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Nonlinear Mittag-Leffler stability of nonlinear fractional partial differential equations

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Abstract

This paper focuses on the application of fractional backstepping control scheme for nonlinear fractional partial differential equation (FPDE). Two types of fractional derivatives are considered in this paper, Caputo and the Grünwald-Letnikov fractional derivatives. Therefore, obtaining highly accurate approximations for this derivative is of a great importance. Here, the discretized approach for the space variable is used to transform the FPDE into a system of fractional differential equations. The convergence of the closed loop system is guaranteed in the sense of Mittag-Leffler stability. An illustrative example is given to demonstrate the effectiveness of the proposed control scheme.

Keywords: Backstepping method, fractional Lyapunov function, fractional derivative, boundary control, fractional partial differential equation.
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1. Introduction

During the last few decades, there has been a great effort from researchers to include fractional order systems in control community. This could be due to two major factors. First, many engineering applications cannot be well described by the ordinary integer order systems [21, 32]. The other reason is that fractional controller has shown that it has more potential and more design freedom comparing to the standard integer order controller [18, 25]. For more details on fractional calculus, please refer to [1, 3, 6, 8, 11, 12, 15, 17, 20, 22, 30].

Relatively, a new class of nonlinear systems has been introduced, which is called fractional-order nonlinear systems (FONSs) [4, 19]. However, the Lyapunov direct method has always been a fundamental stabilizer for the nonlinear systems, there have been few attempts done, recently, to investigate what is known as Lyapunov-like stability for fractional-order systems (FOSs). Lakshmikantham and his collaborates investigated a Lyapunov-like theory for FOSs in early 2008 [10]. Then, there was another attempt in [13, 14] to derive Mittag-Leffler stability and generalized Mittag-Leffler stability for describing Lyapunov-like stability of FOSs. On the other hand, Yu et al. in [31] introduced the generalized Mittag-Leffler stability of multi-variable FOSs.

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However, there are some difficulties arising in using Lyapunov-like theory for stability even though the application is not complicated [14]. One of these difficulties is that the parametric stability conditions seem too parsimonious. Burton proposed some Lyapunov functional to prove the FOSs stability [5]. Additionally, Wang et al. [29] introduced the Hyeres-Ulam-Rassias and Hyeres-Ulam stability of FOSs. Still, it is a tedious task to find an appropriate Lyapunov-like functions for FOSs. However, there are some existing possible Lyapunov-like functions in the literature [2, 28, 33]. On the other hand, some Lyapunov-like functions do not describe the state of the system, Mittag-Leffler stability for example, not the real state of the system [9, 26–28]. The Lyapunov-like function which are constructed from Mittag-Leffler stability are usually called fractional Lyapunov functions, and the techniques are referred to as the direct method of the fractional Lyapunov.

So far, there are not many results on FONSs stabilization with the use of Mittag-Leffler stability have been published. In [30], the authors used the eigenvalue analysis to stabilize a linearized FONSs. Another attempt by Lan et al. [11] was to investigate robust stabilization of fractional order nonlinear complex network using the Lyapunov indirect approach. For more details and examples, please see in [2, 10, 29, 30].

It is well-known that backstepping is an efficient methodology to stabilize nonlinear systems and partial differential equations (PDEs) and it has been widely applied in many particular applications.

In this paper, instead of dealing with classical type of partial differential equations, we propose the fractional-order backstepping method for stabilizing fractional partial differential equation (FPDE). To the best of our knowledge, this is the first time in the literature that the backstepping is being used for stabilizing nonlinear FPDE. The semi-discretized fractional-order backstepping approach will be introduced to find the boundary controller function which stabilizes the FPDE by transforming it into an equivalent stable closed loop. We describe and compare two approximations of fractional derivative namely Caputo and Grünwald-Letnikov fractional derivative for different order of q, $q \in (1, 2]$. The analytic form of feedback control law of stabilizing FPDE with two type of fractional derivative (Caputo and Grünwald-Letnikov) definitions are designed via fractional-order backstepping. An illustrative example will be provided to demonstrate the efficiency of the proposed control scheme. The main aim of this contribution is to derive a systematic method of constructing Mittag-Leffler stable closed-loop systems for FPDEs and a global convergence is built into them.

The rest of this paper is organized as follows; some definitions for fractional order calculus and a class of control fractional Lyapunov functions are listed in Section 2. In Section 3, we illustrate the fractional order backstepping approach to stabilize FPDE based on fractional Lyapunov function. Finally, Section 4 provides a numerical example and the result is illustrated and compared using two types of approximations for fractional derivatives. The conclusions are devoted in the last section.

2. Preliminaries

In this section, we introduce some definitions of fractional calculus and the class of control Lyapunov functions which are used in this paper.

Definition 2.1 ([16]). The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D^q f(x) = \frac{1}{\Gamma(m-q)} \int_0^x (x-s)^{m-q-1} f^{(m)}(s)\, ds$$

for $m - 1 < q < m$, $m \in N$, $x > 0$.

Definition 2.2 ([23]). The Grünwald-Letnikov formula, that is for $q > 0$, is

$$D^q f(x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x^q} \sum_{k=0}^{[\frac{x-a}{\Delta x}]} (-1)^k \binom{q}{k} f(x - k\Delta x).$$

The Grünwald-Letnikov definition is a generalization of the ordinary discretization formulas for integer order derivatives.

Theorem 2.3 ([24]). *Assume that both* $f(u)$ *and* $u(t)$ *are* q *times differentiable with* u *and* t*, respectively. The chain rule of fractional derivative can be described as the following equation*

$$\frac{\partial^q f(u(t))}{\partial t^q} = \Gamma(2-q)u^{q-1}\frac{\partial^q f(u)}{\partial u^q}\frac{\partial^q u(t)}{\partial t^q}.$$

Theorem 2.4 ([14], Mittag-Leffler stability). *Let* $u(t) = 0$ *be the equilibrium point of the FOS* $D^q u = f(u,t), u \in \Omega$*, where* Ω *is a neighborhood region of the origin. Assume that there exists a fractional Lyapunov function* $V(t,u(t)) : [0,\infty) \times \mathcal{R}^n \to \mathcal{R}$ *and* K*-class functions* ξ_i, $i = 1,2,3$*, satisfying*

1. $\xi_1(\|u\|) \leqslant V(t,u(t)) \leqslant \xi_2(\|u\|)$,
2. $D^q V(t,u(t)) \leqslant -\xi_3(\|u\|)$.

Then the FOS is asymptotically Mittag-Leffler stable. Moreover, if $\Omega = \mathcal{R}^n$*, the FOS is globally asymptote Mittag-Leffler stable.*

Definition 2.5 ([7]). A smooth function $V(t,u(t)) : [0,\infty) \times \mathcal{R}^n \to \mathcal{R}$ is called a control fractional Lyapunov function for the FOS $D^q u = f(u,U), u \in \mathcal{R}^n, f(0,0) = 0$ with the control law $U = \alpha(u)$ if there exist three K-class functions $\xi_i, i = 1,2,3$ such that

1. $\xi_1(\|u\|) \leqslant V(t,u(t)) \leqslant \xi_2(\|u\|)$,
2. $D^q V(t,u(t)) \leqslant -\xi_3(\|u\|)$.

Lemma 2.6 ([7]). *Let* $u(t) \in \mathcal{R}$ *be a real continuous differentiable function. Then, for any* $r = 2^n, n \in N$,

$$D^q u^r(t) \leqslant r u^{(r-1)}(t) D^q u(t),$$

where $0 < q \leqslant 1$ *is the fractional order.*

Lemma 2.7 ([7]). *For the FOS* $D^q u = f(u,U), u \in \mathcal{R}, 0 < q \leqslant 1, f(0,0) = 0$ *with the control law* $U = \alpha(u)$ *is asymptotically Mittag-Leffler stable if for* $r = 2^n, n \in N$*, there exist a* K*-class function* ξ*, such that*

$$u^{r-1} D^q u = u^{r-1} f(u,\alpha(u)) \leqslant -\xi(\|u\|).$$

Lemma 2.8 ([7]). *For the FOS* $D^q u = f(u,U), u \in \mathcal{R}, 0 < q \leqslant 1, f(0,0) = 0$ *with the control law* $U = \alpha(u)$ *is stable if for* $r = 2^n, n \in N$,

$$u^{r-1} D^q u = u^{r-1} f(u,\alpha(u)) \leqslant 0,$$

and the system with $U = \alpha(u)$ *is asymptotically Mittag-Leffler stable if* $u^{r-1} f(u,\alpha(u)) < 0$.

3. Nonlinear Mittag-Leffler stability for FPDE

Consider the following nonlinear fractional partial differential equation

$$\frac{\partial^q u(x,t)}{\partial t^q} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} + f(u(x,t)),\ 0 < x < 1,\ t \geqslant 0, \tag{3.1}$$

where the fractional order $q \in (0,1], \beta \in (1,2], u \in L^2(\Omega), \Omega = (0,1) \times [0,T], T > 0$ and f is a nonlinear function of u, such that $f \in C^\infty(\mathcal{R})$, with initial condition

$$u(x,0) = g(x),\ 0 < x < 1. \tag{3.2}$$

The boundary condition at $x = 0$ is homogeneous Dirichlet

$$u(0,t) = 0,\ t \geqslant 0, \tag{3.3}$$

and the boundary condition at the other end

$$u(1,t) = U(t),\ t \geqslant 0, \tag{3.4}$$

where $U(t) : C[0,1] \to \mathcal{R}$ is the unknown nonlinear feedback control function to be designed to achieve stabilization.

We propose the nonlinear Mittag-Leffler stabilization of systems (3.1)-(3.4) via the fractional order back-stepping control technique. A recursive design with Caputo derivative definition for discretizing $\frac{\partial^\beta u(x,t)}{\partial x^\beta}$ is provided in Subsection 3.1 while the shifted Grünwald-Letnikov approximation is provided in Subsection 3.2. The goal convergence of the closed loop control system is analyzed via indirect fractional Lyapunov method.

3.1. Mittag-Leffler stabilization with Caputo approximation for FPDE

In this subsection the back-stepping design technique is applied to obtain the boundary control function $U(t)$. The design procedure is divided into three stages as follows.

In the first stage, the nonlinear fractional partial differential Eq. (3.1) will be semi-discretized into an equivalent nonlinear system of fractional differential equation as follows.

Fix $n \in \mathbb{N}$ and $h = \frac{1}{n+1}$ as the step size of discretization of system (3.1)-(3.4) over the interval of the space variable $x \in (0,1)$. Also, let $u_i(t) = u(ih,t)$ for all $i = 0,1,\dots,n+1$, where it is assumed that $u_0(t)$ is the first boundary condition and $u_{n+1}(t)$ is the control function to be evaluated, such that the original system is asymptotically stable. The semi-discretized version of fractional system (3.1)-(3.4) using the Caputo derivative definition is

$$
\begin{aligned}
&u_0(t) = 0,\\
&\frac{d^q u_i}{dt^q} = \frac{h^{-\beta}}{\Gamma(3-\beta)}\sum_{j=0}^{i-1}\mu_{i,j}(u_{j+2}-2u_{j+1}+u_j) + f(u_i), \quad i = 1,\dots,n,\\
&u_{n+1} = U(t),
\end{aligned}
$$

where $\mu_{i,j} = (i-j)^{2-\beta} - (i-j-1)^{2-\beta}$. We can write the nonlinear semi-discretized system of fractional differential equations as

$$
\begin{aligned}
\frac{d^q u_1}{dt^q} &= \frac{h^{-\beta}\mu_{1,0}}{\Gamma(3-\beta)}u_2 - \frac{2h^{-\beta}\mu_{1,0}}{\Gamma(3-\beta)}u_1 + f(u_1),\\
\frac{d^q u_2}{dt^q} &= \frac{h^{-\beta}\mu_{2,1}}{\Gamma(3-\beta)}u_3 + \frac{h^{-\beta}(\mu_{2,0}-2\mu_{2,1})}{\Gamma(3-\beta)}u_2 + \frac{h^{-\beta}(-2\mu_{2,0}+\mu_{2,1})}{\Gamma(3-\beta)}u_1 + f(u_2),\\
&\vdots\\
\frac{d^q u_n}{dt^q} &= \frac{h^{-\beta}\mu_{n,n-1}}{\Gamma(3-\beta)}U + \frac{h^{-\beta}\mu_{n,0}}{\Gamma(3-\beta)}(u_2-2u_1)\\
&\quad + \frac{h^{-\beta}\mu_{n,1}}{\Gamma(3-\beta)}(u_3-2u_2+u_1) + \dots + \frac{h^{-\beta}\mu_{n,n-1}}{\Gamma(3-\beta)}(-2u_n+u_{n-1}) + f(u_n).
\end{aligned}
\tag{3.5}
$$

In the second stage we will design the needed controller according to the idea of back-stepping. The back-stepping design procedure requires n steps, and the virtual control α_i and the controller U will be constructed. The design procedure is elaborated in the following theorem.

Theorem 3.1. *Consider the plant equation (3.5) with order $0 < q < 1$, $1 < \beta \leqslant 2$, and $\frac{h^{-\beta}}{\Gamma(3-\beta)} \neq 0$ for all $u \in \mathcal{R}^n$ and $U : C[0,1] \to \mathcal{R}$ is the nonlinear control functional. Assume that the control fractional Lyapunov function is taken by*

$$
V(w_1,w_2,\dots,w_n) = \frac{h^\beta}{2}\sum_{i=1}^{n} w_i^2, \tag{3.6}
$$

where $w_1 = u_1$, $w_i = u_i - \alpha_{i-1}$, $i = 2,\dots,n$ and $h = \frac{1}{n+1} > 0$ is the step size of discretization of system (3.1)-(3.4), that is there exists a nonlinear feedback control U which renders the closed loop system asymptotically

Mittag-Leffler stable. The nonlinear feedback control functional can be designed as

$$
\begin{aligned}
U =& (2-k_n\Gamma(3-\beta))\,u_n - 2u_{n-1} - \sum_{j=0}^{n-2}\mu_{n,j}\left(u_{j+2}-2u_{j+1}+u_j\right) - h^{\beta}f(u_n) + \alpha_{n-2} \\
&+ k_n\Gamma(3-\beta)\alpha_{n-1} + h^{\beta}\Gamma(3-\beta)\Gamma(2-q)\sum_{j=1}^{n-1}u_j^{q-1}\frac{\partial^q\alpha_{n-1}}{\partial u_j^q}\frac{\partial^q u_j}{\partial t^q},
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\alpha_i =& \Gamma(3-\beta)\Bigg(-k_iw_i - \frac{1}{\Gamma(3-\beta)}w_{i-1} - \frac{1}{\Gamma(3-\beta)}\sum_{j=0}^{i-2}\mu_{i,j}\left(u_{j+2}-2u_{j+1}+u_j\right) \\
&- \frac{1}{\Gamma(3-\beta)}\left(-2u_i+u_{i-1}\right) - h^{\beta}f(u_i) + h^{\beta}\sum_{j=1}^{i-1}\Gamma(2-q)u_j^{q-1}\frac{\partial^q\alpha_{i-1}}{\partial u_j^q}\frac{\partial^q u_j}{\partial t^q}\Bigg), \quad i=3,\ldots,n-1,
\end{aligned} \tag{3.8}
$$

where $k_1,\ldots,k_n>0$ *are constants,* $\mu_{i,j}=(i-j)^{2-\beta}-(i-j-1)^{2-\beta}$.

Proof. By the use of recursion, we have the following steps

Step 1: We start with the first equation in system (3.5). Design a suitable stabilizing function α_1 to stabilize $w_1(t)$. Select the first fractional Lyapunov function

$$V_1 = \frac{h^{\beta}}{2}w_1^2.$$

Then the q-th order derivative of V_1 is given by

$$D^qV_1 \leqslant -k_1w_1^2 + \frac{1}{\Gamma(3-\beta)}w_1w_2 + w_1\left(\frac{1}{\Gamma(3-\beta)}\alpha_1 + k_1w_1 - \frac{2}{\Gamma(3-\beta)}u_1 + h^{\beta}f(u_1)\right).$$

If the virtual control law α_1 is designed as

$$\alpha_1 = \Gamma(3-\beta)\left(-k_1w_1 + \frac{2}{\Gamma(3-\beta)}u_1 - h^{\beta}f(u_1)\right),$$

where $k_1>0$ is a design parameter, w_2 is to be governed to zero, then, the resulting q-th order derivative is

$$D^qV_1 \leqslant -k_1w_1^2 + \frac{1}{\Gamma(3-\beta)}w_1w_2, \quad k_1>0.$$

Step 2: Study the second equation of system (3.5) by considering α_2 as a virtual control variable. The control objective is to make $w_2\to 0$ as $t\to\infty$. Define a second fractional control Lyapunov function as

$$V_2 = V_1 + \frac{h^{\beta}}{2}w_2^2,$$

and its q-th order time derivative is given by

$$
\begin{aligned}
D^qV_2 \leqslant& -k_1w_1^2 - k_2w_2^2 + \frac{1}{\Gamma(3-\beta)}w_2w_3 + w_2\Bigg(\frac{1}{\Gamma(3-\beta)}\alpha_2 + \frac{1}{\Gamma(3-\beta)}w_1 + k_2w_2 \\
&+ \frac{(\mu_{2,0}-2\mu_{2,1})}{\Gamma(3-\beta)}u_2 + \frac{(-2\mu_{2,0}+\mu_{2,1})}{\Gamma(3-\beta)}u_1 + h^{\beta}f(u_2) - h^{\beta}\Gamma(2-q)u_1^{q-1}\frac{\partial^q\alpha_1}{\partial u_1^q}\frac{\partial^q u_1}{\partial t^q}\Bigg).
\end{aligned}
$$

By selecting

$$
\begin{aligned}
\alpha_2 =& \Gamma(3-\beta)\Bigg(-k_2w_2 - \frac{1}{\Gamma(3-\beta)}w_1 - \frac{(\mu_{2,0}-2\mu_{2,1})}{\Gamma(3-\beta)}u_2 - \frac{(-2\mu_{2,0}+\mu_{2,1})}{\Gamma(3-\beta)}u_1 - h^{\beta}f(u_2) \\
&+h^{\beta}\Gamma(2-q)u_1^{q-1}\frac{\partial^q\alpha_1}{\partial u_1^q}\frac{\partial^q u_1}{\partial t^q}\Bigg),
\end{aligned}
$$

where $k_2>0$ is the design parameter, and w_3 is to be governed to zero, we have

$$D^q V_2 \leqslant -\sum_{i=1}^{2} k_i w_i^2 + \frac{1}{\Gamma(3-\beta)} w_2 w_3.$$

Step i ($i = 3, ..., n-1$): Study the i-th equation of system (3.5) with the virtual control variable α_i. The control fractional Lyapunov function is chosen as

$$V_i = V_{i-1} + \frac{h^\beta}{2} w_i^2.$$

Its q-th time derivative is given by

$$\begin{aligned} D^q V_i \leqslant & -\sum_{j=1}^{i} k_j w_j^2 + \frac{1}{\Gamma(3-\beta)} w_i w_{i+1} + w_i \left(\frac{1}{\Gamma(3-\beta)} \alpha_i + k_i w_i + \frac{1}{\Gamma(3-\beta)} w_{i-1} \right. \\ & + \frac{1}{\Gamma(3-\beta)} \sum_{j=0}^{i-2} \mu_{i,j} (u_{j+2} - 2u_{j+1} + u_j) + \frac{\mu_{i,i-1}}{\Gamma(3-\beta)} (-2u_i + u_{i-1}) + h^\beta f(u_i) \\ & \left. - h^\beta \sum_{j=1}^{i-1} \Gamma(2-q) u_j^{q-1} \frac{\partial^q \alpha_{i-1}}{\partial u_j^q} \frac{\partial^q u_j}{\partial t^q} \right). \end{aligned}$$

If we choose α_i as given in Eq. (3.8), w_{i+1} is governed to zero. Then, the resulting q-th order derivative of V_i is

$$D^q V_i \leqslant -\sum_{j=1}^{i} k_j w_j^2 + \frac{1}{\Gamma(3-\beta)} w_i w_{i+1}.$$

At this point, one can conclude that w_i converges to zero asymptotically.

Step n: In the last step, Step n, the actual control U appears and is at our disposal. The aim is to design a suitable control law to make $w_n \to 0$ as $t \to \infty$, select the fractional Lyapunov function as

$$V_n = V_{n-1} + \frac{h^\beta}{2} w_n^2.$$

Then we can obtain the q-th time derivative as

$$\begin{aligned} D^q V_n \leqslant & -\sum_{j=1}^{n} k_j w_j^2 + w_n \left(\frac{1}{\Gamma(3-\beta)} U + k_n w_n + \frac{1}{\Gamma(3-\beta)} w_{n-1} \right. \\ & + \frac{1}{\Gamma(3-\beta)} \sum_{j=0}^{n-2} \mu_{n,j} (u_{j+2} - 2u_{j+1} + u_j) \\ & \left. + \frac{\mu_{n,n-1}}{\Gamma(3-\beta)} (-2u_n + u_{n-1}) + h^\beta f(u_n) - h^\beta \sum_{j=1}^{n-1} \Gamma(2-q) u_j^{q-1} \frac{\partial^q \alpha_{n-1}}{\partial u_j^q} \frac{\partial^q u_j}{\partial t^q} \right). \end{aligned}$$

One control can be chosen by Eq. (3.7). So, we have

$$D^q V_n \leqslant -\sum_{i=1}^{n} k_i w_i^2.$$

In this stage, it is convenient to consider, according to Lemma 2.7, that the closed-loop system is stable regarding to the classical Lyapunov stability. Then, two cases are considered in our work.

1. When $w \neq 0$, we now have $D^q V_n < 0$. There exists K-class function ξ_1 such that
$$D^q V_n \leqslant -\xi_1\left(\|\overline{w}\|\right), \overline{w} = [w_1, \dots, w_n]^T.$$
2. When $w = 0$, we now have $D^q V_n \leqslant 0$. According to the fractional comparison principle [14], we know that
$$D^q V_n \leqslant D^q k \Rightarrow V_n \leqslant k,$$
where $k = V_n(t=0)$ is a positive constant.

According to the first case in Theorem 2.4, the closed-loop system is defined to be asymptotically Mittag-Leffler stable. □

In the third stage, substitute $U(t)$ evaluated by Eq. (3.7) back into system (3.5), for $i = n$, a system of n nonlinear fractional differential equations is obtained. The solution of resulting system may be solved by using any method for solving nonlinear system of fractional differential equations.

3.2. *Mittag-Leffler stabilization with Grünwald-Letnikov approximation for FPDE*

The design procedure that may be followed in general as we made in Subsection 3.1. Using the shifted Grünwald-Letnikov formula for discretizing $\dfrac{\partial^\beta u(x,t)}{\partial x^\beta}$, we get
$$\begin{aligned}
&u_0(t) = 0,\\
&\frac{d^q u_i}{dt^q} = \frac{1}{h^\beta}\sum_{j=0}^{i+1} \mu_j^{(\beta)} u_{i-j+1} + f(u_i), \quad i = 1, \dots, n,\\
&u_{n+1} = U(t),
\end{aligned}$$
where
$$\mu_j^{(\beta)} = (-1)^j \binom{\beta}{j} = (-1)^j \frac{\beta(\beta-1)\dots(\beta-j+1)}{j!}.$$
Similar to the first approximation, the nonlinear semi-discretized system of fractional differential equations is
$$\begin{aligned}
\frac{d^q u_1}{dt^q} &= \frac{1}{h^\beta}u_2 + \frac{\mu_1^{(\beta)}}{h^\beta}u_1 + f(u_1),\\
\frac{d^q u_2}{dt^q} &= \frac{1}{h^\beta}u_3 + \frac{\mu_1^{(\beta)}}{h^\beta}u_2 + \frac{\mu_2^{(\beta)}}{h^\beta}u_1 + f(u_2),\\
&\ \vdots\\
\frac{d^q u_n}{dt^q} &= \frac{1}{h^\beta}U + \frac{\mu_1^{(\beta)}}{h^\beta}u_n + \frac{\mu_2^{(\beta)}}{h^\beta}u_{n-1} + \dots + \frac{\mu_n^{(\beta)}}{h^\beta}u_1 + f(u_n).
\end{aligned} \tag{3.9}$$
We will design the needed controller according to the idea of back-stepping in the following theorem.

Theorem 3.2. *Consider the plant equation (3.9) with order $0 < q < 1$, $1 < \beta \leqslant 2$ and $\frac{1}{h^\beta} \neq 0$ for all $u \in \mathcal{R}^n$ and $U : C[0,1] \to \mathcal{R}$ is the nonlinear control functional. If the control fractional Lyapunov function is taken by Eq. (3.6), then, there exists a nonlinear feedback control U which renders the closed loop system asymptotically Mittag-Leffler stable. The nonlinear control functional can be chosen by*
$$\begin{aligned}
U = &-k_n u_n - u_{n-1} - \sum_{j=1}^{n} \mu_j^{(\beta)} u_{n-j+1} - h^\beta f(u_n) + k_n \alpha_{n-1} + \alpha_{n-2}\\
&+ h^\beta \Gamma(2-q) \sum_{j=1}^{n-1} u_j^{q-1} \frac{\partial^q \alpha_{n-1}}{\partial u_j^q} \frac{\partial^q u_j}{\partial t^q},
\end{aligned} \tag{3.10}$$

$$\begin{aligned}\alpha_i &= -k_i w_i - w_{i-1} - \sum_{j=1}^{i} \mu_j^{(\beta)} u_{i-j+1} - h^\beta f(u_i) \\ &\quad + h^\beta \Gamma(2-q) \sum_{j=1}^{i-1} u_j^{q-1} \frac{\partial^q \alpha_{i-1}}{\partial u_j^q} \frac{\partial^q u_j}{\partial t^q}, \quad i = 3, \dots, n-1,\end{aligned} \tag{3.11}$$

where $k_1, \dots, k_n > 0$ *are constants, and* $\mu_j^{(\beta)} = (-1)^j \frac{\beta(\beta-1)\dots(\beta-j+1)}{j!}$.

Proof. The design procedure is given in the following steps.

Step 1: The first fractional Lyapunov functional is $V_1 = \frac{h^\beta}{2} w_1^2$, and its q-th order time derivative is

$$D^q V_1 < -k_1 w_1^2 + w_1 w_2 + w_1 \left(\alpha_1 + k_1 w_1 + \mu_1^{(\beta)} u_1 + h^\beta f(u_1) \right).$$

By selecting

$$\alpha_1 = -k_1 w_1 - \mu_1^{(\beta)} u_1 - h^\beta f(u_1),$$

w_2 is to be governed to zero. Then, $D^q V_1 < -k_1 w_1^2 + w_1 w_2, k_1 > 0$.

Step 2: Consider the fractional Lyapunov function $V_2 = V_1 + \frac{h^\beta}{2} w_2^2$, and its q-th order time derivative is given by

$$\begin{aligned} D^q V_2 \leqslant &- k_1 w_1^2 - k_2 w_2^2 + w_2 w_3 + w_2 \Big(\alpha_2 + w_1 + k_2 w_2 + \mu_1^{(\beta)} u_2 + \mu_2^{(\beta)} u_1 + h^\beta f(u_2) \\ &- h^\beta \Gamma(2-q) u_1^{q-1} \frac{\partial^q \alpha_1}{\partial u_1^q} \frac{\partial^q u_1}{\partial t^q} \Big). \end{aligned}$$

If the virtual control law α_2 is designed as

$$\alpha_2 = -w_1 - k_2 w_2 - \mu_1^{(\beta)} u_2 - \mu_2^{(\beta)} u_1 - h^\beta f(u_2) + h^\beta \Gamma(2-q) u_1^{q-1} \frac{\partial^q \alpha_1}{\partial u_1^q} \frac{\partial^q u_1}{\partial t^q},$$

w_3 is governed to zero. Then, $D^q V_2 \leqslant -k_1 w_1^2 - k_2 w_2^2 + w_2 w_3$, where $k_i > 0$, $i = 1, 2$ are the design parameters.

Step i ($i = 3, \dots, n-1$): Study the i-th equation of Eq. (3.9) with the virtual control variable α_i. The fractional control Lyapunov function is chosen as

$$V_i = V_{i-1} + \frac{h^\beta}{2} w_i^2.$$

Its q-th order time derivative is given by

$$\begin{aligned} D^q V_i \leqslant &- \sum_{j=1}^{i} k_j w_j^2 + w_i w_{i+1} + w_i \Bigg(\alpha_i + k_i w_i + w_{i-1} + \sum_{j=1}^{i} \mu_j^{(\beta)} u_{i-j+1} + h^\beta f(u_i) \\ &- h^\beta \sum_{j=1}^{i-1} \Gamma(2-q) u_j^{q-1} \frac{\partial^q \alpha_{i-1}}{\partial u_j^q} \frac{\partial^q u_j}{\partial t^q} \Bigg). \end{aligned}$$

If we choose α_i as given by Eq. (3.11), w_{i+1} is governed to zero. Then, we have

$$D^q V_i \leqslant -\sum_{j=1}^{i} k_j w_j^2 + w_i w_{i+1}.$$

Step n: In the last step, Step n, the actual control U appears and is at our disposal. The aim is to design a suitable control law to make $w_n \to 0$ as $t \to \infty$, select the fractional control Lyapunov function as

$$V_n = V_{n-1} + \frac{h^\beta}{2} w_n^2.$$

Then, we can obtain the q-th order time derivative as

$$D^q V_n \leqslant -\sum_{j=1}^{n} k_j w_j^2 + w_n \Bigg(U + k_n w_n + w_{n-1} + \sum_{j=1}^{n} \mu_j^{(\beta)} u_{n-j+1} + h^\beta f(u_n) \\ - h^\beta \sum_{j=1}^{n-1} \Gamma(2-q) u_j^{q-1} \frac{\partial^q \alpha_{n-1}}{\partial u_j^q} \frac{\partial^q u_j}{\partial t^q} \Bigg).$$

The controller can be chosen by Eq. (3.10). Then, the resulting q-th order time derivative of V_n is

$$D^q V_n \leqslant -\sum_{i=1}^{n} k_i w_i^2.$$

The closed loop system is asymptotically Mittag-Leffler stable. □

Finally, substitute $U(t)$ evaluated by Eq. (3.10) back into system (3.9), for $i = n$, a system of n nonlinear fractional differential equations is obtained.

4. Simulation results

In this section, one example is presented to illustrate and compare the obtained controller by using Caputo and Grünwald-Letnikov approximations.

Consider the following nonlinear fractional order partial differential equation

$$\frac{\partial^q u(x,t)}{\partial t^q} = \frac{\partial^\beta u(x,t)}{\partial x^\beta} + u^3(x,t), \quad 0 < x < 1, \quad t > 0, \quad 0 < q \leqslant 1, \quad 1 < \beta \leqslant 2, \tag{4.1}$$

$$u(x,0) = (3x+1)\cos(x), \tag{4.2}$$

$$u(0,t) = 0, \quad u(1,t) = U(t). \tag{4.3}$$

The open loop system (4.1)-(4.3) with $u(1,t) = 0$ is unstable (see Fig. 1).

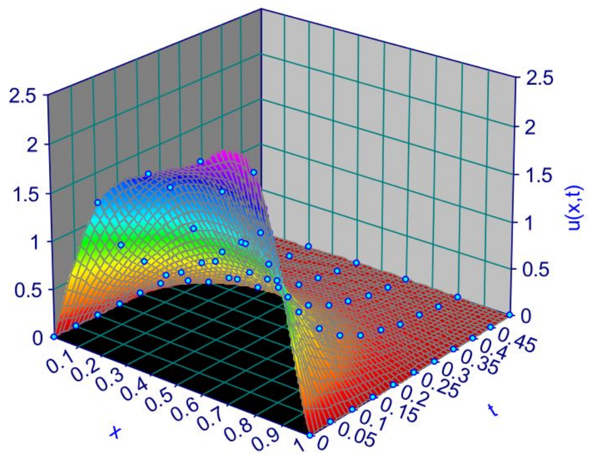

Figure 1: Solution of system (4.1)-(4.3) without control.

Using the Caputo derivative semi-discretization for the space variable will give

$$
\begin{aligned}
\frac{d^q u_1}{dt^q} =& \frac{(0.2)^{-\beta}}{\Gamma(3-\beta)}u_2 - \frac{2(0.2)^{-\beta}}{\Gamma(3-\beta)}u_1 + u_1^3,\\
\frac{d^q u_2}{dt^q} =& \frac{(0.2)^{-\beta}}{\Gamma(3-\beta)}u_3 + \frac{(0.2)^{-\beta}(2^{2-\beta}-3)}{\Gamma(3-\beta)}u_2 + \frac{(0.2)^{-\beta}(-2^{3-\beta}+3)}{\Gamma(3-\beta)}u_1 + u_2^3,\\
\frac{d^q u_3}{dt^q} =& \frac{(0.2)^{-\beta}}{\Gamma(3-\beta)}u_4 + \frac{(0.2)^{-\beta}(2^{2-\beta}-3)}{\Gamma(3-\beta)}u_3 + \frac{(0.2)^{-\beta}(3^{2-\beta}-2^{2-\beta}-2^{3-\beta}+3)}{\Gamma(3-\beta)}u_2\\
&+ \frac{(0.2)^{-\beta}(-2\times 3^{2-\beta}+2^{2-\beta}+2^{3-\beta}-1)}{\Gamma(3-\beta)}u_1 + u_3^3,\\
\frac{d^q u_4}{dt^q} =& \frac{(0.2)^{-\beta}}{\Gamma(3-\beta)}U + \frac{(0.2)^{-\beta}(2^{2-\beta}-3)}{\Gamma(3-\beta)}u_4 + \frac{(0.2)^{-\beta}(3^{2-\beta}-2^{2-\beta}-2^{3-\beta}+3)}{\Gamma(3-\beta)}u_3\\
&+ \frac{(0.2)^{-\beta}(4^{2-\beta}-3^{2-\beta}-2\times 3^{2-\beta}+2^{2-\beta}+2^{3-\beta}-1)}{\Gamma(3-\beta)}u_2\\
&+ \frac{(0.2)^{-\beta}(-2\times 4^{2-\beta}+2\times 3^{2-\beta}+3^{2-\beta}-2^{2-\beta})}{\Gamma(3-\beta)}u_1 + u_4^3.
\end{aligned}
\tag{4.4}
$$

The feedback control law can be designed as

$$
\begin{aligned}
U(t) =& (2-k_4\Gamma(3-\beta)-2^{2-\beta}+3)u_4 - (3^{2-\beta}-2^{2-\beta}-2^{3-\beta}+5\\
&+ k_4\Gamma(3-\beta)(-k_3\Gamma(3-\beta)+\Gamma(3-\beta)(-k_1-k_2)-2^{3-\beta}+8))u_3\\
&+ (\Gamma(3-\beta)(-k_1-k_2)-2^{2-\beta}+6-4^{2-\beta}+3^{2-\beta}+2\times 3^{2-\beta}-2^{3-\beta}-2^{2-\beta}\\
&+ k_4\Gamma(3-\beta)(k_3\Gamma(3-\beta)(\Gamma(3-\beta)(-k_1-k_2)-2^{2-\beta}+5)-3^{2-\beta}+2^{2-\beta}+2^{3-\beta}\\
&+ \Gamma(3-\beta)(-k_1k_2\Gamma(3-\beta)+2k_1+2k_2)+2^{3-\beta}-12+(2^{2-\beta}-3)(\Gamma(3-\beta)(-k_1-k_2)\\
&- 2^{2-\beta}+5)))u_2 + (\Gamma(3-\beta)(-k_1k_2\Gamma(3-\beta)+2k_1+2k_2)+2^{3-\beta}-8\\
&+ 2\times 4^{2-\beta}-2\times 3^{2-\beta}-3^{2-\beta}+2^{2-\beta}+k_4\Gamma(3-\beta)(k_3\Gamma(3-\beta)(\Gamma(3-\beta)(-k_1k_2\Gamma(3-\beta)\\
&+ 2k_1+2k_2)+2^{3-\beta}-8)-k_1\Gamma(3-\beta)+19+2\times 3^{2-\beta}-2^{3-\beta}+2^{2-\beta}-2\Gamma(3-\beta)\\
&\times(-k_1k_2\Gamma(3-\beta)+2k_1+2k_2)-2^{4-\beta}+(-2^{3-\beta}+3)(\Gamma(3-\beta)(-k_1-k_2)\\
&- 2^{2-\beta}+5)))u_1 + \Bigg(-(0.2)^{\beta}k_2(\Gamma(3-\beta))^2+(0.2)^{\beta}\Gamma(3-\beta)(2-k_1\Gamma(3-\beta))\\
&+ \frac{12(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)} + k_4\Gamma(3-\beta)\Bigg(k_3\Gamma(3-\beta)\Bigg(-(0.2)^{\beta}k_2(\Gamma(3-\beta))^2\\
&+ (0.2)^{\beta}\Gamma(3-\beta)(2-k_1\Gamma(3-\beta)) + \frac{12(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}\Bigg) - \frac{24(0.2)^{\beta}\Gamma(3-\beta)\Gamma(2-q)}{\Gamma(4-q)}\\
&- \frac{12(0.2)^{\beta}(\Gamma(3-\beta))^2\Gamma(2-q)(k_1+k_2)}{\Gamma(4-q)} - \frac{144(0.2)^{\beta}(\Gamma(2-q))^2\Gamma(3-\beta)}{(\Gamma(4-q))^2}\\
&+ (0.2)^{\beta}(\Gamma(3-\beta))^2(-k_1k_2\Gamma(3-\beta)+2k_1+2k_2)+2^{3-\beta}(0.2)^{\beta}(\Gamma(3-\beta))^2\\
&- 8(0.2)^{\beta}\Gamma(3-\beta) - \frac{6(-2^{3-\beta}+3)(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)} - (0.2)^{\beta}\Gamma(3-\beta)\Bigg)\Bigg)u_1^3\\
&+ \Bigg(-(0.2)^{\beta}\Gamma(3-\beta)+k_4\Gamma(3-\beta)\Bigg(-(0.2)^{\beta}(\Gamma(3-\beta))^2k_3+\Gamma(3-\beta)(-k_1-k_2)
\end{aligned}
$$

$$
\begin{aligned}
&-2^{2-\beta}+5-\frac{2(2^{2-\beta}-3)(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(3-q)}\bigg)\bigg)u_2^3\\
&+\bigg(-\frac{6(0.2)^{2\beta}\Gamma(2-q)(\Gamma(3-\beta))^2}{\Gamma(4-q)}+k_4\Gamma(3-\beta)\bigg(-\frac{6(0.2)^{2\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_3}{\Gamma(4-q)}\\
&+\frac{12(0.2)^{2\beta}(\Gamma(3-\beta))^2\Gamma(2-q)}{\Gamma(4-q)}-\frac{6(0.2)^{2\beta}(\Gamma(3-\beta))^3\Gamma(2-q)(k_1+k_2)}{\Gamma(4-q)}\\
&+\frac{72(0.2)^{2\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^2}{(\Gamma(4-q))^2}+\frac{1440(0.2)^{2\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^2}{\Gamma(4-q)\Gamma(6-q)}\bigg)\bigg)u_1^5\\
&-\frac{2(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_4}{\Gamma(3-q)}u_2^5-\frac{720(0.2)^{3\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^4k_4}{\Gamma(4-q)\Gamma(6-q)}u_1^7\\
&+\bigg(-\frac{6(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}+\frac{(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_4}{\Gamma(4-q)}\bigg(12-6\Gamma(3-\beta)(k_1+k_2)\\
&+\frac{72\Gamma(2-q)}{\Gamma(4-q)}+\frac{24\Gamma(2-q)}{\Gamma(3-q)}-6(2^{2-\beta}-3)\bigg)\bigg)u_1^2u_2-(0.2)^{\beta}(\Gamma(3-\beta))^2k_4u_3^3-(0.2)^{\beta}u_4^3\\
&+\bigg(-\frac{720(0.2)^{\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^3k_4}{\Gamma(4-q)\Gamma(6-q)}-\frac{12(0.2)^{2\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^3k_4}{\Gamma(3-q)\Gamma(4-q)}\bigg)u_1^4u_2\\
&+\bigg(-\frac{12(0.2)^{\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^2k_4}{\Gamma(3-q)\Gamma(4-q)}-\frac{2(-2^{3-\beta}+3)(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_4}{\Gamma(3-q)}\bigg)u_1u_2^2\\
&-\frac{6(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_4}{\Gamma(4-q)}u_1^2u_3-\frac{2(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_4}{\Gamma(3-q)}u_2^2u_3\\
&-\frac{6(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^2k_4}{\Gamma(4-q)}u_1^2u_2^3\\
&+\bigg(k_3\Gamma(3-\beta)(\Gamma(3-\beta)(-k_1k_2\Gamma(3-\beta)+2k_1+2k_2)+2^{3-\beta}-8)-k_1\Gamma(3-\beta)+19\\
&+2\times3^{2-\beta}-2^{3-\beta}+2^{2-\beta}-2\Gamma(3-\beta)(-k_1k_2\Gamma(3-\beta)+2k_1+2k_2)-2^{4-\beta}\\
&+(-2^{3-\beta}+3)(\Gamma(3-\beta)(-k_1-k_2)-2^{2-\beta}+5) \qquad (4.5)\\
&+\frac{6\Gamma(2-q)}{\Gamma(4-q)}\bigg(k_3\Gamma(3-\beta)-(0.2)^{\beta}k_2(\Gamma(3-\beta))^2+(0.2)^{\beta}\Gamma(3-\beta)(2-k_1\Gamma(3-\beta))\\
&+\frac{12(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}(1-\Gamma(3-\beta)(k_1+k_2))-\frac{24(0.2)^{\beta}\Gamma(3-\beta)\Gamma(2-q)}{\Gamma(4-q)}\\
&-\frac{144(0.2)^{\beta}(\Gamma(2-q))^2\Gamma(3-\beta)}{(\Gamma(4-q))^2}+(0.2)^{\beta}(\Gamma(3-\beta))^2(-k_1k_2\Gamma(3-\beta)\\
&+2k_1+2k_2)+2^{3-\beta}(0.2)^{\beta}(\Gamma(3-\beta))^2-8(0.2)^{\beta}\Gamma(3-\beta)-(0.2)^{\beta}\Gamma(3-\beta)\\
&-\frac{6(-2^{3-\beta}+3)(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}\bigg)\bigg)u_1^2\\
&+\frac{120\Gamma(2-q)}{\Gamma(6-q)}\bigg(-\frac{6(0.2)^{2\beta}\Gamma(2-q)(\Gamma(3-\beta))^3k_3}{\Gamma(4-q)}+\frac{12(0.2)^{2\beta}(\Gamma(3-\beta))^2\Gamma(2-q)}{\Gamma(4-q)}\\
&-\frac{6(0.2)^{2\beta}(\Gamma(3-\beta))^3\Gamma(2-q)(k_1+k_2)}{\Gamma(4-q)}+\frac{72(0.2)^{2\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^2}{(\Gamma(4-q))^2}\\
&+\frac{1440(0.2)^{2\beta}(\Gamma(2-q))^2(\Gamma(3-\beta))^2}{\Gamma(4-q)\Gamma(6-q)}\bigg)u_1^4-\frac{720(0.2)^{3\beta}\Gamma(8)(\Gamma(2-q))^3(\Gamma(3-\beta))^3}{\Gamma(4-q)\Gamma(6-q)\Gamma(8-q)}u_1^6
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\Gamma(2-q)}{\Gamma(3-q)}\Big(-\frac{6(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^{2}k_{3}}{\Gamma(4-q)}+\frac{12(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}\\
&-\frac{6(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^{2}(k_{1}+k_{2})}{\Gamma(4-q)}+\frac{72(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{(\Gamma(4-q))^{2}}\\
&+\frac{24(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(4-q)}-\frac{6(0.2)^{\beta}(2^{2-\beta}-3)\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}u_{1}u_{2}\\
&+\frac{24\Gamma(2-q)}{\Gamma(5-q)}\Big(-\frac{720(0.2)^{\beta}(\Gamma(2-q))^{2}(\Gamma(3-\beta))^{2}}{\Gamma(4-q)\Gamma(6-q)}-\frac{12(0.2)^{2\beta}(\Gamma(2-q))^{2}(\Gamma(3-\beta))^{2}}{\Gamma(3-q)\Gamma(4-q)}\Big)u_{1}^{3}u_{2}\\
&+\Big(\frac{12(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(4-q)}-\frac{2(-2^{3-\beta}+3)(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(3-q)}u_{2}^{2}\\
&-\frac{12(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(4-q)}u_{1}u_{3}-\frac{12(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(4-q)}u_{1}u_{2}^{3}\Big)\\
&\times(u_{2}-2u_{1}+(0.2)^{\beta}\Gamma(3-\beta)u_{1}^{3})\\
&+\Big(k_{3}\Gamma(3-\beta)(\Gamma(3-\beta)(-k_{1}-k_{2})-2^{2-\beta}+5)-3^{2-\beta}+2^{2-\beta}+2^{3-\beta}\\
&+\Gamma(3-\beta)(-k_{1}k_{2}\Gamma(3-\beta)+2k_{1}+2k_{2})+2^{3-\beta}-12+(2^{2-\beta}-3)\Gamma(3-\beta)(-k_{1}+k_{2})\\
&-2^{2-\beta}+5+\frac{6\Gamma(2-q)}{\Gamma(4-q)}\Big((-0.2)^{\beta}(\Gamma(3-\beta))^{2}k_{3}+\Gamma(3-\beta)(-k_{1}-k_{2})-2^{2-\beta}+5\\
&-\frac{2(2^{2-\beta}-3)(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(3-q)}\Big)u_{2}^{2}-\frac{240(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(6-q)}u_{2}^{4}\\
&+\Big(-\frac{6(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^{2}k_{3}}{\Gamma(4-q)}+\frac{12(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}\\
&-\frac{6(0.2)^{\beta}\Gamma(2-q)(\Gamma(3-\beta))^{2}(k_{1}+k_{2})}{\Gamma(4-q)}+\frac{72(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{(\Gamma(4-q))^{2}}\\
&+\frac{24(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(4-q)}-\frac{6(0.2)^{\beta}(2^{2-\beta}-3)\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}\Big)u_{1}^{2}\\
&+\Big(-\frac{720(0.2)^{\beta}(\Gamma(2-q))^{2}(\Gamma(3-\beta))^{2}}{\Gamma(4-q)\Gamma(6-q)}-\frac{12(0.2)^{2\beta}(\Gamma(2-q))^{2}(\Gamma(3-\beta))^{2}}{\Gamma(3-q)\Gamma(4-q)}u_{1}^{4}\\
&+\frac{2\Gamma(2-q)}{\Gamma(3-q)}\Big(-\frac{12(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{\Gamma(3-q)\Gamma(4-q)}-\frac{2(-2^{3-\beta}+3)(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(3-q)}\Big)u_{1}u_{2}\\
&-\frac{4(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{(\Gamma(3-q))^{2}}u_{2}u_{3}-\frac{36(0.2)^{\beta}(\Gamma(2-q))^{2}\Gamma(3-\beta)}{(\Gamma(4-q))^{2}}u_{1}^{2}u_{2}^{2}\Big)\\
&\times(u_{3}+(2^{2-\beta}-3)u_{2}+(-2^{3-\beta}+3)u_{1}+(0.2)^{\beta}\Gamma(3-\beta)u_{2}^{3})\\
&+\Big(-k_{3}\Gamma(3-\beta)+\Gamma(3-\beta)(-k_{1}-k_{2})-2^{3-\beta}+8-\frac{6(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}u_{3}^{2}\\
&-\frac{6(0.2)^{\beta}\Gamma(2-q)\Gamma(3-\beta)}{\Gamma(4-q)}u_{1}^{2}-\frac{2(0.2)^{\beta}\Gamma(3-\beta)\Gamma(2-q)}{\Gamma(3-q)}u_{2}^{2}\\
&\times\Big(u_{4}+(2^{2-\beta}-3)u_{3}+(3^{2-\beta}-2^{2-\beta}-2^{3-\beta}+3)u_{2}\\
&+(-2\times 3^{2-\beta}+2^{2-\beta}+2^{3-\beta}-1)u_{1}+(0.2)^{\beta}\Gamma(3-\beta)u_{3}^{3}\Big).
\end{aligned}
$$

Fig. 2 illustrates the solution of $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$, while the controlled function $U(t)$ is presented in Fig. 3 and Fig. 4, illustrating the solution of system (4.4) with the initial condition $u(x, 0) = (3x+1)\cos(x)$, which is equivalent to the solution of the original FPDE (4.1) with order $q = 0.75$

and $\beta = 1.5$. In the simulation, the design parameters are set as $k_1 = 1$, $k_2 = 4$, $k_3 = 2$, and $k_4 = 1$.

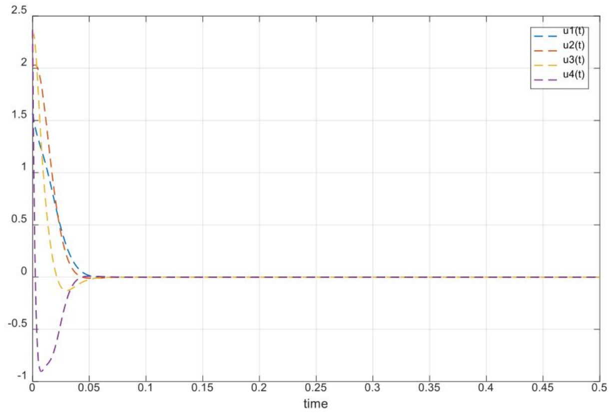

Figure 2: The solutions of $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$ of system (4.4) with $q = 0.75$, $\beta = 1.5$.

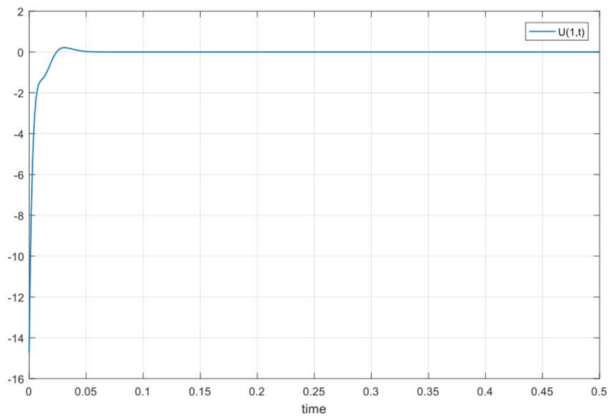

Figure 3: The controller $U(t)$ of Eq. (4.5) with $q = 0.75$, $\beta = 1.5$.

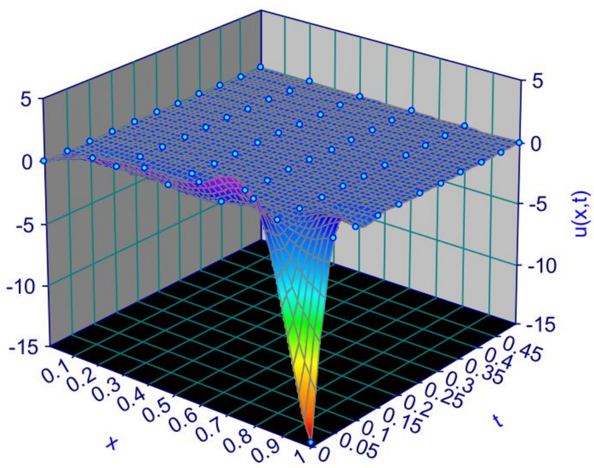

Figure 4: Closed-loop response with controller when $q = 0.75$, $\beta = 1.5$.

Now, we will stabilize system (4.4) by using the shifted Grünwald-Letnikov formula for discretization of the space variable, obtaining

$$\begin{aligned}
\frac{d^q u_1}{dt^q} &= \frac{1}{(0.2)^\beta}u_2 + \frac{\mu_1^{(\beta)}}{(0.2)^\beta}u_1 + u_1^3,\\
\frac{d^q u_2}{dt^q} &= \frac{1}{(0.2)^\beta}u_3 + \frac{\mu_1^{(\beta)}}{(0.2)^\beta}u_2 + \frac{\mu_2^{(\beta)}}{(0.2)^\beta}u_1 + u_2^3,\\
\frac{d^q u_3}{dt^q} &= \frac{1}{(0.2)^\beta}u_4 + \frac{\mu_1^{(\beta)}}{(0.2)^\beta}u_3 + \frac{\mu_2^{(\beta)}}{(0.2)^\beta}u_2 + \frac{\mu_3^{(\beta)}}{(0.2)^\beta}u_1 + u_3^3,\\
\frac{d^q u_4}{dt^q} &= \frac{1}{(0.2)^\beta}U + \frac{\mu_1^{(\beta)}}{(0.2)^\beta}u_4 + \frac{\mu_2^{(\beta)}}{(0.2)^\beta}u_3 + \frac{\mu_3^{(\beta)}}{(0.2)^\beta}u_2 + \frac{\mu_4^{(\beta)}}{(0.2)^\beta}u_1 + u_4^3.
\end{aligned} \tag{4.6}$$

The feedback control law can be designed as

$$\begin{aligned}
U(t) =& (-k_4 - \mu_1^{(\beta)})u_4 + (-1 - \mu_2^{(\beta)} - k_1k_4 - k_2k_4 - k_3k_4 - 3\mu_1^{(\beta)}k_4)u_3\\
&+ (-\mu_3^{(\beta)} - k_1 - k_2 - 2\mu_1^{(\beta)} - k_1k_3k_4 - k_2k_3k_4 - 2\mu_1^{(\beta)}k_3k_4 - 2k_4 - 2\mu_2^{(\beta)}k_4\\
&- 2k_1k_4\mu_1^{(\beta)} - 2k_2k_4\mu_1^{(\beta)} - 3(\mu_1^{(\beta)})^2k_4 - k_1k_2k_4)u_2\\
&+ (-\mu_4^{(\beta)} - 1 - \mu_2^{(\beta)} - k_1k_2 - \mu_1^{(\beta)}k_2 - \mu_1^{(\beta)}k_1 - (\mu_1^{(\beta)})^2 - k_3k_4 - \mu_2^{(\beta)}k_3k_4\\
&- k_1k_2k_3k_4 - \mu_1^{(\beta)}k_2k_3k_4 - \mu_1^{(\beta)}k_1k_3k_4 - (\mu_1^{(\beta)})^2k_3k_4 - k_1k_4 - 2\mu_1^{(\beta)}k_4\\
&+ \mu_3^{(\beta)}k_4 - \mu_1^{(\beta)}\mu_2^{(\beta)}k_4 - k_1k_2k_4\mu_1^{(\beta)} - (\mu_1^{(\beta)})^2k_2k_4 - (\mu_1^{(\beta)})^2k_1k_4\\
&- (\mu_1^{(\beta)})^3k_4 - \mu_2^{(\beta)}k_1k_4 - \mu_2^{(\beta)}k_2k_4 - 2\mu_1^{(\beta)}\mu_2^{(\beta)}k_4)u_1\\
&+ \Bigg(- (0.2)^\beta k_2 + (0.2)^\beta(-k_1 - \mu_1^{(\beta)}) - \frac{6\mu_1^{(\beta)}(0.2)^\beta\Gamma(2-q)}{\Gamma(4-q)} - (0.2)^\beta k_2k_3k_4\\
&+ (0.2)^\beta k_4(-k_1k_3 - \mu_1^{(\beta)}k_3) - \frac{6\mu_1^{(\beta)}k_3k_4(0.2)^\beta\Gamma(2-q)}{\Gamma(4-q)} - \frac{6(0.2)^\beta\mu_2^{(\beta)}k_4\Gamma(2-q)}{\Gamma(4-q)}\\
&+ \frac{6\Gamma(2-q)\mu_1^{(\beta)}k_4}{\Gamma(4-q)}\Bigg(- (0.2)^\beta k_2 + (0.2)^\beta(-k_1 - \mu_1^{(\beta)}) - \frac{6\mu_1^{(\beta)}(0.2)^\beta\Gamma(2-q)}{\Gamma(4-q)}\Bigg)\\
&+ (0.2)^\beta k_4(-2 - \mu_2^{(\beta)} - k_1k_2 - \mu_1^{(\beta)}k_2 - k_1\mu_1^{(\beta)} - (\mu_1^{(\beta)})^2)\Bigg)u_1^3\\
&+ \Bigg(- (0.2)^\beta - (0.2)^\beta k_3k_4 - \frac{6\mu_1^{(\beta)}(0.2)^\beta k_4\Gamma(2-q)}{\Gamma(4-q)} + (0.2)^\beta(-k_1k_4 - k_2k_4 - 2\mu_1^{(\beta)}k_4)\Bigg)u_2^3\\
&- (0.2)^\beta k_4u_3^3 - (0.2)^\beta u_4^3 + \Bigg(\frac{-6(0.2)^{2\beta}\Gamma(2-q)}{\Gamma(4-q)} - \frac{6(0.2)^{2\beta}k_3k_4\Gamma(2-q)}{\Gamma(4-q)}\\
&- \frac{720\mu_1^{(\beta)}(0.2)^{2\beta}k_4(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)} - \frac{6\mu_1^{(\beta)}(0.2)^\beta\Gamma(2-q)k_4}{\Gamma(4-q)}\Bigg)u_1^5\\
&- \frac{6(0.2)^{2\beta}k_4\Gamma(2-q)}{\Gamma(4-q)}u_2^5 - \frac{720k_4(0.2)^{3\beta}(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)}u_1^7\\
&+ \Bigg(- \frac{6(0.2)^\beta\Gamma(2-q)}{\Gamma(4-q)} - \frac{6(0.2)^\beta\Gamma(2-q)k_3k_4}{\Gamma(4-q)} - \frac{6\mu_1^{(\beta)}k_4(0.2)^\beta\Gamma(2-q)}{\Gamma(4-q)}\\
&+ \frac{6\Gamma(2-q)k_4}{\Gamma(4-q)}\Bigg(- (0.2)^\beta k_2 + (0.2)^\beta(-k_1 - \mu_1^{(\beta)}) - \frac{6\mu_1^{(\beta)}(0.2)^\beta\Gamma(2-q)}{\Gamma(4-q)}\Bigg)\\
&- \frac{12\mu_1^{(\beta)}(0.2)^\beta(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}\Bigg)u_1^2u_2\\
&+ \Bigg(- \frac{720(0.2)^{2\beta}k_4(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)} - \frac{12(0.2)^{2\beta}k_4(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}\Bigg)u_1^4u_2
\end{aligned}$$

$$
\begin{aligned}
&+\Big(-\frac{12(0.2)^{\beta}(\Gamma(2-q))^2k_4}{\Gamma(3-q)\Gamma(4-q)}-\frac{6\mu_2^{(\beta)}(0.2)^{\beta}k_4\Gamma(2-q)}{\Gamma(4-q)}\Big)u_1u_2\\
&-\frac{6(0.2)^{\beta}k_4\Gamma(2-q)}{\Gamma(4-q)}u_1^2u_3-\frac{6(0.2)^{\beta}k_4\Gamma(2-q)}{\Gamma(4-q)}u_2^2u_3\\
&-\frac{6(0.2)^{2\beta}k_4\Gamma(2-q)}{\Gamma(4-q)}u_1^2u_2^3+\Big(-k_3-\mu_2^{(\beta)}k_3-k_1k_2k_3-\mu_1^{(\beta)}k_2k_3-\mu_1^{(\beta)}k_1k_3\\
&-(\mu_1^{(\beta)})^2k_3-k_1-2\mu_1^{(\beta)}+\mu_3^{(\beta)}-\mu_1^{(\beta)}\mu_2^{(\beta)}-k_1k_2\mu_1^{(\beta)}-(\mu_1^{(\beta)})^2k_2-(\mu_1^{(\beta)})^2k_1 \qquad (4.7)\\
&-(\mu_1^{(\beta)})^3-k_1\mu_2^{(\beta)}-k_2\mu_2^{(\beta)}-2\mu_1^{(\beta)}\mu_2^{(\beta)}+\frac{6\Gamma(2-q)}{\Gamma(4-q)}\Big(-(0.2)^{\beta}k_2k_3+(0.2)^{\beta}(-k_1k_3\\
&-\mu_1k_3)-\frac{6\mu_1^{(\beta)}k_3(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}-\frac{6(0.2)^{\beta}\mu_2\Gamma(2-q)}{\Gamma(4-q)}+\frac{6\Gamma(2-q)\mu_1^{(\beta)}}{\Gamma(4-q)}\Big(-(0.2)^{\beta}k_2\\
&+(0.2)^{\beta}(-k_1-\mu_1^{(\beta)})-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\Big)\\
&+(0.2)^{\beta}(-2-\mu_2^{(\beta)}-k_1k_2-\mu_1^{(\beta)}k_2-k_1\mu_1^{(\beta)}-(\mu_1^{(\beta)})^2)\Big)u_1^2\\
&+\frac{120\Gamma(2-q)}{\Gamma(6-q)}\Big(-\frac{6(0.2)^{2\beta}k_3\Gamma(2-q)}{\Gamma(4-q)}-\frac{720\mu_1^{(\beta)}(0.2)^{2\beta}(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)}-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\Big)u_1^4\\
&-\frac{720\Gamma(8)(0.2)^{3\beta}(\Gamma(2-q))^3}{\Gamma(4-q)\Gamma(6-q)\Gamma(8-q)}u_1^6+\frac{2\Gamma(2-q)}{\Gamma(3-q)}\Big(-\frac{6(0.2)^{\beta}\Gamma(2-q)k_3}{\Gamma(4-q)}\\
&-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}+\frac{6\Gamma(2-q)}{\Gamma(4-q)}\Big(-(0.2)^{\beta}k_2+(0.2)^{\beta}(-k_1-\mu_1^{(\beta)})-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\Big)\\
&-\frac{12\mu_1^{(\beta)}(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}\Big)u_1u_2\\
&+\frac{24\Gamma(2-q)}{\Gamma(5-q)}\Big(-\frac{720(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)}-\frac{12(0.2)^{2\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}\Big)u_1^3u_2\\
&+\Big(-\frac{12(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}-\frac{6\mu_2^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\Big)u_2^2-\frac{12(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}u_1u_3\\
&-\frac{12(0.2)^{2\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}u_1u_2^3\Big)(u_2+\mu_1^{(\beta)}u_1+(0.2)^{\beta}u_1^3)\\
&+\Big(-k_1k_3-k_2k_3-2\mu_1^{(\beta)}k_3-2-2\mu_2^{(\beta)}-2k_1\mu_1^{(\beta)}-2k_2\mu_1^{(\beta)}-3(\mu_1^{(\beta)})^2-k_1k_2\\
&+\frac{6\Gamma(2-q)}{\Gamma(4-q)}\Big(-(0.2)^{\beta}k_3-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}+(0.2)^{\beta}(-k_1-k_2-2\mu_1^{(\beta)})\Big)u_2^2\\
&-\frac{720(0.2)^{2\beta}(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)}u_2^4+\Big(-\frac{6(0.2)^{\beta}\Gamma(2-q)k_3}{\Gamma(4-q)}-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\\
&+\frac{6\Gamma(2-q)}{\Gamma(4-q)}\Big(-(0.2)^{\beta}k_2+(0.2)^{\beta}(-k_1-\mu_1^{(\beta)})-\frac{6\mu_1^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\Big)\\
&-\frac{12\mu_1^{(\beta)}(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}\Big)u_1^2+\Big(-\frac{720(0.2)^{2\beta}(\Gamma(2-q))^2}{\Gamma(4-q)\Gamma(6-q)}-\frac{12(0.2)^{2\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}\Big)u_1^4\\
&+\frac{2\Gamma(2-q)}{\Gamma(3-q)}\Big(-\frac{12(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}-\frac{6\mu_2^{(\beta)}(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}\Big)u_1u_2
\end{aligned}
$$

$$-\frac{12(0.2)^{\beta}(\Gamma(2-q))^2}{\Gamma(3-q)\Gamma(4-q)}u_2u_3-\frac{36(0.2)^{2\beta}(\Gamma(2-q))^2}{(\Gamma(4-q))^2}u_1^2u_2^2\Big)\Big(u_3+\mu_1^{(\beta)}u_2$$
$$+\mu_2^{(\beta)}u_1+(0.2)^{\beta}u_2^3\Big)+\Big(-k_1-k_2-k_3-3\mu_1^{(\beta)}-\frac{(0.2)^{\beta}6\Gamma(2-q)}{\Gamma(4-q)}u_3^2$$
$$-\frac{6(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}u_1^2-\frac{6(0.2)^{\beta}\Gamma(2-q)}{\Gamma(4-q)}u_2^2\Big)\Big(u_4+\mu_1^{(\beta)}u_3+\mu_2^{(\beta)}u_2+\mu_3^{(\beta)}u_1+(0.2)^{\beta}u_3^3\Big).$$

Fig. 5 illustrates the solution of $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$, while the controlled function $U(t)$ is presented in Figs. 6 and 7, illustrating the solution of system (4.6).

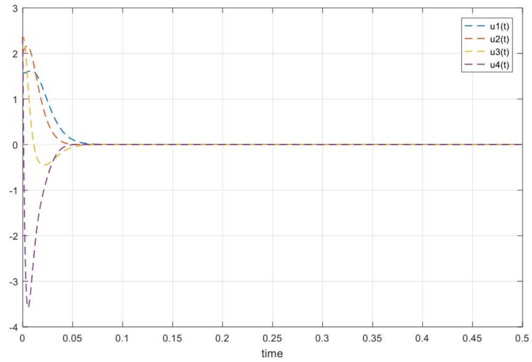

Figure 5: The solutions of $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$ of system (4.6) with $q = 0.75$, $\beta = 1.5$.

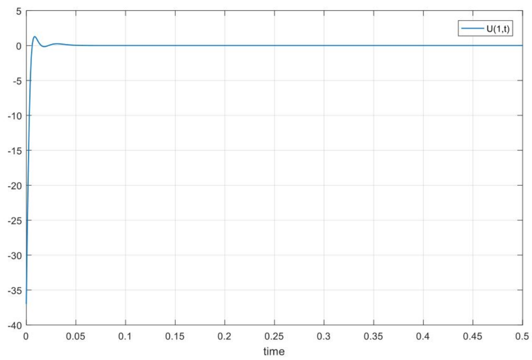

Figure 6: The controller $U(t)$ of Eq. (4.7) with $q = 0.75$, $\beta = 1.5$.

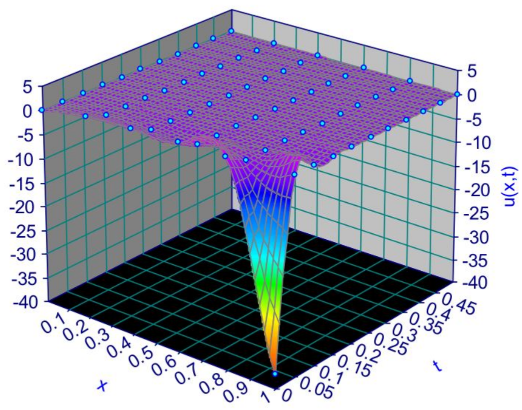

Figure 7: Closed-loop response with controller by using Grünwald-Letnikov approximations when $q = 0.75$, $\beta = 1.5$.

Fig. 8 illustrates the solution of $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$, while the controlled function $U(t)$ is presented in Figs. 9 and 10, illustrating the solution of the system, which is equivalent to the solution of the original FPDE (4.1) with order $q = 1$, $\beta = 2$.

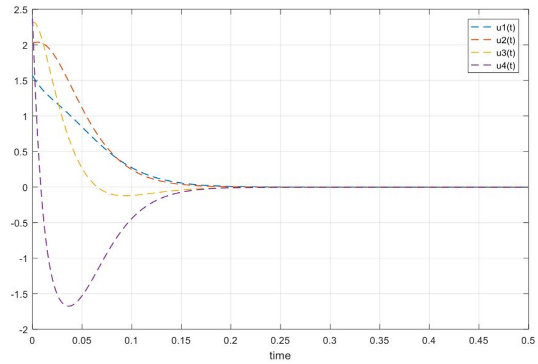

Figure 8: The solutions of $u_1(t)$, $u_2(t)$, $u_3(t)$, and $u_4(t)$ of system (4.4) with $q = 1$, $\beta = 2$.

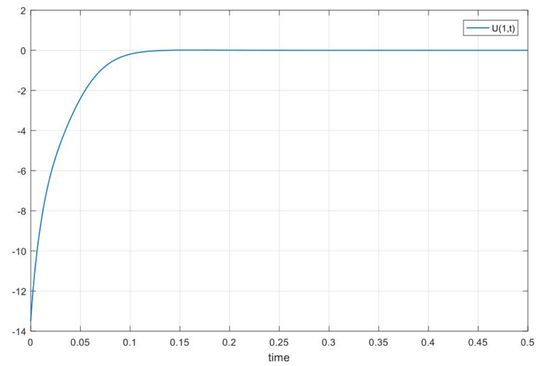

Figure 9: The controller $U(t)$ with $q = 1$, $\beta = 2$.

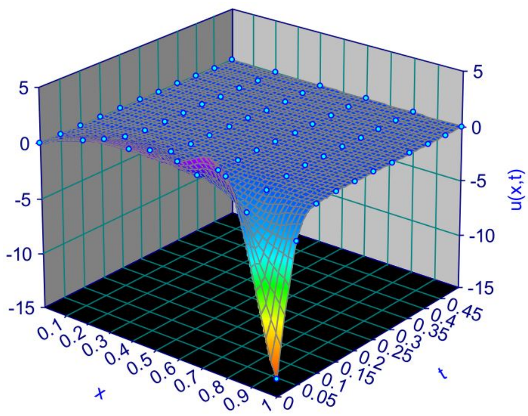

Figure 10: Closed-loop response with controller when $q = 1$, $\beta = 2$.

5. Conclusions

In this paper, the discretized fractional-order back-stepping technique has been developed for stabilizing the nonlinear FPDE with two types of fractional derivatives (Caputo and Grünwald-Letnikov) definitions. With this technique an effective boundary controller can be designed for FPDE. The design procedures consist of three stages are constructed such that the boundary controller can always be constructed with appropriate choices of some design parameters. The analytical forms of control law by using two types of fractional derivatives are presented. Through simulation, it has been established that our results are in excellent agreement for two types of fractional derivatives, where the result obtained by using Caputo approximation performs better, but the symbolic calculation of the virtual control becomes demanding computationally for increasing values of n and it is depending on the complexity of the nonlinear function.

For future work, one can assume more applications of the proposed procedure for other types of FPDEs, such as fractional hyperbolic and fractional elliptic partial differential equations.

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