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On $S_{\alpha}^{\beta}(\theta)$-convergence and strong $N_{\alpha}^{\beta}(\theta,p)$-summability

Hacer Şengül

Department of Mathematics, Siirt University 56100, Siirt, Turkey.

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Abstract

In the papers [M. Et, H. Şengül, Filomat, **28** (2014), 1593–1602] and [H. Şengül, M. Et, Acta Math. Sci. Ser. B Engl. Ed., **34** (2014), 473–482], we defined the spaces of $S^{\alpha}(\theta)$-convergent and strongly $N^{\alpha}(\theta,p)$-summable sequences. In this paper these spaces are generalized to the space of $S_{\alpha}^{\beta}(\theta)$-convergent sequences and the space of strongly $N_{\alpha}^{\beta}(\theta,p)$-summable sequences and are given some inclusion relationships among these spaces.

Keywords: Lacunary sequence, statistical convergence, Cesàro summability.
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1. Introduction

In 1951, Steinhaus [27] and Fast [15] introduced the concept of statistical convergence and later in 1959, Schoenberg [24] reintroduced independently. Caserta et al. [4], Çakallı [3], Connor [9], Çolak [6], Et et al. [5, 11–13], Fridy [17], Salat [21], Altınok et al. [2], Çolak et al. [7, 8], and many others investigated some arguments related to this notion.

Çolak [6] studied statistical convergence order α by giving the definition as follows: we say that the sequence $x = (x_k)$ is statistically convergent of order α to L if there is a complex number L such that

$$\lim_{n\to\infty} \frac{1}{n^{\alpha}} |\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}| = 0.$$

Let $0 < \alpha \leqslant \beta \leqslant 1$. We define the (α, β)-density of the subset A of $\mathbb{N}$ (set of natural numbers) by

$$\delta_{\alpha}^{\beta}(A) = \lim_{n} \frac{1}{n^{\alpha}} |\{k \leqslant n : k \in A\}|^{\beta}$$

provided the limit exists (finite or infinite), where $|\{k \leqslant n : k \in A\}|^{\beta}$ denotes the βth power of number of elements of A not exceeding n.

Throughout this paper w indicates the space of sequences of real number.

Email address: hacer.sengul@hotmail.com (Hacer Şengül)

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Let $0 < \beta \leqslant 1, 0 < \alpha \leqslant 1$, $\alpha \leqslant \beta$, and $x = (x_k) \in w$. Space of sequences of S_α^β-convergent (or statistically convergent sequences of order (α, β)) is defined by

$$S_\alpha^\beta = \left\{ x = (x_k) : \lim_{n\to\infty} \frac{1}{n^\alpha} \left|\{k \leqslant n : |x_k - L| \geqslant \varepsilon\}\right|^\beta = 0 \right\},$$

where there exists a real number L. This convergence is indicated by $S_\alpha^\beta - \lim x_k = L$ (see [25]).

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$ and $\alpha \in (0, 1]$. Through this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Subsequently lacunary sequences have been studied in ([1, 10, 16, 18–20, 22, 23]).

2. Main results

In this Section, we define the concepts of $S_\alpha^\beta(\theta)$-convergence and strong $N_\alpha^\beta(\theta, p)$-summability of sequences of complex (or real) numbers for $0 < \alpha \leqslant \beta \leqslant 1$. Furthermore, we mention the inclusion relationships among the set of $S_\alpha^\beta(\theta)$-convergent sequences and the set of strongly $N_\alpha^\beta(\theta, p)$-summable sequences for different α and β values.

Definition 2.1. Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leqslant \beta \leqslant 1$. h_r^α denotes the αth power $(h_r)^\alpha$ of h_r, and $|\{k \leqslant n : k \in A\}|^\beta$ denotes the βth power of number of elements of A not exceeding n. $S_\alpha^\beta(\theta)$-convergent (or lacunary statistically convergent sequences of order (α, β)) sequences spaces is defined by

$$S_\alpha^\beta(\theta) = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}\right|^\beta = 0 \right\},$$

where there exists a real number L. In the present case this convergence is indicated by $S_\alpha^\beta(\theta) - \lim x_k = L$. $S_\alpha^\beta(\theta)$ will indicate the set of all $S_\alpha^\beta(\theta)$-statistically convergent sequences. If $\theta = (2^r)$, then we will write S_α^β in the place of $S_\alpha^\beta(\theta)$. If $\alpha = \beta = 1$ and $\theta = (2^r)$, then we will write S in the place of $S_\alpha^\beta(\theta)$.

The $S_\alpha^\beta(\theta)$-convergent is in a good way defined for $\alpha \leqslant \beta$, but it is not usually in a good way defined for $\beta < \alpha$. Define $x = (x_k)$ by for $r = 1, 2, \ldots$,

$$x_k = \begin{cases} 1, & \text{if} \quad k = 2r, \\ 0, & \text{if} \quad k \neq 2r. \end{cases}$$

Then for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - 1| \geqslant \varepsilon\}\right|^\beta \leqslant \lim_{r\to\infty} \frac{h_r^\beta}{2^\beta h_r^\alpha} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \left|\{k \in I_r : |x_k - 0| \geqslant \varepsilon\}\right|^\beta \leqslant \lim_{r\to\infty} \frac{h_r^\beta}{2^\beta h_r^\alpha} = 0$$

for $\beta < \alpha$. So $S_\alpha^\beta(\theta) - \lim x_k = 1$ and $S_\alpha^\beta(\theta) - \lim x_k = 0$. But this is impossible.

Theorem 2.2. *Let $0 < \alpha \leqslant \beta \leqslant 1$ and $x = (x_k)$, $y = (y_k)$ be sequences of real numbers, then*

(i) *if $S_\alpha^\beta(\theta) - \lim x_k = x_0$ and $c \in \mathbb{C}$, then $S_\alpha^\beta(\theta) - \lim (cx_k) = cx_0$;*

(ii) *if $S_\alpha^\beta(\theta) - \lim x_k = x_0$ and $S_\alpha^\beta(\theta) - \lim y_k = y_0$, then $S_\alpha^\beta(\theta) - \lim (x_k + y_k) = x_0 + y_0$.*

Definition 2.3. Let $\theta = (k_r)$ be a lacunary sequence, $0 < \alpha \leqslant \beta \leqslant 1$, and p be a positive real number. Space of sequences of strongly $N_\alpha^\beta(\theta)$-summable (or strong $N(\theta, p)$-summability of order (α, β)) is defined by

$$N_\alpha^\beta(\theta) = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \left(\sum_{k\in I_r} |x_k - L|^p \right)^\beta = 0 \right\},$$

where there exists a real number L. In the present case we write $N_\alpha^\beta(\theta, p) - \lim x_k = L$. $N_\alpha^\beta(\theta, p)$ will denote the set of all strongly $N(\theta, p)$-summable of order (α, β). If $\alpha = \beta = 1$, then we will write $N(\theta, p)$ in the place of $N_\alpha^\beta(\theta, p)$. If $\theta = (2^r)$, then we will write $w_\alpha^\beta(p)$ in the place of $N_\alpha^\beta(\theta, p)$. If $L = 0$, then we will write $w_{\alpha,0}^\beta(p)$ in the place of $w_\alpha^\beta(p)$. $N_{\alpha,0}^\beta(\theta, p)$ will denote the set of all strongly $N_\theta(p)$-summable of order (α, β) to 0.

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leqslant \beta \leqslant 1$. If $\liminf_r q_r > 1$, then $w_\alpha^\beta(p) - \lim x_k = L$ implies $N_\alpha^\beta(\theta, p) - \lim x_k = L$.*

Proof. If $\liminf_r q_r > 1$, then $1 + \delta \leqslant q_r$ for all $r \geqslant 1$, where there is a $\delta > 0$. Then for $x \in w_{\alpha,0}^\beta(p)$, we write

$$\begin{aligned}
\tau_{r,\beta}^\alpha = \frac{1}{h_r^\alpha} \left(\sum_{i\in I_r} |x_i|^p \right)^\beta &= \frac{1}{h_r^\alpha} \left(\sum_{i=1}^{k_r} |x_i|^p - \sum_{i=1}^{k_{r-1}} |x_i|^p \right)^\beta \\
&\leqslant \frac{1}{h_r^\alpha} \left(\sum_{i=1}^{k_r} |x_i|^p \right)^\beta + \frac{1}{h_r^\alpha} \left(\sum_{i=1}^{k_{r-1}} |x_i|^p \right)^\beta \\
&= \frac{k_r^\alpha}{h_r^\alpha} \frac{1}{k_r^\alpha} \left(\sum_{i=1}^{k_r} |x_i|^p \right)^\beta + \frac{k_{r-1}^\alpha}{h_r^\alpha} \frac{1}{k_{r-1}^\alpha} \left(\sum_{i=1}^{k_{r-1}} |x_i|^p \right)^\beta.
\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r^\alpha}{h_r^\alpha} \leqslant \frac{(1+\delta)^\alpha}{\delta^\alpha} \quad \text{and} \quad \frac{k_{r-1}^\alpha}{h_r^\alpha} \leqslant \frac{1}{\delta^\alpha}.$$

The terms $\frac{1}{k_r^\alpha}\left(\sum_{i=1}^{k_r} |x_i|^p\right)^\beta$ and $\frac{1}{k_{r-1}^\alpha}\left(\sum_{i=1}^{k_{r-1}} |x_i|^p\right)^\beta$ both converge to 0, and it follows that $\tau_{r,\beta}^\alpha$ converges to 0. Hence $x \in N_{\alpha,0}^\beta(\theta, p)$. □

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence and $0 < \alpha \leqslant \beta \leqslant 1$. If $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then $N(\theta, p) \subset w_\alpha^\beta(p)$.*

Proof. The proof can be seen easily. □

Theorem 2.6. *If $x \in w_\alpha^\beta \cap N_\alpha^\beta(\theta)$ and $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$, then $N_\alpha^\beta(\theta) - \lim x_k = w_\alpha^\beta - \lim x_k$.*

Proof. Let $N_\alpha^\beta(\theta) - \lim x_k = L$ and $w_\alpha^\beta - \lim x_k = L_1$, and assume that $L \neq L_1$. Since $\limsup_r \frac{k_r}{k_{r-1}^\alpha} < \infty$ by Theorem 2.5, we have $N_0(\theta, p) \subset w_{\alpha,0}^\beta(p)$. Since $(x - L_1) \in N_0(\theta, p)$, we get $(x - L_1) \in w_{\alpha,0}^\beta(p)$ and thus $\frac{1}{t^\alpha}\left(\sum_{i=1}^t |x_i - L_1|\right)^\beta \to 0$. Then we have

$$\frac{1}{t^\alpha} \left(\sum_{i=1}^t |x_i - L_1| \right)^\beta + \frac{1}{t^\alpha} \left(\sum_{i=1}^t |x_i - L| \right)^\beta \geqslant \frac{1}{t^\alpha} \left(\sum_{i=1}^t |x_i - L_1| + \sum_{i=1}^t |x_i - L| \right)^\beta \geqslant \frac{1}{t^\alpha} (|L - L_1|)^\beta > 0.$$

This is a contradiction, and the proof is completed. □

Theorem 2.7. *Let* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$ *and* $\theta = (k_r)$ *be a lacunary sequence. For* $0 < p < \infty$, $N_{\alpha_1}^{\beta_2}(\theta, p) \subset S_{\alpha_2}^{\beta_1}(\theta)$ *and the inclusion is certain for some* $\alpha_1, \alpha_2, \beta_1$, *and* β_2.

Proof. Let $x = (x_k) \in w$ and $\varepsilon > 0$. We can write

$$\left(\sum_{k\in I_r} |x_k - L|^p\right)^{\beta_2} = \left(\sum_{\substack{k\in I_r \\ |x_k-L|\geqslant\varepsilon}} |x_k - L|^p + \sum_{\substack{k\in I_r \\ |x_k-L|<\varepsilon}} |x_k - L|^p\right)^{\beta_2} \geqslant \left(\sum_{\substack{k\in I_r \\ |x_k-L|\geqslant\varepsilon}} |x_k - L|^p\right)^{\beta_2}$$
$$\geqslant |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_2} \varepsilon^{p\beta_2}$$

and so that

$$\frac{1}{h_r^{\alpha_1}} \left(\sum_{k\in I_r} |x_k - L|^p\right)^{\beta_2} \geqslant \frac{1}{h_r^{\alpha_1}} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_2} \varepsilon^{p\beta_2} \geqslant \frac{1}{h_r^{\alpha_2}} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \varepsilon^{p\beta_1}.$$

Hence $N_{\alpha_1}^{\beta_2}(\theta, p) \subset S_{\alpha_2}^{\beta_1}(\theta)$ for $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$.

We demonstrate the certainty of the inclusion $N_{\alpha_1}^{\beta_2}(\theta, p) \subset S_{\alpha_2}^{\beta_1}(\theta)$ for $p = 1$ and $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$. Define $x = (x_k)$ by

$$x_k = \begin{cases} \left[\sqrt{h_r}\right], & k_{r-1} < k \leqslant k_{r-1} + \left[\sqrt{h_r}\right], \\ 0, & \text{otherwise.} \end{cases}$$

We get for $\frac{1}{4} < \alpha_2 < 1$ and $\beta_1 = \frac{1}{2}$,

$$\frac{1}{h_r^{\alpha_2}} |\{k \in I_r : |x_k - 0| \geqslant \varepsilon\}|^{\beta_1} = \frac{\left(\left[\sqrt{h_r}\right]\right)^{\beta_1}}{h_r^{\alpha_2}} \to 0, \text{ as } r \to \infty,$$

i.e., $S_{\alpha_2}^{\beta_1}(\theta) - \lim x_k = 0$. For $0 < \alpha_1 < \frac{1}{4}$ and $\beta_2 = 1$,

$$\frac{1}{h_r^{\alpha_1}} \left(\sum_{k\in I_r} |x_k|\right)^{\beta_2} = \frac{\left(\left[\sqrt{h_r}\right]\left[\sqrt{h_r}\right]\right)^{\beta_2}}{h_r^{\alpha_1}} \to \infty.$$

For $0 < \alpha_1 < \frac{1}{4}$ and $\beta_2 = 1$, $N_{\alpha_1}^{\beta_2}(\theta, p) - \lim x_k \neq 0$. Therefore, $N_{\alpha_1}^{\beta_2}(\theta, p) \subset S_{\alpha_2}^{\beta_1}(\theta)$ is certain for $0 < \alpha_1 < \frac{1}{4}$, $\beta_2 = 1$, $\frac{1}{4} < \alpha_2 < 1$, and $\beta_1 = \frac{1}{2}$. □

The following result is a consequence of Theorem 2.7.

Corollary 2.8. *Let* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$ *and* $0 < p < \infty$.

i) *If* $\beta_2 = 1$, *then* $N_{\alpha_1}(\theta, p) \subset S_{\alpha_2}^{\beta_1}(\theta)$ *for* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant 1$.
ii) *If* $\beta_1 = \beta_2 = 1$, *then* $N_{\alpha_1}(\theta, p) \subset S_{\alpha_2}(\theta)$ *for* $0 < \alpha_1 \leqslant \alpha_2 \leqslant 1$.

Theorem 2.9. *Let* $0 < \alpha \leqslant \beta \leqslant 1$ *and* $\theta = (k_r)$ *be a lacunary sequence. If* $\liminf_r q_r > 1$, *then* $S_\alpha^\beta \subset S_\alpha^\beta(\theta)$.

Proof. If $\liminf_r q_r > 1$, then $q_r \geqslant 1 + \delta$ for sufficiently large r where there is a $\delta > 0$. We can write

$$\frac{h_r}{k_r} \geqslant \frac{\delta}{1+\delta} \Longrightarrow \frac{1}{k_r^\alpha} \geqslant \frac{\delta^\alpha}{(1+\delta)^\alpha} \frac{1}{h_r^\alpha}.$$

If $S_\alpha^\beta - \lim x_k = L$, then for every $\varepsilon > 0$, we have

$$\frac{1}{k_r^\alpha} |\{k \leqslant k_r : |x_k - L| \geqslant \varepsilon\}|^\beta \geqslant \frac{1}{k_r^\alpha} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^\beta \geqslant \frac{\delta^\alpha}{(1+\delta)^\alpha} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^\beta.$$

Therefore $S_\alpha^\beta - \lim x_k = L$ implies $S_\alpha^\beta(\theta) - \lim x_k = L$. □

Theorem 2.10. *Let* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$. *If*

$$\lim_{r\to\infty}\inf\left(\frac{h_r}{k_r}\right)^{\alpha_2} > 0,$$

then $S_{\alpha_1} \subset S_{\alpha_2}^{\beta_1}(\theta)$.

Proof. For a given $\varepsilon > 0$, we have

$$\{k \leqslant k_r : |x_k - L| \geqslant \varepsilon\} \supset \{k \in I_r : |x_k - L| \geqslant \varepsilon\}.$$

Therefore,

$$\frac{1}{k_r^{\alpha_1}}|\{k \leqslant k_r : |x_k - L| \geqslant \varepsilon\}| \geqslant \frac{1}{k_r^{\alpha_2}}|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_2} \geqslant \frac{h_r^{\alpha_2}}{k_r^{\alpha_2}}\frac{1}{h_r^{\alpha_2}}|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1}.$$

Since $\lim_{r\to\infty}\inf\left(\frac{h_r}{k_r}\right)^{\alpha_2} > 0$ and $x = (x_k) \in S_{\alpha_1}$, we have $x \in S_{\alpha_2}^{\beta_1}(\theta)$. □

Theorem 2.11. *Let* $\theta = (k_r)$ *and* $\theta' = (s_r)$ *be two lacunary sequences such that* $I_r \subset J_r$ *for all* $r \in \mathbb{N}$ *and* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$,

(i) *if*

$$\lim_{r\to\infty}\inf\frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} > 0, \tag{2.1}$$

then $S_{\alpha_2}^{\beta_2}\left(\theta'\right) \subseteq S_{\alpha_1}^{\beta_1}(\theta)$;

(ii) *if*

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^{\alpha_2}} = 1, \tag{2.2}$$

then $S_{\alpha_1}^{\beta_2}(\theta) \subseteq S_{\alpha_2}^{\beta_1}\left(\theta'\right)$.

Proof.

(i). Suppose that $I_r \subset J_r$ for all $r \in \mathbb{N}$. We can write

$$\{k \in J_r : |x_k - L| \geqslant \varepsilon\} \supseteq \{k \in I_r : |x_k - L| \geqslant \varepsilon\}$$

and so

$$\frac{1}{\ell_r^{\alpha_2}}|\{k \in J_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_2} \geqslant \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}}\frac{1}{h_r^{\alpha_1}}|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1}$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $\ell_r = s_r - s_{r-1}$. Thus we get $S_{\alpha_2}^{\beta_2}\left(\theta'\right) \subseteq S_{\alpha_1}^{\beta_1}(\theta)$.

(ii). Let $x = (x_k) \in S_{\alpha_1}^{\beta_2}(\theta)$ and (2.2) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}}|\{k \in J_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} &= \frac{1}{\ell_r^{\alpha_2}}|\{s_{r-1} < k \leqslant k_{r-1} : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \\
&\quad + \frac{1}{\ell_r^{\alpha_2}}|\{k_r < k \leqslant s_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} + \frac{1}{\ell_r^{\alpha_2}}|\{k_{r-1} < k \leqslant k_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \\
&\leqslant \frac{1}{\ell_r^{\alpha_2}}(k_{r-1} - s_{r-1})^{\beta_1} + \frac{1}{\ell_r^{\alpha_2}}(s_r - k_r)^{\beta_1} + \frac{1}{\ell_r^{\alpha_2}}|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \\
&\leqslant \frac{k_{r-1} - s_{r-1}}{\ell_r^{\alpha_2}} + \frac{s_r - k_r}{\ell_r^{\alpha_2}} + \frac{1}{\ell_r^{\alpha_2}}|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\ell_r - h_r}{\ell_r^{\alpha_2}} + \frac{1}{\ell_r^{\alpha_2}} \left|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}\right|^{\beta_1} \\
&\leqslant \frac{\ell_r - h_r^{\alpha_2}}{h_r^{\alpha_2}} + \frac{1}{h_r^{\alpha_1}} \left|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}\right|^{\beta_2} \\
&\leqslant \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1\right) + \frac{1}{h_r^{\alpha_1}} \left|\{k \in I_r : |x_k - L| \geqslant \varepsilon\}\right|^{\beta_2}
\end{aligned}
$$

for all $r \in \mathbb{N}$. This implies that $S_{\alpha_1}^{\beta_2}(\theta) \subseteq S_{\alpha_2}^{\beta_1}\left(\theta'\right)$. □

Theorem 2.12. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, $0 < \alpha_1 \leqslant \alpha_2 \leqslant \beta_1 \leqslant \beta_2 \leqslant 1$, and $0 < p < \infty$. Then we have*

(i) *if (2.1) is provided, then $w_{\alpha_2}^{\beta_2}\left(\theta', p\right) \subset w_{\alpha_1}^{\beta_1}(\theta, p)$;*

(ii) *if (2.2) is provided and $x \in \ell_\infty$, then $w_{\alpha_1}^{\beta_2}(\theta, p) \subset w_{\alpha_2}^{\beta_1}\left(\theta', p\right)$;*

(iii) *if (2.1) is provided, then $w_{\alpha_2}^{\beta_2}\left(\theta', p\right) \subset S_{\alpha_1}^{\beta_1}(\theta)$;*

(iv) *if (2.2) is provided and $x \in \ell_\infty$, then $S_{\alpha_1}^{\beta_2}(\theta) \subset w_{\alpha_2}^{\beta_1}\left(\theta', p\right)$.*

Proof.

(i). The proof can be seen easily.

(ii). Let $x = (x_k) \in w_{\alpha_1}^{\beta_2}(\theta, p)$ and assume that (2.2) is provided. Since $x = (x_k) \in \ell_\infty$, then we obtain $|x_k - L| \leqslant M$ for all k where there is an $M > 0$. Now, since $I_r \subseteq J_r$ and $h_r \leqslant \ell_r$ for all $r \in \mathbb{N}$, we may write

$$
\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in J_r} |x_k - L|^p\right)^{\beta_1} &= \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in J_r - I_r} |x_k - L|^p\right)^{\beta_1} + \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in I_r} |x_k - L|^p\right)^{\beta_1} \\
&\leqslant \left(\frac{\ell_r - h_r}{\ell_r^{\alpha_2}}\right)^{\beta_1} M^{p\beta_1} + \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k \in I_r} |x_k - L|^p\right)^{\beta_1} \\
&\leqslant \left(\frac{\ell_r - h_r^{\alpha_2}}{h_r^{\alpha_2}}\right) M^{p\beta_1} + \frac{1}{h_r^{\alpha_2}} \left(\sum_{k \in I_r} |x_k - L|^p\right)^{\beta_2} \\
&\leqslant \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1\right) M^{p\beta_1} + \frac{1}{h_r^{\alpha_1}} \left(\sum_{k \in I_r} |x_k - L|^p\right)^{\beta_2}
\end{aligned}
$$

for every $r \in \mathbb{N}$. Therefore $\ell_\infty \cap w_{\alpha_1}^{\beta_2}(\theta, p) \subset w_{\alpha_2}^{\beta_1}\left(\theta', p\right)$.

(iii). Let $x = (x_k) \in w_{\alpha_2}^{\beta_2}\left(\theta', p\right)$ and $\varepsilon > 0$, we can write

$$
\begin{aligned}
\left(\sum_{k \in J_r} |x_k - L|^p\right)^{\beta_2} &= \left(\sum_{\substack{k \in J_r \\ |x_k - L| \geqslant \varepsilon}} |x_k - L|^p + \sum_{\substack{k \in J_r \\ |x_k - L| < \varepsilon}} |x_k - L|^p\right)^{\beta_2} \\
&\geqslant \left(\sum_{\substack{k \in I_r \\ |x_k - L| \geqslant \varepsilon}} |x_k - L|^p + \sum_{\substack{k \in I_r \\ |x_k - L| < \varepsilon}} |x_k - L|^p\right)^{\beta_1}
\end{aligned}
$$

$$\geqslant \left(\sum_{\substack{k\in I_r \\ |x_k-L|\geqslant\varepsilon}} |x_k - L|^p \right)^{\beta_1} \geqslant |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \varepsilon^{p\beta_1}$$

and so that

$$\frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k\in J_r} |x_k - L|^p \right)^{\beta_2} \geqslant \frac{1}{\ell_r^{\alpha_2}} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \varepsilon^{p\beta_1} \geqslant \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}} \frac{1}{h_r^{\alpha_1}} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_1} \varepsilon^{p\beta_1}.$$

Therefore $x \in S_{\alpha_1}^{\beta_1}(\theta)$.

(iv). Assume that $x = (x_k) \in S_{\alpha_1}^{\beta_2}(\theta)$ and $x \in \ell_\infty$. Then we obtain $|x_k - L| \leqslant M$ for all k where there is an $M > 0$. We may write

$$\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k\in J_r} |x_k - L|^p \right)^{\beta_1} &= \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k\in J_r - I_r} |x_k - L|^p \right)^{\beta_1} + \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k\in I_r} |x_k - L|^p \right)^{\beta_1} \\
&\leqslant \left(\frac{\ell_r - h_r}{\ell_r^{\alpha_2}} \right) M^{p\beta_1} + \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k\in I_r} |x_k - L|^p \right)^{\beta_1} \\
&\leqslant \left(\frac{\ell_r - h_r^{\alpha_2}}{\ell_r^{\alpha_2}} \right) M^{p\beta_1} + \frac{1}{\ell_r^{\alpha_2}} \left(\sum_{k\in I_r} |x_k - L|^p \right)^{\beta_2} \\
&\leqslant \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1 \right) M^{p\beta_1} + \frac{1}{h_r^{\alpha_1}} \left(\sum_{\substack{k\in I_r \\ |x_k-L|\geqslant\varepsilon}} |x_k - L|^p \right)^{\beta_2} + \frac{1}{h_r^{\alpha_2}} \left(\sum_{\substack{k\in I_r \\ |x_k-L|<\varepsilon}} |x_k - L|^p \right)^{\beta_2} \\
&\leqslant \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1 \right) M^{p\beta_1} + \frac{M^{p\beta_2}}{h_r^{\alpha_1}} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_2} + \frac{h_r^{\beta_2}}{h_r^{\alpha_2}} \varepsilon^{p\beta_2} \\
&\leqslant \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1 \right) M^{p\beta_1} + \frac{M^{p\beta_2}}{h_r^{\alpha_1}} |\{k \in I_r : |x_k - L| \geqslant \varepsilon\}|^{\beta_2} + \frac{\ell_r}{h_r^{\alpha_2}} \varepsilon^{p\beta_2}
\end{aligned}$$

for every $\varepsilon > 0$ and all $r \in \mathbb{N}$. Thus $x = (x_k) \in w_{\alpha_2}^{\beta_1}(\theta', p)$. □

3. Conclusion

In this paper, the set of $S^\alpha(\theta)$-convergent sequences and the set of strongly $N^\alpha(\theta, p)$-summable sequences were generalized to the sets of order (α, β). We can obtain different inclusions from Theorems 2.10, 2.11, and 2.12 for $\beta_2 = 1$, $\beta_1 = \beta_2 = 1$, $\alpha_2 = \beta_1 = \beta_2 = 1$, $\beta_1 = \beta_2$, and $\alpha_1 = \alpha_2$. If we look closely, then we have the set of $S^\alpha(\theta)$-convergent sequences and the set of strongly $N^\alpha(\theta, p)$-summable sequences for $\beta_1 = \beta_2 = 1$.

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