i \in [x_k, \to)$, for $i = 1, 2$. Hence $\{x_n\} \subset (\leftarrow, z_i]$, for $i \in \{1, 2\}$, holds which implies for the same reason $z_j \in (\leftarrow, z_i]$, for $i, j \in \{1, 2\}$. Hence $z_1 = z_2$, which implies that $\{x_n\}$ has at most one τ-cluster point. □

Note that if we assume C is τ-compact, then $\{x_n\}$ is τ- convergent to some $z \in C$ and either $x_n \preceq z$ or $z \preceq x_n$, for any $n \in \mathbb{N}$. Before, we state the main result of this section, define $\mathcal{K}(C)$ to be the set of all nonempty compact subsets of C.

Theorem 3.5. *Let* $(X, \|.\|, \preceq)$ *and* τ *be as described above. Assume* X *satisfies the monotone* τ*-Opial condition. Let* C *be a* τ*-compact, convex and bounded nonempty subset of* X *not reduced to one point. Let* $T : C \to \mathcal{K}(C)$ *be a monotone nonexpansive multivalued mapping with nonempty values. Assume* C_T *is not empty. Then* T *has a fixed point.*

Proof. Since C_T is not empty, pick $x_0 \in C_T$. Fix $\lambda \in (0, 1)$. Without loss of generality, assume there exists $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. Consider the sequences $\{x_n\}$ and $\{y_n\}$ in C generated by λ, x_0 and the iteration (3.1). Since C is τ-compact, we know that $\{x_n\}$ is τ-convergent to some point $z \in C$. Since $\{x_n\}$ is monotone increasing, then we have $x_n \preceq z$, for all $n \in \mathbb{N}$. Since T is monotone multivalued nonexpansive and $y_n \in T(x_n)$, there exists $z_n \in T(z)$ such that $y_n \preceq z_n$ and

$$\|y_n - z_n\| \leqslant \|x_n - z\|, \quad \forall n \in \mathbb{N}.$$

Since $T(z)$ is compact, there exists a subsequence $\{z_{\phi(n)}\}$ of $\{z_n\}$ such that $\{z_{\phi(n)}\}$ converges to $\nu \in T(z)$. Moreover, we have $\lim\limits_{n\to+\infty} \|x_n - y_n\| = 0$, since C is bounded. Hence, we have

$$\limsup_{n\to\infty} \|y_{\phi(n)} - z_{\phi(n)}\| = \limsup_{n\to\infty} \|y_{\phi(n)} - \nu\| = \limsup_{n\to\infty} \|x_{\phi(n)} - \nu\|,$$

which implies

$$\limsup_{n\to\infty} \|x_{\phi(n)} - \nu\| \leqslant \limsup_{n\to\infty} \|x_{\phi(n)} - z\|.$$

Let us prove that $z \preceq \nu$. Indeed, we have $x_n \preceq y_n \preceq z_n$, for any $n \in \mathbb{N}$. Since $\{x_n\}$ is increasing and the order intervals are closed, we conclude that $x_n \preceq \nu$. Finally since $\{x_n\}$ τ-converges to z and the order intervals are τ-closed, we obtain $z \preceq \nu$. Since X satisfies the monotone τ-Opial condition, we conclude that $z = \nu \in T(z)$, i.e., z is a fixed point of T. □

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