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Commutators of multilinear Calderón-Zygmund operators with Dini type kernels on some function spaces

Jie Sun, Pu Zhang*

Department of Mathematics, Mudanjiang Normal University, Mudanjiang, Heilongjiang 157011, P. R. China.

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Abstract

In this paper, we establish some new boundedness for commutators of multilinear Calderón-Zygmund operators with kernels of type ω from product of Lebesgue spaces into Lebesgue spaces, Lipschitz spaces, and Triebel-Lizorkin spaces, which extend some previous results. ©2017 All rights reserved.

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1. Introduction and results

In 1975, Coifman and Meyer [2] studied the bilinear singular integral operators. Then many researchers were interested in bilinear or multilinear singular integrals (see [3, 5–11, 21, 22]). During the same period some generalizations of Calderón-Zygmund operators were also studied by many authors such as Calderón-Zygmund operators with kernels of type ω which was first studied by Yabuta [20] in 1985. Maldonado and Naibo [12] studied the bilinear Calderón-Zygmund operators of type ω in 2009. Recently, Lu and Zhang [11] considered the multilinear case.

Throughout this paper, we always assume that $\omega(t) : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $0 < \omega(1) < \infty$. For $a > 0$, we say $\omega \in \mathrm{Dini}(a)$ if

$$|\omega|_{\mathrm{Dini}(a)} = \int_0^1 \frac{\omega^a(t)}{t} dt < \infty.$$

Definition 1.1. A locally integrable function $K(x, y_1, \cdots, y_m)$, defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, is called an m-linear Calderón-Zygmund kernel of type $\omega(t)$, if there exists a constant $A > 0$ such that

$$|K(x, y_1, \cdots, y_m)| \leqslant \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \tag{1.1}$$

*Corresponding author
Email addresses: sj800816@163.com (Jie Sun), puzhang@sohu.com (Pu Zhang)

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for all $(x, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some $j \in \{1, 2, \cdots, m\}$, and

$$|K(x, y_1, \cdots, y_m) - K(x', y_1, \cdots, y_m)| \leqslant \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\left(\frac{|x-x'|}{|x-y_1| + \cdots + |x-y_m|}\right), \quad (1.2)$$

whenever $|x - x'| \leqslant \frac{1}{2} \max_{1 \leqslant j \leqslant m} |x - y_j|$, and

$$\begin{aligned} &|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x, y_1, \cdots, y_j', \cdots, y_m)| \\ &\leqslant \frac{A}{(|x-y_1| + \cdots + |x-y_m|)^{mn}} \omega\left(\frac{|y_j - y_j'|}{|x-y_1| + \cdots + |x-y_m|)}\right), \end{aligned} \quad (1.3)$$

whenever $|y_j - y_j'| \leqslant \frac{1}{2} \max_{1 \leqslant j \leqslant m} |x - y_j|$.

We say $T : \mathscr{S}(\mathbb{R}^n) \times \cdots \times \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is an m-linear operator with an m-linear Calderón-Zygmund kernel $K(x, y_1, \cdots, y_m)$ of type $\omega(t)$, if

$$T(f_1, \cdots, f_m) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) dy_1 \cdots dy_m,$$

whenever $f_1, \cdots, f_m \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \cap_{j=1}^m \mathrm{supp} f_j$.

If T can be extended to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $1 < q, q_1, \cdots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1/q$, or, from $L^{q_1}(\mathbb{R}^n) \times \cdots \times L^{q_m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for some $1 < q_1, \cdots, q_m < \infty$ with $1/q_1 + \cdots + 1/q_m = 1$, then T is called an m-linear Calderón-Zygmund operator of type ω, abbreviated to m-linear ω-CZO.

Obviously, when $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, the m-linear ω-CZO is exactly the multilinear Calderón-Zygmund operator studied by Grafakos and Torres [6] and Lerner et al. [9].

To shorten the notation, we denote by $\vec{f} = (f_1, \cdots, f_m)$ and $d\vec{y} = dy_1 \cdots dy_m$ in the following.

In 2014, Lu and Zhang gave the endpoint estimate for the m-linear ω-CZO under some weaker assumptions of $\omega(t)$, and also got the following multiple weighted estimates.

Theorem 1.2 ([11]). *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Let $1/p = 1/p_1 + \cdots + 1/p_m$ and $\vec{w} \in A_{\vec{P}}$.*

(1) *If $1 < p_j < \infty$ for all $j = 1, \cdots, m$, then*

$$\|T(\vec{f})\|_{L^p(\nu_{\vec{w}})} \leqslant C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

(2) *If $1 \leqslant p_j < \infty$ for all $j = 1, \cdots, m$, and at least one of the $p_j = 1$, then*

$$\|T(\vec{f})\|_{L^{p,\infty}(\nu_{\vec{w}})} \leqslant C \prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}.$$

Let $\vec{b} = (b_1, \cdots, b_m)$ be a collection of locally integrable functions, the commutator generated by m-linear ω-CZO and $\vec{b}$ is defined by:

$$T_{\Sigma \vec{b}}(f_1, f_2, \cdots, f_m)(x) = \sum_{j=1}^m T_{b_j}^j (f_1, f_2, \cdots, f_m)(x)$$

where

$$T_{b_j}^j(\vec{f})(x) = [b_j, T]_j(\vec{f})(x) = b_j(x) T(f_1, \cdots, f_j, \cdots, f_m) - T(f_1, \cdots, f_j b_j, \cdots, f_m)(x), j = 1, \cdots, m.$$

The iterated commutator $T_{\Pi\vec{b}}(\vec{f})$ is defined as follows:

$$T_{\Pi\vec{b}}(\vec{f})(x) = [b_1, [b_2, \cdots [b_{m-1}, [b_m, T]_m]_{m-1}, \cdots]_2]_1(\vec{f})(x),$$

which can also be given formally by

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{m} (b_j(x) - b_j(y_j)) K(x, \vec{y}) f_1(y_1) \cdots f_m(y_m) d\vec{y}.$$

When $m = 1$, $T_{\Sigma\vec{b}}(\vec{f}) = T_{\Pi\vec{b}}(\vec{f}) = [b, T]f = bT(f) - T(bf)$, which is the well-known classical commutator studied in [4]. These multilinear commutators are early appeared in the reference [19] by Xu.

Now we recall the following definition of Lipschitz functions.

Definition 1.3 ([14])**.** Let $0 < \beta \leqslant 1$ and b be a locally integrable function on $\mathbb{R}^n$. We say b belongs to the Lipschitz space Lip_β if there is a constant $C > 0$ such that

$$|b(x) - b(y)| \leqslant C|x - y|^\beta \tag{1.4}$$

for almost every x and y in $\mathbb{R}^n$. The minimal constant C in (1.4) is the Lip_β norm of b and is denoted simply by $\|b\|_{\mathrm{Lip}_\beta}$.

In 1995, Paluszynski [14] proved that the commutator $[b, T]$ generated by Calderón-Zygmund operators T with classical kernel and Lipschitz functions b is bounded from L^p to L^q whenever $0 < \beta < 1, 1/q = 1/p - \beta/n$ and $1 < p < q < \infty$, and from L^p to homogenous Triebel-Lizorkin spaces $\dot{F}_p^{\beta,\infty}$ which is defined in [16].

For the commutators generated by the multilinear Calderón-Zygmund operators with the kernel of standard estimates, Wang and Xu [17] and Mo and Lu [13] got the boundedness from product of Lebesgue spaces to Lebesgue space and to homogenous Triebel-Lizorkin space, respectively.

In this paper, we will discuss the mapping properties of multilinear commutators generated by m-linear ω-CZO and Lipschitz functions on some function spaces. We obtain the following results.

Theorem 1.4. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Suppose $b_j \in \mathrm{Lip}_{\beta_j}$ with $0 < \beta_j < 1$ for $j = 1, \cdots, m$ and $\beta = \beta_1 + \cdots + \beta_m$. If $1 < p_1, \cdots, p_m < \infty$, $0 < q < \infty$ and $1/p_j > \beta_j/n$ with $1/q = 1/p_1 + \cdots + 1/p_m - \beta/n$, then $T_{\Pi\vec{b}}(\vec{f})$ can be extended to a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ into L^q.*

Theorem 1.5. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Suppose $b_j \in \mathrm{Lip}_{\beta_j}$ with $0 < \beta_j < 1$ for $j = 1, \cdots, m$ and $\beta = \beta_1 + \cdots + \beta_m$. If $1 < p_1, \cdots, p_m < \infty$, $0 < 1/p_j < \beta_j/n$ and $0 < \beta - n/p < 1$ with $1/p = 1/p_1 + \cdots + 1/p_m$, and ω satisfies*

$$\int_0^1 \frac{\omega(t)}{t^{1+\beta-n/p}} dt < \infty,$$

then $T_{\Pi\vec{b}}(\vec{f})$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into $\mathrm{Lip}_{\beta-n/p}$.

Theorem 1.6. *Let T be an m-linear ω-CZO with $\omega \in \mathrm{Dini}(1)$. Suppose $b_j \in \mathrm{Lip}_{\beta_j}$ with $0 < \beta_j < 1$ for $j = 1, \cdots, m$ and $\beta = \beta_1 + \cdots + \beta_m$. If $1 < p, p_1, \cdots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, and ω satisfies*

$$\int_0^1 \frac{\omega(t)}{t^{1+\beta}} dt < \infty,$$

then $T_{\Pi\vec{b}}(\vec{f})$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into Triebel-Lizorkin space $\dot{F}_p^{\beta,\infty}$.

Remark 1.7. Theorems 1.4-1.6 extend the corresponding results in [17] and [13].

Remark 1.8. Theorems 1.4-1.6 are also true for each $T^j_{b_j}(\vec{f})$, $j = 1, \cdots, m$.

Remark 1.9. Similar to [11] and [12], by applying our above results, one can obtain the similar results for commutators of paraproduct and pseudo-differential operators with mild regularity. We omit the details to the readers.

The rest of this paper is organized as follows. After recalling some notations and lemmas in Section 2, we will prove our results in Section 3.

Throughout this paper, we denote by p' the conjugate index of p, that is $1/p + 1/p' = 1$. The letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence but is independent of the main parameters.

2. Preliminaries and lemmas

For a function $f \in L_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal and the sharp maximal functions are defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

and

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_B| dy \approx \sup_{B \ni x} \inf_C \frac{1}{|Q|} \int_Q |f(y) - C| dy,$$

where f_Q denotes the average of f over cube Q, that is, $f_Q = \frac{1}{|Q|} \int_Q f(x) dx$.

For $\delta > 0$, we denote $M_\delta(f)$ and $M^\sharp_\delta(f)$ by $M_\delta(f) = M(|f|^\delta)^{1/\delta}$ and $M^\sharp_\delta(f) = [M^\sharp(|f|^\delta)]^{1/\delta}$. For $0 < \beta < n/r$, we define the following fractional maximal operator

$$M_{r,\beta} f(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\beta/n}} \int_Q |f(y)|^r dy \right)^{1/r},$$

when $\beta = 0$, we denote $M_{r,\beta}$ simply by M_r. It is well-known if $r < p < \infty$, M_r is bounded on L^p spaces.

Lemma 2.1 ([15])**.** *Let $0 < p, \delta < \infty$, $\omega \in A_\infty$, then there exists a constant C, such that*

$$\int_{\mathbb{R}^n} M_\delta f(x)^p \omega(x) dx \leqslant C \int_{\mathbb{R}^n} M^\sharp_\delta f(x)^p \omega(x) dx$$

for any function f for which the left-hand side is finite.

Lemma 2.2 ([14])**.**

(1) *For $0 < \beta < 1, 1 \leqslant q < \infty$, we have*

$$\|f\|_{Lip_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f - f_Q|^q \right)^{1/q}.$$

(2) *For $0 < \beta < 1, 1 \leqslant p < \infty$, we have*

$$\|f\|_{\dot{F}^{\beta,\infty}_p} \approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f - f_Q| \right\|_{L^p}.$$

Lemma 2.3 ([14])**.** *Let $b \in Lip_\beta, 0 < \beta < 1$. For any cubes Q, Q' in $\mathbb{R}^n$ and $Q' \subset Q$, then*

$$|b_{Q'} - b_Q| \leqslant C \|b\|_{Lip_\beta} |Q|^{\beta/n}.$$

Lemma 2.4 ([1])**.** *If $0 < \beta < n$, $0 < r < p < n/\beta$ and $1/q = 1/p - \beta/n$, then $\|M_{r,\beta} f\|_{L^q} \leqslant C \|f\|_{L^p}$.*

3. Proof of theorems

For simplicity, we only consider the case $m = 2$. Similar argument also works for general m.

Lemma 3.1. *Let* T *be a* 2*-linear* ω*-CZO with* $\omega \in \text{Dini}(1)$*. Suppose that* $b_j \in \text{Lip}_\beta, 0 < \beta < 1$ *and* $0 < \delta < \varepsilon < 1/2 < 1 < n/\beta$*, then for* $j = 1, 2$*,*

$$M_\delta^\sharp\left(T_{b_j}^j(f_1, f_2)\right)(x) \leqslant C\|b_j\|_{\text{Lip}_\beta}\left\{M_{\varepsilon,\beta}(T(f_1, f_2))(x) + M_{1,\beta}(f_j)(x)M(f_k)(x)\right\},$$

$j \neq k, k = 1, 2$.

Proof. We only estimate $M_\delta^\sharp\left(T_{b_1}^1(f_1, f_2)\right)$ and write $b_1 = b$ for simplicity. A similar argument also works for $M_\delta^\sharp\left(T_{b_2}^2(f_1, f_2)\right)$.

Fix $x \in \mathbb{R}^n$, for any cube $Q(x_Q, l)$ containing x with side-length l, set $Q^* = 8\sqrt{n}Q = Q(x_Q, 8\sqrt{n}l)$. We decompose $f_j = f_j^0 + f_j^\infty$, where $f_j^0 = f_j\chi_{Q^*}$ and $f_j^\infty = f\chi_{\mathbb{R}^n\setminus Q^*}, j = 1, 2$. Since $0 < \delta < 1/2$, then for any constant c, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left||T_b^1(f_1, f_2)(z)|^\delta - |c|^\delta\right| dz\right)^{1/\delta} \\
&\leqslant \left(\frac{1}{|Q|}\int_Q |T_b^1(f_1, f_2)(z) - c|^\delta dz\right)^{1/\delta} \\
&\leqslant C\left(\frac{1}{|Q|}\int_Q |(b(z) - b_{Q^*})T(f_1, f_2)(z)|^\delta dz\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{Q^*})f_1^0, f_2^0)(z)|^\delta dz\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{Q^*})f_1^\infty, f_2^0)(z)|^\delta dz\right)^{1/\delta} + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{Q^*})f_1^0, f_2^\infty)(z)|^\delta dz\right)^{1/\delta} \\
&\quad + C\left(\frac{1}{|Q|}\int_Q |T((b - b_{Q^*})f_1^\infty, f_2^\infty)(z) - c|^\delta dz\right)^{1/\delta} \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For $y \in Q$, since $b \in \text{Lip}_\beta(\mathbb{R}^n)$, by the definition of Lipschitz function, we have

$$|b(y) - b_Q| \leqslant C|Q|^{\beta/n}\|b\|_{\text{Lip}_\beta}. \tag{3.1}$$

Since $b \in \text{Lip}_\beta$, $0 < \delta < \varepsilon < 1/2$, by (3.1) and Hölder's inequality, we get

$$\begin{aligned}
I_1 &\leqslant C|Q|^{\beta/n}\|b\|_{\text{Lip}_\beta}\left(\frac{1}{|Q|}\int_Q |T(f_1, f_2)(z)|^\delta dz\right)^{1/\delta} \\
&\leqslant C\|b\|_{\text{Lip}_\beta} M_{\delta,\beta}(T(f_1, f_2))(x) \leqslant C\|b\|_{\text{Lip}_\beta} M_{\varepsilon,\beta}(T(f_1, f_2))(x).
\end{aligned}$$

For the second term I_2, since $0 < \delta < 1/2$, by Kolmogorov's inequality (see [9, 18]), Theorem 1.2, and (3.1), we obtain

$$\begin{aligned}
I_2 &= C|Q|^{-\delta}\|T((b - b_{Q^*})f_1^0, f_2^0)(z)\|_{L^\delta(Q)} \\
&\leqslant C|Q|^{-2}\|T((b - b_{Q^*})f_1^0, f_2^0)(z)\|_{L^{1/2,\infty}(Q)} \\
&\leqslant C|Q|^{-2}\|T((b - b_{Q^*})f_1^0, f_2^0)(z)\|_{L^{1/2,\infty}(\mathbb{R}^n)} \\
&\leqslant C\left(\frac{1}{|Q^*|}\int_{Q^*} |(b(z) - b_{Q^*})f_1(z)|dz\right)\left(\frac{1}{|Q^*|}\int_{Q^*} |f_2(z)|dz\right) \\
&\leqslant C\|b\|_{\text{Lip}_\beta}|Q^*|^{\beta/n}\left(\frac{1}{|Q^*|}\int_{Q^*} |f_1(z)|dz\right)\left(\frac{1}{|Q^*|}\int_{Q^*} |f_2(z)|dz\right) \\
&\leqslant C\|b\|_{\text{Lip}_\beta} M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}$$

For any $y \in \mathbb{R}^n \setminus Q^*$, there exists an integer $i > 1$ such that $y \in 2^i Q^* \setminus 2^{i-1} Q^*$, and $|y - x_Q| \sim |2^i Q^*|^{1/n}$, then by (3.1) and Lemma 2.3, we have

$$|b(y) - b_{Q^*}| \leqslant |b(y) - b_{2^i Q^*}| + |b_{2^i Q^*} - b_{Q^*}| \leqslant C\|b\|_{\mathrm{Lip}_\beta} |y - x_Q|^\beta. \tag{3.2}$$

For the term I_3, noting the fact that $|z - y_1| \sim |y_1 - x_Q|$ for any $y_1 \in (Q^*)^c$ and $z \in Q$, then by (1.1) and (3.2), we obtain

$$\begin{aligned}
I_3 &\leqslant \frac{C}{|Q|} \int_Q |T((b - b_{Q^*}) f_1^\infty, f_2^0)(z)| dz \\
&\leqslant \frac{C}{|Q|} \int_Q \int_{Q^*} \int_{(Q^*)^c} \frac{A}{(|z - y_1| + |z - y_2|)^{2n}} |b(y_1) - b_{Q^*}| |f_1(y_1)| |f_2(y_2)| dy_1 dy_2 dz \\
&\leqslant C \int_{(Q^*)^c} \frac{|b(y_1) - b_{Q^*}|}{|y_1 - x_Q|^{2n}} |f_1(y_1)| dy_1 \int_{Q^*} |f_2(y_2)| dy_2 \\
&\leqslant C|Q^*| \sum_{k=1}^\infty \int_{2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q} \frac{|b(y_1) - b_{Q^*}|}{|y_1 - x_Q|^{2n}} |f_1(y_1)| dy_1 M(f_2)(x) \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} |Q^*| \sum_{k=1}^\infty \int_{2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q} |y_1 - x_Q|^{\beta - 2n} |f_1(y_1)| dy_1 M(f_2)(x) \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} |Q^*| \sum_{k=1}^\infty |2^k Q^*|^{\beta/n - 2} \int_{2^k Q^*} |f_1(y_1)| dy_1 M(f_2)(x) \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^\infty 2^{-k} \frac{1}{|2^k Q^*|^{1 - \beta/n}} \int_{2^k Q^*} |f_1(y_1)| dy_1 M(f_2)(x) \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} M_{1,\beta}(f_1)(x) M(f_2)(x).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_4 &\leqslant \frac{C}{|Q|} \int_Q |T((b - b_{Q^*}) f_1^0, f_2^\infty)(z)| dz \\
&\leqslant \frac{C}{|Q|} \int_Q \int_{Q^*} \int_{(Q^*)^c} \frac{A}{(|z - y_1| + |z - y_2|)^{2n}} |b(y_1) - b_{Q^*}| |f_1(y_1)| |f_2(y_2)| dy_2 dy_1 dz \\
&\leqslant C \int_{Q^*} |b(y_1) - b_{Q^*}| |f_1(y_1)| dy_1 \int_{(Q^*)^c} \frac{|f_2(y_2)|}{|y_2 - x_Q|^{2n}} dy_2 \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^\infty \int_{2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q} \frac{|f_2(y_2)|}{|y_2 - x_Q|^{2n}} dy_2 \int_{Q^*} |y_1 - x_Q|^\beta |f_1(y_1)| dy_1 \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^\infty |2^k Q^*|^{-2} |Q^*| \int_{2^{k+3}\sqrt{n}Q \setminus 2^{k+2}\sqrt{n}Q} |f_2(y_2)| dy_2 M_{1,\beta}(f_1)(x) \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^\infty 2^{-k} \frac{1}{|2^k Q^*|} \int_{2^k Q^*} |f_2(y_2)| dy_2 M_{1,\beta}(f_1)(x) \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} M_{1,\beta}(f_1)(x) M(f_2)(x).
\end{aligned}$$

For the last term I_5, since $(\mathbb{R}^n \setminus Q^*)^2 \subseteq \mathbb{R}^{2n} \setminus (Q^*)^2 \subseteq \cup_{k=1}^\infty (2^{k+3}\sqrt{n}Q)^2 \setminus (2^{k+2}\sqrt{n}Q)^2$, we will make use of the assumptions (1.2) and (3.2), we have

$$I_5 \leqslant \frac{C}{|Q|} \int_Q |T((b - b_{Q^*}) f_1^\infty, f_2^\infty)(z) - T((b - b_{Q^*}) f_1^\infty, f_2^\infty)(x_Q)| dz$$

$$
\begin{aligned}
&\leqslant \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}|K(z,y_1,y_2)-K(x_Q,y_1,y_2)||(b(y_1)-b_{Q^*})f_1^\infty(y_1)f_2^\infty(y_2)|dy_1dy_2dz\\
&\leqslant \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|z-x_Q|}{|x-y_1|+|x-y_2|}\right)\\
&\quad\times|(b(y_1)-b_{Q^*})f_1^\infty(y_1)f_2^\infty(y_2)|dy_1dy_2dz\\
&\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2}\frac{|(b(y_1)-b_{Q^*})f_1^\infty(y_1)f_2^\infty(y_2)|}{|2^{k+3}\sqrt{n}Q|^2}\omega(2^{-k})dy_1dy_2dz\\
&\leqslant \frac{C\|b\|_{Lip_\beta}}{|Q|}\int_Q\sum_{k=1}^\infty\int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2}\frac{|y_1-x_Q|^\beta|f_1^\infty(y_1)f_2^\infty(y_2)|}{|2^kQ^*|^2}\omega(2^{-k})dy_1dy_2dz\\
&\leqslant \frac{C\|b\|_{Lip_\beta}}{|Q|}\int_Q\sum_{k=1}^\infty|2^kQ^*|^{\beta/n-2}\int_{(2^kQ^*)^2}|f_1^\infty(y_1)f_2^\infty(y_2)|\omega(2^{-k})dy_1dy_2dz\\
&\leqslant C\|b\|_{Lip_\beta}\sum_{k=1}^\infty\omega(2^{-k})\frac{1}{|2^kQ^*|^{1-\beta/n}}\int_{2^kQ^*}|f_1(y_1)|dy_1\frac{1}{|2^kQ^*|}\int_{2^kQ^*}|f_2(y_2)|dy_2\\
&\leqslant C\|b\|_{Lip_\beta}M_{1,\beta}(f_1)(x)M(f_2)(x).
\end{aligned}
$$

The proof of Lemma 3.1 is finished. □

Lemma 3.2. *Let* T *be a* 2*-linear* ω*-CZO with* $\omega\in \text{Dini}(1)$*. Suppose* $b_1\in Lip_{\beta_1}$ *and* $b_2\in Lip_{\beta_2}$ *with* $0<\beta_1,\beta_2<1$*. Let* $\beta_1+\beta_2=\beta$ *and* $0<\delta<\varepsilon<1/2$*, then*

$$
\begin{aligned}
M_\delta^\sharp\left(T_{\Pi\vec{b}}(f_1,f_2)\right)(x)\leqslant C\Big\{&\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}M_{\varepsilon,\beta}(T(f_1,f_2))(x)+\|b_2\|_{Lip_{\beta_2}}M_{\varepsilon,\beta_2}(T^1_{b_1}(f_1,f_2))(x)\\
&+\|b_1\|_{Lip_{\beta_1}}M_{\varepsilon,\beta_1}(T^2_{b_2}(f_1,f_2))(x)+\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x)\Big\}.
\end{aligned}
$$

Proof. Fix $x\in\mathbb{R}^n$. Let $Q, x_Q, f_j^0, f_j^\infty, j=1,2$ and Q^* be as in the proof of Lemma 3.1. Let $\lambda_j=(b_j)_{Q^*}$, $j=1,2$. Since

$$
\begin{aligned}
T_{\Pi\vec{b}}(\vec{f})(z)&=(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)-(b_1(z)-\lambda_1)T(f_1,(b_2-\lambda_2)f_2)(z)\\
&\quad-(b_2(z)-\lambda_2)T((b_1-\lambda_1)f_1,f_2)(z)+T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z)\\
&=-(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)+(b_1(z)-\lambda_1)T^2_{b_2}(f_1,f_2)(z)\\
&\quad+(b_2(z)-\lambda_2)T^1_{b_1}(f_1,f_2)(z)+T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z).
\end{aligned}
$$

For $0<\delta<\varepsilon<1/2$, we have

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q\left||T_{\Pi\vec{b}}(f_1,f_2)(z)|^\delta-|c|^\delta\right|dz\right)^{1/\delta}&\leqslant\left(\frac{1}{|Q|}\int_Q|T_{\Pi\vec{b}}(f_1,f_2)(z)-c|^\delta dz\right)^{1/\delta}\\
&\leqslant C\left(\frac{1}{|Q|}\int_Q|(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)|^\delta dz\right)^{1/\delta}\\
&\quad+C\left(\frac{1}{|Q|}\int_Q|(b_1(z)-\lambda_1)T^2_{b_2}(f_1,f_2)(z)|^\delta dz\right)^{1/\delta}\\
&\quad+C\left(\frac{1}{|Q|}\int_Q|(b_2(z)-\lambda_2)T^1_{b_1}(f_1,f_2)(z)|^\delta dz\right)^{1/\delta}\\
&\quad+C\left(\frac{1}{|Q|}\int_Q|T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z)-c|^\delta dz\right)^{1/\delta}\\
&=J_1+J_2+J_3+J_4.
\end{aligned}
$$

For the term J_1, since $0<\delta<\varepsilon<1/2$, by (3.1) and Hölder's inequality, we get

$$\begin{aligned}J_1 &\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}|Q|^{\beta_1/n}|Q|^{\beta_2/n}\left(\frac{1}{|Q|}\int_Q |T(f_1,f_2)(z)|^\delta dz\right)^{1/\delta}\\ &\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}M_{\varepsilon,\beta}(T(f_1,f_2))(x).\end{aligned}$$

For the term J_2, notice that $0<\delta<\varepsilon<1/2$, then by (3.1) and Hölder's inequality, we have

$$\begin{aligned}J_2 &\leqslant C\|b_1\|_{Lip_{\beta_1}}|Q|^{\beta_1/n}\left(\frac{1}{|Q|}\int_Q |T^2_{b_2}(f_1,f_2)(z)|^\delta dz\right)^{1/\delta}\\ &\leqslant C\|b_1\|_{Lip_{\beta_1}}|Q^*|^{\beta_1/n}\left(\frac{1}{|Q|}\int_Q |T^2_{b_2}(f_1,f_2)(z)|^\varepsilon dz\right)^{1/\varepsilon}\\ &\leqslant C\|b_1\|_{Lip_{\beta_1}}M_{\varepsilon,\beta_1}(T^2_{b_2}(f_1,f_2))(x).\end{aligned}$$

Similarly, we have

$$J_3 \leqslant C\|b_2\|_{Lip_{\beta_2}}M_{\varepsilon,\beta_2}(T^1_{b_1}(f_1,f_2))(x).$$

Now we turn to estimate J_4. For each j, we decompose $f_j=f_j^0+f_j^\infty$, where $f_j^0=f_j\chi_{Q^*}, j=1,2$, then we get

$$\begin{aligned}J_4 &\leqslant C\left(\frac{1}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)(z)|^\delta dz\right)^{1/\delta}\\ &+C\left(\frac{1}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(z)|^\delta dz\right)^{1/\delta}\\ &+C\left(\frac{1}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^0)(z)|^\delta dz\right)^{1/\delta}\\ &+C\left(\frac{1}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(z)-c|^\delta dz\right)^{1/\delta}\\ &:=J_{41}+J_{42}+J_{43}+J_{44}.\end{aligned}$$

We first estimate J_{41}. Similar to the proof of I_2 in Lemma 3.1, we obtain

$$\begin{aligned}J_{41} &\leqslant C|Q|^{-2}\|T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)\|_{L^{1/2,\infty}(Q)}\\ &\leqslant C\left(\frac{1}{|Q^*|}\int_{Q^*}|(b_1(z)-\lambda_1)f_1(z)|dz\right)\left(\frac{1}{|Q^*|}\int_{Q^*}|(b_2(z)-\lambda_2)f_2(z)|dz\right)\\ &\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x).\end{aligned}$$

Next we consider the term J_{42}. Similar to the proof of I_3 in Lemma 3.1, note that the fact $|z-y_2|\sim|y_2-x_Q|$ for any $y_2\in(Q^*)^c$ and $z\in Q$, then by (1.1), (3.1), and (3.2), we have

$$\begin{aligned}J_{42} &\leqslant \frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^\infty)(z)|dz\\ &\leqslant \frac{C}{|Q|}\int_Q\int_{\mathbb{R}^n\setminus Q^*}\int_Q \frac{|b_1(y_1)-\lambda_1||f_1^0(y_1)||b_2(y_2)-\lambda_2||f_2^\infty(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}}dy_1dy_2dz\\ &\leqslant C\int_Q |b_1(y_1)-\lambda_1||f_1^0(y_1)|dy_1\int_{\mathbb{R}^n\setminus Q^*}\frac{|b_2(y_2)-\lambda_2||f_2^\infty(y_2)|}{(|y_2-x_Q|)^{2n}}dy_2\\ &\leqslant C\|b_1\|_{Lip_{\beta_1}}M_{1,\beta_1}(f_1)(x)|Q|\sum_{k=1}^\infty\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|b_2(y_2)-\lambda_2||f_2^\infty(y_2)|}{(|y_2-x_Q|)^{2n}}dy_2\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{1,\beta_1}(f_1)(x)|Q|\sum_{k=1}^{\infty}\frac{1}{|2^{k+3}\sqrt{n}Q|^2}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}|b_2(y_2)-\lambda_2||f_2^{\infty}(y_2)|dy_2\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}} M_{1,\beta_1}(f_1)(x)M_{1,\beta_2}(f_2)(x)\sum_{k=1}^{\infty}2^{-k}\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{1,\beta_1}(f_1)(x)\|b_2\|_{\mathrm{Lip}_{\beta_2}} M_{1,\beta_2}(f_2)(x).
\end{aligned}
$$

Similarly, we can estimate

$$J_{43}\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{1,\beta_1}(f_1)(x)\|b_2\|_{\mathrm{Lip}_{\beta_2}} M_{1,\beta_2}(f_2)(x).$$

Finally we consider the term J_{44}. Similar to the estimate of I_5 in Lemma 3.1, we will use (1.2) and (3.2). Now set $c = T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(x_Q)$, then we have

$$
\begin{aligned}
J_{44} &\leqslant \frac{C}{|Q|}\int_Q |T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(z)-T((b_1-\lambda_1)f_1^{\infty},(b_2-\lambda_2)f_2^{\infty})(x_Q)|dz\\
&\leqslant \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}|K(z,y_1,y_2)-K(x_Q,y_1,y_2)|\\
&\quad\times |(b_1(y_1)-\lambda_1)f_1^{\infty}(y_1)(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)|dy_1dy_2dz\\
&\leqslant \frac{C}{|Q|}\int_Q\sum_{k=1}^{\infty}\frac{1}{|2^{k+3}\sqrt{n}Q|^2}\int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2}|(b_1(y_1)-\lambda_1)f_1^{\infty}(y_1)|\\
&\quad\times |(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)|\omega(2^{-k})dy_1dy_2dz\\
&\leqslant \sum_{k=1}^{\infty}\frac{C}{|2^{k+3}\sqrt{n}Q|^2}\int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2}|(b_1(y_1)-\lambda_1)f_1^{\infty}(y_1)|\\
&\quad\times |(b_2(y_2)-\lambda_2)f_2^{\infty}(y_2)|\omega(2^{-k})dy_1dy_2\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}} M_{1,\beta_1}(f_1)(x)\|b_2\|_{\mathrm{Lip}_{\beta_2}} M_{1,\beta_2}(f_2)(x).
\end{aligned}
$$

The proof of Lemma 3.2 is finished. □

Proof. First we assume that for some $0<\delta<\varepsilon<1/2$, we have

$$\int_{\mathbb{R}^n}\left|M_\delta\left(T_{\Pi\vec{b}}(f_1,f_2)\right)(x)\right|^q dx<\infty, \tag{3.3}$$

$$\int_{\mathbb{R}^n}\left|M_\delta\left(T^j_{b_j}(f_1,f_2)\right)(x)\right|^q dx<\infty,\quad j=1,2, \tag{3.4}$$

where f_1, f_2 belong to C_c^{∞}.

By the assumption in Theorem 1.4, for $0<\delta<1/2<q$, $1<p_1<q_1$, $1<p_2<q_2$, and by Lemmas 2.1 and 3.2, we have

$$
\begin{aligned}
\|T_{\Pi\vec{b}}(f_1,f_2)\|_{L^q} &\leqslant C\|M_\delta(T_{\Pi\vec{b}}(f_1,f_2))\|_{L^q}\\
&\leqslant C\|M_\delta^{\sharp}(T_{\Pi\vec{b}}(f_1,f_2))\|_{L^q}\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|M_{\varepsilon,\beta}(T(f_1,f_2))\|_{L^q}\\
&\quad + C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|M_{\varepsilon,\beta_1}(T^2_{b_2}(f_1,f_2))\|_{L^q}+C\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|M_{\varepsilon,\beta_2}(T^1_{b_1}(f_1,f_2))\|_{L^q}\\
&\quad + C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|M_{1,\beta_1}(f_1)M_{1,\beta_2}(f_2)\|_{L^q}.
\end{aligned}
$$

For $1/q=1/p-\beta/n$ and $1/p=1/p_1+1/p_2$, by Lemma 2.4 and Theorem 1.2, we have

$$\|M_{\varepsilon,\beta}(T(f_1,f_2))\|_{L^q}\leqslant C\|T(f_1,f_2)\|_{L^p}\leqslant C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

For $1/q = 1/q_1^* + 1/q_2^*$, $1/q_1^* = 1/p_1 - \beta_1/n$ and $1/q_2^* = 1/p_2 - \beta_2/n$, we get

$$\|M_{1,\beta_1}(f_1)M_{1,\beta_2}(f_2)\|_{L^q} \leqslant C\|M_{1,\beta_1}(f_1)\|_{L^{q_1^*}}\|M_{1,\beta_2}(f_2)\|_{L^{q_2^*}} \leqslant C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

For $t > 0$ with $1/q = 1/t - \beta_1/n$, by Lemmas 2.1, 2.4, and 3.1, we obtain

$$\begin{aligned}\|M_{\varepsilon,\beta_1}(T^2_{b_2}(f_1,f_2))\|_{L^q} &\leqslant C\|T^2_{b_2}(f_1,f_2)\|_{L^t}\\ &\leqslant C\|M_\delta(T^2_{b_2}(f_1,f_2))\|_{L^t}\\ &\leqslant C\|M^\sharp_\delta(T^2_{b_2}(f_1,f_2))\|_{L^t}\\ &\leqslant C\|b_2\|_{Lip_{\beta_2}}\Big\{\|M_{\varepsilon,\beta_2}(T(f_1,f_2))\|_{L^t} + \|M(f_1)M_{\varepsilon,\beta_2}(f_2)\|_{L^t}\Big\}.\end{aligned}$$

For $1/t = 1/p - \beta_2/n$,

$$\|M_{\varepsilon,\beta_2}(T(f_1,f_2))\|_{L^t} \leqslant C\|T(f_1,f_2)\|_{L^p} \leqslant C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

And for $1/t = 1/p - \beta_2/n = 1/p_1 + 1/p_2 - \beta_2/n$, denote $1/t_2 = 1/p_2 - \beta_2/n$, and by Lemma 2.4, we have

$$\|M(f_1)M_{\varepsilon,\beta_2}(f_2)\|_{L^t} \leqslant C\|M(f_1)\|_{L^{p_1}}\|M_{\varepsilon,\beta_2}(f_2)\|_{L^{t_2}} \leqslant C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Thus we get

$$\|M_{\varepsilon,\beta_1}(T^2_{b_2}(f_1,f_2))\|_{L^q} \leqslant C\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Similarly we have

$$\|M_{\varepsilon,\beta_2}(T^1_{b_1}(f_1,f_2))\|_{L^q} \leqslant C\|b_1\|_{Lip_{\beta_1}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Then we have

$$\|T_{\Pi\vec{b}}(f_1,f_2)\|_{L^q} \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

In order to estimate (3.3) and (3.4), by the definition of M_δ, we only need to prove

$$\int_{\mathbb{R}^n}|T_{\Pi\vec{b}}(f_1,f_2)(x)|^q dx < \infty, \int_{\mathbb{R}^n}|T^j_{b_j}(f_1,f_2)(x)|^q dx < \infty, j = 1,2.$$

For simplicity we only prove $\int_{\mathbb{R}^n}|T^1_{b_1}(f_1,f_2)(x)|^q dx < \infty$.

Case (i). Assume that b_1, b_2 are bounded functions. Suppose $\operatorname{supp} f_j \subset B(0,R_j), j = 1,2$, since $f \in C_c^\infty$, for all $x \in B(0,R_j)$, we have $|f_j(x)| \leqslant C$, then we get

$$\begin{aligned}\int_{\mathbb{R}^n}|T^1_{b_1}(f_1,f_2)(x)|^q dx &\leqslant C\|b_1\|^q_\infty\int_{\mathbb{R}^n}|T(f_1,f_2)(x)|^q dx + \int_{\mathbb{R}^n}|T(b_1f_1,f_2)(x)|^q dx\\ &\leqslant C\|b_1\|^q_\infty\left(\int_{\mathbb{R}^n}|f_1(x)|^{q_1^*}dx\right)^{q/q_1^*}\left(\int_{\mathbb{R}^n}|f_2(x)|^{q_2^*}dx\right)^{q/q_2^*}\\ &\quad + \left(\int_{\mathbb{R}^n}|b_1(x)f_1(x)|^{q_1^*}dx\right)^{q/q_1^*}\left(\int_{\mathbb{R}^n}|f_2(x)|^{q_2^*}dx\right)^{q/q_2^*}\\ &\leqslant C\|b_1\|^q_\infty\left(\int_{\mathbb{R}^n}|f_1(x)|^{q_1^*}dx\right)^{q/q_1^*}\left(\int_{\mathbb{R}^n}|f_2(x)|^{q_2^*}dx\right)^{q/q_2^*}\\ &\leqslant C\|b_1\|^q_\infty\left(\int_{B(0,R_1)}|f_1(x)|^{q_1^*}dx\right)^{q/q_1^*}\left(\int_{B(0,R_2)}|f_2(x)|^{q_2^*}dx\right)^{q/q_2^*} < \infty.\end{aligned}$$

Thus when b_1, b_2 are bounded functions, Theorem 1.4 holds true.

Case (ii). For the general case, we will truncate the b_i as follows

$$b_i^N(x)=\begin{cases} N, & b_i(x)>N,\\ b_i(x), & |b_i(x)|\leqslant N,\\ -N, & b_i(x)<-N,\end{cases}$$

where N is a non-negative integer. Observe that in our case

$$\|b_j^N\|_{Lip_{\beta_j}}\leqslant C\|b_j\|_{Lip_{\beta_j}}.$$

Denote by $\vec{b}^N=(b_1^N,b_2^N)$, then

$$\int_{\mathbb{R}^n}\left|T_{\Pi\vec{b}^N}(f_1,f_2)(x)\right|^q dx\leqslant C\|b_1\|^q_{Lip_{\beta_1}}\|b_2\|^q_{Lip_{\beta_2}}\|f_1\|^q_{L^{p_1}}\|f_2\|^q_{L^{p_2}}.$$

Since f_1,f_2 are smooth functions with compact support, modifying the argument in [15], one can obtain that $\{T_{\Pi\vec{b}^N}(f_1,f_2)\}_{N=1}^{\infty}$ converges pointwise almost everywhere to $T_{\Pi\vec{b}}(f_1,f_2)$ as $N\to\infty$, and by Fatou's lemma we conclude that the theorem holds for this general case. □

Proof of Theorem 1.5. For any cube Q, by the definition of $Lip_{\beta-n/p}$, for any cube Q, we have

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q |T_{\Pi\vec{b}}(f_1,f_2)(z)-(T_{\Pi\vec{b}}(f_1,f_2))_Q|dz\\ &\leqslant\frac{2}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|T_{\Pi\vec{b}}(f_1,f_2)(z)-c|dz:=K.\end{aligned}$$

Let $x_Q,f_j^0,f_j^\infty,j=1,2$ and Q^* be as in the proof of Lemma 3.1 and let $\lambda_j=(b_j)_Q,j=1,2$. Set $c=c_1+c_2+c_3$, we have

$$\begin{aligned}K\leqslant&\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|T_{\Pi\vec{b}}(f_1^0,f_2^0)(z)|dz+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|T_{\Pi\vec{b}}(f_1^0,f_2^\infty)(z)-c_1|dz\\ &+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|T_{\Pi\vec{b}}(f_1^\infty,f_2^0)(z)-c_2|dz+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|T_{\Pi\vec{b}}(f_1^\infty,f_2^\infty)(z)-c_3|dz\\ &:=K_1+K_2+K_3+K_4.\end{aligned}$$

For $1<q_j<n/\beta_j<p_j,j=1,2$ and $q>1$ with $1/q=1/q_1+1/q_2-(\beta_1+\beta_2)/n$, it follows from Hölder's inequality and Theorem 1.4 that

$$\begin{aligned}K_1&\leqslant\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\left(\int_Q|T_{\Pi\vec{b}}(f_1^0,f_2^0)(z)|^q dz\right)^{1/q}|Q|^{1-1/q}\\ &\leqslant\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}|Q|^{1-1/q}\|f_1^0\|_{L^{q_1}}\|f_2^0\|_{L^{q_2}}\leqslant C\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.\end{aligned}$$

For the second term K_2, we take $c_1=T((b_1-\lambda_1)f_1^0,f_2^\infty)(x_Q)$, then we have

$$\begin{aligned}K_2\leqslant&\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)K(z,\vec{y})f_1(y_1)f_2(y_2)d\vec{y}\right|dz\\ &+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}(b_1(z)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,\vec{y})f_1(y_1)f_2(y_2)d\vec{y}\right|dz\\ &+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}(b_1(y_1)-\lambda_1)(b_2(z)-\lambda_2)K(z,\vec{y})f_1(y_1)f_2(y_2)d\vec{y}\right|dz\end{aligned}$$

$$
\begin{aligned}
&+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}(b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)[K(z,\vec{y})\right.\\
&\left.-K(x_Q,\vec{y})]f_1(y_1)f_2(y_2)d\vec{y}\right|dz\\
&:=K_{21}+K_{22}+K_{23}+K_{24}.
\end{aligned}
$$

We first estimate K_{21}. By (1.1) and (3.1), we have

$$
\begin{aligned}
K_{21}&\leqslant\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)|\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\frac{A}{(|z-y_1|+|z-y_2|)^{2n}}\\
&\quad\times|f_1(y_1)||f_2(y_2)|dy_2dy_1dz\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}|Q|^{1/p}\int_{Q^*}|f_1(y_1)|dy_1\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus2^{k+2}\sqrt{n}Q}\frac{|f_2(y_2)|}{|y_2-x_Q|^{2n}}dy_2\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}|Q|^{1/p}\int_{Q^*}|f_1(y_1)|dy_1\\
&\quad\times\sum_{k=1}^{\infty}\frac{1}{|2^{k+3}\sqrt{n}Q|^2}\int_{2^{k+3}\sqrt{n}Q\setminus2^{k+2}\sqrt{n}Q}|f_2(y_2)|dy_2\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^{\infty}2^{kn(-1-1/p_2)}\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Since $-1-\beta_2/n+n/p_2<0$, then by (1.1), (3.1), and (3.2), we have

$$
\begin{aligned}
K_{22}&\leqslant\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q|b_1(z)-\lambda_1|\\
&\quad\times\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\frac{|b_2(y_2)-\lambda_2||f_1(y_1)||f_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}}dy_1dy_2dz\\
&\leqslant\frac{C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}}{|Q|^{\beta_2/n-1/p}}\int_{Q^*}|f_1(y_1)|dy_1\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus2^{k+2}\sqrt{n}Q}\frac{|f_2(y_2)|}{|y_2-x_Q|^{2n-\beta_2}}dy_2\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^{\infty}2^{kn(-1-1/p_2+\beta_2/n)}\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Similarly, we have

$$
K_{23}\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
$$

Now we estimate K_{24}. By using the assumptions (1.3), (3.1), (3.2) and noticing the fact $1-\beta_2/n+1/p_2>0$, we have

$$
\begin{aligned}
K_{24}&\leqslant\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\int_{Q^*}\int_{\mathbb{R}^n\setminus Q^*}\frac{|b_1(y_1)-\lambda_1||b_2(y_2)-\lambda_2||f_1(y_1)||f_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}}\\
&\quad\times\omega\Big(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}\Big)dy_2dy_1dz\\
&\leqslant\frac{C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_2/n-1/p}}\int_Q\int_{Q^*}\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus2^{k+2}\sqrt{n}Q}\frac{|f_1(y_1)||f_2(y_2)|}{(|x_Q-y_2|)^{2n-\beta_2}}\times\omega\Big(\frac{|z-x_Q|}{|x_Q-y_2|}\Big)dy_2dy_1dz
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{\beta_2/n-1/p}}\int_{Q^*}|f_1(y_1)|dy_1 \times \sum_{k=1}^{\infty}\frac{\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^{2-\beta_2/n}}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}|f_2(y_2)|dy_2\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}\sum_{k=1}^{\infty}\omega(2^{-k})2^{-kn(1-\beta_2/n+1/p_2)}\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Thus

$$K_2 \leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

And similarly, we have

$$K_3 \leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.$$

Now we turn to estimate K_4. We will use the scheme for the estimate of K_4 which is similar to the discussion for K_2. For $z \in Q$ and by (3.1), after simple deductions, we have

$$|(b_1(z)-\lambda_1)T(f_1^\infty,(b_2-\lambda_2)f_2^\infty)(z)-c_{31}| \leqslant C|\|b_1\|_{\mathrm{Lip}_{\beta_1}}|Q|^{\beta_1/n}T(f_1^\infty,(b_2-\lambda_2)f_2^\infty)(z)-c'_{31}|,$$

and

$$|(b_2(z)-\lambda_2)T((b_1-\lambda_1)f_1^\infty,f_2^\infty)(z)-c_{32}| \leqslant C|\|b_2\|_{\mathrm{Lip}_{\beta_2}}|Q|^{\beta_2/n}T((b_1-\lambda_1)f_1^\infty,f_2^\infty)(z)-c'_{32}|,$$

where $c_3 = -c_{31}-c_{32}+c_{33}$, and take

$$
\begin{aligned}
c'_{31} &= C\|b_1\|_{\mathrm{Lip}_{\beta_1}}|Q|^{\beta_1/n}T(f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x_Q),\\
c'_{32} &= C\|b_2\|_{\mathrm{Lip}_{\beta_2}}|Q|^{\beta_2/n}T((b_1-\lambda_1)f_1^\infty,f_2^\infty)(x_Q),\\
c_{33} &= T((b_1-\lambda_1)f_1^\infty,(b_2-\lambda_2)f_2^\infty)(x_Q).
\end{aligned}
$$

Thus

$$
\begin{aligned}
K_4 &\leqslant \frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{(\mathbb{R}^n\setminus Q^*)^2}(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2\right|dz\\
&+\frac{C\|b_1\|_{\mathrm{Lip}_{\beta_1}}|Q|^{\beta_1/n}}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{(\mathbb{R}^n\setminus Q^*)^2}(b_2(y_2)-\lambda_2)[K(z,y_1,y_2)-K(x_Q,y_1,y_2)]f_1(y_1)f_2(y_2)dy_1dy_2\right|dz\\
&+\frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}|Q|^{\beta_2/n}}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{(\mathbb{R}^n\setminus Q^*)^2}(b_1(y_1)-\lambda_1)[K(z,y_1,y_2)\right.\\
&\left.-K(x_Q,y_1,y_2)]f_1(y_1)f_2(y_2)dy_1dy_2\right|dz\\
&+\frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\left|\int_{(\mathbb{R}^n\setminus Q^*)^2}(b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)[K(z,y_1,y_2)\right.\\
&\left.-K(x_Q,y_1,y_2)]f_1(y_1)f_2(y_2)dy_1dy_2\right|dz\\
&:= K_{41}+K_{42}+K_{43}+K_{44}.
\end{aligned}
$$

For K_{41}, by (1.1) and (3.1), we have

$$
\begin{aligned}
K_{41} &\leqslant \frac{C}{|Q|^{1+\beta_1/n+\beta_2/n-1/p}}\int_Q\int_{(\mathbb{R}^n\setminus Q^*)^2}|b_1(z)-\lambda_1||b_2(z)-\lambda_2|\frac{|f_1(y_1)||f_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n}}dy_1dy_2dz\\
&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}|Q|^{1/p}\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}\frac{|f_1(y_1)|}{|y_1-x_Q|^n}dy_1
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{i=1}^{\infty} \int_{2^{i+3}\sqrt{n}Q \backslash 2^{i+2}\sqrt{n}Q} \frac{|f_2(y_2)|}{|y_2 - x_Q|^n} dy_2 \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} |Q|^{1/p} \sum_{k=1}^{\infty} |2^{k+3}\sqrt{n}Q|^{-1/p_1} \sum_{i=1}^{\infty} |2^{i+3}\sqrt{n}Q|^{-1/p_2} \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Since $0 < \beta_2/n - 1/p_2 < \beta/n - 1/p$, it follows from (1.2), (3.1), and (3.2) that

$$
\begin{aligned}
K_{42} &\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} |Q|^{-\beta_2/n + 1/p} \sum_{k=1}^{\infty} \int_{2^{k+3}\sqrt{n}Q \backslash 2^{k+2}\sqrt{n}Q} \frac{|f_1(y_1)|}{|y_1 - x_Q|^n} dy_1 \\
&\quad \times \sum_{i=1}^{\infty} \int_{2^{i+3}\sqrt{n}Q \backslash 2^{i+2}\sqrt{n}Q} \frac{|f_2(y_2)|}{|y_2 - x_Q|^{n-\beta_2}} \omega(2^{-i}) dy_2 \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{k=1}^{\infty} |2^{k+3}\sqrt{n}Q|^{-1/p_1} |Q|^{1/p_1} \\
&\quad \times \sum_{i=1}^{\infty} \omega(2^{-i}) |2^{i+3}\sqrt{n}Q|^{\beta_2/n - 1/p_2} |Q|^{-\beta_2/n + 1/p_2} \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{i=1}^{\infty} \omega(2^{-i}) 2^{in(\beta_2/n - 1/p_2)} \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{i=1}^{\infty} \omega(2^{-i}) 2^{in(\beta/n - 1/p)} \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Similarly, we have

$$
K_{43} \leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
$$

Now we turn to estimate K_{44}, by (3.2) and (1.2), we obtain

$$
\begin{aligned}
K_{44} &\leqslant \frac{C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}}}{|Q|^{\beta_1/n + \beta_2/n - 1/p}} \sum_{k=1}^{\infty} \int_{(2^{k+3}\sqrt{n}Q)^2 \backslash (2^{k+2}\sqrt{n}Q)^2} \frac{|f_1(y_1)||f_2(y_2)|}{(|y_2 - x_Q|)^{2n-\beta_1-\beta_2}} \times \omega(\frac{|z - x_Q|}{|y_2 - x_Q|}) dy_1 dy_2 \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{k=1}^{\infty} \omega(2^{-k}) 2^{-kn(1/p - \beta_1/n - \beta_2/n)} \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \sum_{k=1}^{\infty} \omega(2^{-k}) 2^{kn(\beta/n - 1/p)} \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}} \|b_2\|_{Lip_{\beta_2}} \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Thus we finish the proof of Theorem 1.5. □

Proof of Theorem 1.6. We will use the same discussion as in Lemma 3.1, for $z \in Q$, let $x_Q, f_j^0, f_j^\infty, j = 1, 2, Q^*$ be as in the proof of Lemma 3.1 and let $\lambda_j = (b_j)_Q, j = 1, 2$. Similar to the proof of Lemma 3.2 , we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T_{\Pi\vec{b}}(f_1, f_2)(z) - (T_{\Pi\vec{b}}(f_1, f_2))_Q \right| dz \\
&\leqslant \frac{2}{|Q|^{1+\beta/n}} \int_Q \left| T_{\Pi\vec{b}}(f_1, f_2)(z) - c \right| dz
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{C}{|Q|^{1+\beta/n}}\int_Q \Big|(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)\Big|dz\\
&\quad+\frac{C}{|Q|^{1+\beta/n}}\int_Q \Big|(b_2(z)-\lambda_2)T^1_{b_1}(f_1,f_2)(z)-c_1\Big|dz\\
&\quad+\frac{C}{|Q|^{1+\beta/n}}\int_Q \Big|(b_1(z)-\lambda_1)T^2_{b_2}(f_1,f_2)(z)-c_2\Big|dz\\
&\quad+\frac{C}{|Q|^{1+\beta/n}}\int_Q \Big|T((b_1-\lambda_1)f_1,(b_2-\lambda_2)f_2)(z)-c_3\Big|dz\\
&:= L_1+L_2+L_3+L_4,
\end{aligned}
$$

where $c=c_1+c_2+c_3$.

First we estimate L_1. For $1<r<p$ and then by (3.1), we have

$$L_1 \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}M_r(T(f_1,f_2))(x).$$

Next we estimate L_2. We will use the same method in the proof of K_4 in Theorem 1.5,

$$L_2 \leqslant \frac{C}{|Q|^{1+\beta/n}}\int_Q \Big|\|b_2\|_{Lip_{\beta_2}}|Q|^{\beta_2/n}T^1_{b_1}(f_1,f_2)(z)-c_1'\Big|dz.$$

Since

$$
\begin{aligned}
T^1_b(f_1,f_2)(z) = (b_1(z)-\lambda_1)T(f_1,f_2)(z) &- T((b_1-\lambda_1)f_1^0,f_2^0)(z)\\
&-\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^\infty(y_1)f_2^0(y_2)K(z,y_1,y_2)dy_1dy_2\\
&-\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^0(y_1)f_2^\infty(y_2)K(z,y_1,y_2)dy_1dy_2\\
&-\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^\infty(y_1)f_2^\infty(y_2)K(z,y_1,y_2)dy_1dy_2.
\end{aligned}
$$

Take

$$
\begin{aligned}
c_1' = &\|b_2\|_{Lip_{\beta_2}}|Q|^{\beta_2/n}\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^\infty(y_1)f_2^0(y_2)K(x_Q,y_1,y_2)dy_1dy_2\\
&+\|b_2\|_{Lip_{\beta_2}}|Q|^{\beta_2/n}\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^0(y_1)f_2^\infty(y_2)K(x_Q,y_1,y_2)dy_1dy_2\\
&+\|b_2\|_{Lip_{\beta_2}}|Q|^{\beta_2/n}\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^\infty(y_1)f_2^\infty(y_2)K(x_Q,y_1,y_2)dy_1dy_2.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
L_2 &\leqslant \frac{C\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q|(b_1(z)-\lambda_1)T(f_1,f_2)(z)|dz+\frac{C\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q|T((b_1-\lambda_1)f_1^0,f_2^0)(z)|dz\\
&\quad+\frac{C\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\Big|\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^0(y_1)f_2^\infty(y_2)[K(z,y_1,y_2)-K(x_Q,y_1,y_2)]dy_1dy_2\Big|dz\\
&\quad+\frac{C\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\Big|\int_{\mathbb{R}^{2n}}(b(y_1)-\lambda_1)f_1^\infty(y_1)f_2^0(y_2)[K(z,y_1,y_2)-K(x_Q,y_1,y_2)]dy_1dy_2\Big|dz\\
&\quad+\frac{C\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\Big|\int_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)f_1^\infty(y_1)f_2^\infty(y_2)[K(z,y_1,y_2)-K(x_Q,y_1,y_2)]dy_1dy_2\Big|dz\\
&:= L_{21}+L_{22}+L_{23}+L_{24}+L_{25}.
\end{aligned}
$$

We make use of Lemma 2.1 and Hölder's inequality for $1<r<p$, then we get

$$\begin{aligned}L_{21} &\leqslant C\|b_2\|_{\mathrm{Lip}_{\beta_2}}\left(\frac{1}{|Q|^{r'\beta_1/n+1}}\int_Q |b_1(z)-\lambda_1|^{r'}dz\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T(f_1,f_2)(z)|^r dz\right)^{1/r}\\&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}M_r(T(f_1,f_2))(x).\end{aligned}$$

Take $1<q_1<p_1, 1<q_2<p_2$ and $1<q<\infty$, such that $1/q=1/q_1+1/q_2$, then by Hölder's inequality, Theorem 1.2, and (3.1), we get

$$\begin{aligned}L_{22} &\leqslant \frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{\beta_1/n+1/q}}\left(\int_Q |T((b_1-\lambda_1)f_1^0,f_2^0)(z)|^q dz\right)^{1/q}\\&\leqslant \frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{\beta_1/n+1/q}}\|(b_1-\lambda_1)f_1^0\|_{L^{q_1}}\|f_2^0\|_{L^{q_2}}\\&\leqslant \frac{C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{1/q}}\|f_1^0\|_{L^{q_1}}\|f_2^0\|_{L^{q_2}}\\&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}M_{q_1}(f_1)(x)M_{q_2}(f_2)(x).\end{aligned}$$

For $y_2\in(Q^*)^c$, $|y_2-x_Q|\sim|y_2-z|$, $|z-x_Q|<\frac{|y_2-z|}{2}\leqslant\frac{1}{2}\max\{|z-y_1|,|z-y_2|\}$, then by (1.2) and (3.2), we have

$$\begin{aligned}L_{23} &\leqslant \frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\int_Q\int_{Q^*}\int_{(Q^*)^c}|b(y_1)-\lambda_1||f_1(y_1)||f_2(y_2)||K(z,y_1,y_2)-K(x_Q,y_1,y_2)|dy_2dy_1dz\\&\leqslant \frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\|b_1\|_{\mathrm{Lip}_{\beta_1}}\int_Q\int_{Q^*}\int_{(Q^*)^c}|y_1-x_Q|^{\beta_1}|f_1(y_1)||f_2(y_2)|\\&\quad\times|K(z,y_1,y_2)-K(x_Q,y_1,y_2)|dy_2dy_1dz\\&\leqslant \frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\|b_1\|_{\mathrm{Lip}_{\beta_1}}\int_Q\int_{Q^*}\int_{(Q^*)^c}|f_1(y_1)||f_2(y_2)|\frac{A|y_1-x_Q|^{\beta_1}}{(|z-y_1|+|z-y_2|)^{2n}}\\&\quad\times\omega(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|})dy_2dy_1dz\\&\leqslant \frac{C\|b_2\|_{\mathrm{Lip}_{\beta_2}}}{|Q|^{1+\beta_1/n}}\|b_1\|_{\mathrm{Lip}_{\beta_1}}\int_Q\int_{Q^*}\int_{(Q^*)^c}\frac{|f_1(y_1)||f_2(y_2)|}{(|z-y_1|+|z-y_2|)^{2n-\beta_1}}\times\omega(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|})dy_2dy_1dz\\&\leqslant \frac{C}{|Q|}\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\int_Q\int_{Q^*}|f_1(y_1)|\int_{(Q^*)^c}\frac{|f_2(y_2)|}{(|z-y_2|)^{2n}}\omega(\frac{|z-x_Q|}{|z-y_2|})dy_2dy_1dz\\&\leqslant C\frac{1}{|Q|}\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\int_Q\int_{Q^*}|f_1(y_1)|\\&\quad\times\sum_{k=1}^{\infty}\int_{2^{k+3}\sqrt{n}Q\setminus 2^{k+2}\sqrt{n}Q}|f_2(y_2)||2^k\sqrt{n}Q|^{-2}\omega(2^{-k})dy_2dy_1dz\\&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}\frac{1}{|Q|}\int_{Q^*}|f_1(y_1)|dy_1\\&\quad\times\sum_{k=1}^{\infty}|Q||2^{k+3}Q|^{-1}\omega(2^{-k})\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_2(y_2)|dy_2\\&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}M(f_1)(x)\sum_{k=1}^{\infty}2^{-k}\omega(2^{-k})\frac{1}{|2^{k+3}\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_2(y_2)|dy_2\\&\leqslant C\|b_1\|_{\mathrm{Lip}_{\beta_1}}\|b_2\|_{\mathrm{Lip}_{\beta_2}}M(f_1)(x)M(f_2)(x).\end{aligned}$$

Employing the same idea for the proof of L_{23}, with some minor changes, we may get

$$L_{24} \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} M(f_1)(x)M(f_2)(x).$$

Since $y_1, y_2 \in (Q^*)^c, z \in Q$, we have $|y_1 - x_Q| \sim |z-y_1|, |z-y_2| \sim |y_2 - x_Q|$. Then by (1.2) and (3.2), we obtain

$$\begin{aligned}
L_{25} &\leqslant \frac{C\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(Q^*)^c}\int_{(Q^*)^c} |b_1(y_1)-\lambda_1||f_1(y_1)||f_2(y_2)| \\
&\quad \times |K(z,y_1,y_2) - K(x_Q,y_1,y_2)| dy_2 dy_1 dz \\
&\leqslant \frac{C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(Q^*)^c}\int_{(Q^*)^c} |y_1 - x_Q|^{\beta_1}|f_1(y_1)||f_2(y_2)| \\
&\quad \times |K(z,y_1,y_2) - K(x_Q,y_1,y_2)| dy_2 dy_1 dz \\
&\leqslant \frac{C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(Q^*)^c}\int_{(Q^*)^c} |f_1(y_1)||f_2(y_2)| \frac{A|y_1-x_Q|^{\beta_1}}{(|z-y_1|+|z-y_2|)^{2n}} \\
&\quad \times \omega(\frac{|z-x_Q|}{|z-y_1|+|z-y_2|}) dy_2 dy_1 dz \\
&\leqslant \frac{C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \int_{(R\setminus Q^*)^2} \frac{|f_1(y_1)||f_2(y_2)|}{(|y_1-x_Q|)^{2n-\beta_1}} \omega(\frac{|z-x_Q|}{|z-y_1|}) dy_1 dy_2 dz \\
&\leqslant \frac{C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}}{|Q|^{1+\beta_1/n}} \int_Q \sum_{k=1}^{\infty} \int_{(2^{k+3}\sqrt{n}Q)^2\setminus(2^{k+2}\sqrt{n}Q)^2} \frac{|f_1(y_1)||f_2(y_2)|}{(|y_1-x_Q|)^{2n-\beta_1}} \times \omega(\frac{|z-x_Q|}{|z-y_1|}) dy_1 dy_2 dz \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} \sum_{k=1}^{\infty} \frac{2^{k\beta_1}\omega(2^{-k})}{|2^{k+3}\sqrt{n}Q|^2} \int_{2^{k+3}\sqrt{n}Q}\int_{2^{k+3}\sqrt{n}Q} |f_1(y_1)||f_2(y_2)| dy_1 dy_2 \\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} M(f_1)(x)M(f_2)(x).
\end{aligned}$$

In conclusion, it follows from the estimates for $L_{21}, L_{22}, L_{23}, L_{24}, L_{25}$ that

$$L_2 \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} \Big\{ M_r(T(f_1,f_2))(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x) \Big\}.$$

And similarly, we have

$$L_3 \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} \Big\{ M_r(T(f_1,f_2))(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x) \Big\}.$$

Since the estimate for L_4 is similar to the discussion in L_{25}, then we have

$$L_4 \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} \Big\{ M_r(T(f_1,f_2))(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x) \Big\}.$$

Thus

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta/n}} \int_Q \Big| T_{\Pi\vec{b}}(f_1,f_2)(z) - (T_{\Pi\vec{b}}(f_1,f_2))_Q \Big| dz \\
&\quad \leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}} \Big\{ M_r(T(f_1,f_2))(x) + M_{q_1}(f_1)(x)M_{q_2}(f_2)(x) + M(f_1)(x)M(f_2)(x) \Big\}.
\end{aligned}$$

Since $1 < r < p$ and $1 < q_1 < p_1, 1 < q_2 < p_2$, then by Minkowski's inequality and Theorem 1.2, we arrive at

$$\begin{aligned}
&\|T_{\Pi\vec{b}}(f_1,f_2)\|_{\dot{F}_p^{\beta,\infty}} \\
&\quad \approx \Big\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q \Big| T_{\Pi\vec{b}}(f_1,f_2)(z) - (T_{\Pi\vec{b}}(f_1,f_2))_Q \Big| dz \Big\|_{L^p}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\Big\{\|M_r(T(f_1,f_2))\|_{L^p}+\|M_{q_1}(f_1)M_{q_2}(f_2)\|_{L^p}+\|M(f_1)(x)M(f_2)(x)\|_{L^p}\Big\}\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\Big\{\|T(f_1,f_2))\|_{L^p}+\|M_{q_1}(f_1)\|_{L^{p_1}}\|M_{q_2}(f_2)\|_{L^{p_2}}+\|M(f_1)\|_{L^{p_1}}\|M(f_2)\|_{L^{p_2}}\Big\}\\
&\leqslant C\|b_1\|_{Lip_{\beta_1}}\|b_2\|_{Lip_{\beta_2}}\|f_1\|_{L^{p_1}}\|f_2\|_{L^{p_2}}.
\end{aligned}
$$

Thus we finish the proof of Theorem 1.6. □

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