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A fixed point theorem for F-Khan-contractions on complete metric spaces and application to integral equations

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Abstract

In this article, we introduce a new concept of contraction called F-Khan-contractions and prove a fixed point theorem concerning this contraction which generalizes the results announced by Khan [M. S. Khan, Rend. Inst. Math. Univ. Trieste., **8** (1976), 69–72], Fisher [B. Fisher, Riv. Math. Univ. Parma., **4** (1978), 135–137], and Piri et al. [H. Piri, S. Rahrovi, P. Kumam, J. Math. Computer Sci., **17** (2017), 76–83]. An example and application for the solution of certain integral equations are given to illustrate the usability of the obtained results.

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1. Introduction

The Banach contraction principle is one of the most fundamental and important results in modern mathematics which is widely applied in many other branches of science and applied science. The Banach contraction principle provides a constructive method of finding a unique solution for models involving various types of differential and integral equations. This principle is generalized by several authors in various directions (see [1, 6–8] and references therein).

In recent years an interesting but different generalization of Banach-contraction theorem has been given by Wardowski [9]. He introduced a new contraction called F-contraction and established a fixed

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point result as a generalization of the Banach contraction principle in a dissimilar way than in the other acknowledged results from the literature.

On the other hand, some generalizations of Banach contraction principle are obtained by contraction conditions containing rational expressions. In this direction, in 1973, Geraghty [5] introduced a contraction in which the contraction constant was replaced by a function having some specific properties. Since then, several papers which dealt with fixed point theory for rational Geraghty contractive mappings have appeared (see, e.g., [2, 10] and references therein). One of the well-known works in this direction was established by Khan [6] and revised by Fisher [4] as follows.

Theorem 1.1 ([4]). *Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$d(Tx, Ty) \leqslant \begin{cases} k\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(Tx,y)}, & \text{if } d(x, Ty) + d(Tx, y) \neq 0, \\ 0, & \text{if } d(x, Ty) + d(Tx, y) = 0, \end{cases} \tag{1.1}$$

where $k \in [0, 1)$ and $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.

Recently, Piri et al., [8] extended the results of Khan [6] and Fisher [4] by introducing a new general contractive condition with rational expressions as follows:

Theorem 1.2 ([8]). *Let (X, d) be a complete metric space and $T : X \to X$ satisfy*

$$d(Tx, Ty) \leqslant \begin{cases} k\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}}, & \text{if } \max\{d(x, Ty), d(Tx, y)\} \neq 0, \\ 0, & \text{if } \max\{d(x, Ty), d(Tx, y)\} = 0 \end{cases} \tag{1.2}$$

for some $k \in [0, 1)$ and $x, y \in X$. Then T has a unique fixed point $x^* \in X$. Moreover, for all $x \in X$, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^* .

Also by providing some examples, Piri et al. [8] showed that their results are a proper generalization of Fisher [4] and Khan [6].

Following this direction of research, in the present article, we will present some fixed point results of F-Khan-type self-mappings on complete metric spaces. Moreover, an example and application for the solution of certain integral equations are given to illustrate the usability of the obtained results.

2. Preliminaries

Definition 2.1 ([9]). Let $\mathcal{F}$ be the family of all functions $F: (0, \infty) \to \mathbb{R}$ such that

(F1) F is strictly increasing, i.e. for all $x, y \in (0, \infty)$ such that $x < y$, $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 2.2 ([9]). Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction on (X, d), if there exist $F \in \mathcal{F}$ and $\tau \in (0, \infty)$ such that

$$\forall x, y \in X, \quad [d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leqslant F(d(x, y))].$$

Wardowski [9] stated a modified version of Banach contraction principle as follows.

Theorem 2.3 ([9]). *Let (X, d) be a complete metric space and $T : X \to X$ an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Definition 2.4. Let $\mathcal{F}_K$ be the family of all increasing functions $F\colon (0,\infty) \to \mathbb{R}$, i.e. for all $x, y \in (0,\infty)$, if $x < y$, then $F(x) \leqslant F(y)$.

Definition 2.5. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-Khan-contraction if there exists $\tau \in (0,\infty)$ and $F \in \mathcal{F}_K$ such that for all $x, y \in X$ if $\max\{d(x, Ty), d(Tx, y)\} \neq 0$, then $Tx \neq Ty$ and

$$\tau + F(d(Tx, Ty)) \leqslant F\left(\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)\}}\right), \tag{2.1}$$

and if $\max\{d(x, Ty), d(Tx, y)\} = 0$, then $Tx = Ty$.

Example 2.6. Let $F_1(\alpha) = \ln(\alpha)$, $\alpha > 0$. Obviously $F \in \mathcal{F}_K$ and for F_1-Khan-contraction T, for all $x, y \in X$ such that $\max\{d(x, Ty), d(Tx, y)\} \neq 0$, the following condition holds:

$$d(Tx, Ty) \leqslant e^{-\tau}\frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)\}}. \tag{2.2}$$

Example 2.7. Let $F_2(\alpha) = \frac{-1}{\alpha}$, $\alpha > 0$. Obviously $F_2 \in \mathcal{F}_K$ and for F_2-Khan-contraction T, for all $x, y \in X$ such that $\max\{d(x, Ty), d(Tx, y)\} \neq 0$, the following condition holds:

$$d(Tx, Ty) \leqslant \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\tau\,[d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)] + \max\{d(x, Ty), d(Tx, y)\}}. \tag{2.3}$$

Example 2.8. Let $F_3 : (0,\infty) \to \mathbb{R}$ be given by the formula $F_3(\alpha) = [\alpha]$, where $[\alpha]$ denotes the integer part of α. Obviously $F_3 \in \mathcal{F}_K$ and for F_3-Khan-contraction T, for all $x, y \in X$ such that $\max\{d(x, Ty), d(Tx, y)\} \neq 0$, the following condition holds:

$$\tau + d(Tx, Ty) \leqslant 1 + \frac{d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx)}{\max\{d(x, Ty), d(Tx, y)\}}. \tag{2.4}$$

Example 2.9. Let $X = \{0, 1, 2, 3\}$ and $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Let $T : X \to X$ be defined by

$$T(2) = T(1) = 1,\ T(0) = 2, T(3) = 0.$$

Now we consider the following cases:

Case 1. Let $x = 0$ and $y = 1$, then

$$\begin{aligned} d(Tx, Ty) &= d(2, 1) = 1,\\ d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx) &= d(0, 2)d(0, 1) + d(1, 1)d(1, 2) = 2,\\ \max\{d(x, Ty), d(Tx, y)\} &= \max\{d(0, 1), d(2, 1)\} = 1. \end{aligned}$$

Case 2. Let $x = 0$ and $y = 2$, then

$$\begin{aligned} d(Tx, Ty) &= d(2, 1) = 1,\\ d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx) &= d(0, 2)d(0, 1) + d(2, 1)d(2, 2) = 2,\\ \max\{d(x, Ty), d(Tx, y)\} &= \max\{d(0, 1), d(2, 2)\} = 1. \end{aligned}$$

Case 3. Let $x = 0$ and $y = 3$, then

$$\begin{aligned} d(Tx, Ty) &= d(2, 0) = 2,\\ d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx) &= d(0, 2)d(0, 0) + d(3, 0)d(3, 2) = 3,\\ \max\{d(x, Ty), d(Tx, y)\} &= \max\{d(0, 0), d(2, 3)\} = 1. \end{aligned}$$

Case 4. Let $x = 1$ and $y = 2$, then

$$\begin{aligned} d(Tx, Ty) &= d(1, 1) = 0, \\ d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx) &= d(1, 1)d(1, 1) + d(2, 1)d(2, 1) = 1, \\ \max\{d(x, Ty), d(Tx, y)\} &= \max\{d(1, 1), d(2, 1)\} = 1. \end{aligned}$$

Case 5. Let $x = 1$ and $y = 3$, then

$$\begin{aligned} d(Tx, Ty) &= d(1, 0) = 1, \\ d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx) &= d(1, 1)d(1, 0) + d(3, 0)d(3, 1) = 6, \\ \max\{d(x, Ty), d(Tx, y)\} &= \max\{d(1, 0), d(1, 3)\} = 2. \end{aligned}$$

Case 6. Let $x = 2$ and $y = 3$, then

$$\begin{aligned} d(Tx, Ty) &= d(1, 0) = 1, \\ d(x, Tx)d(x, Ty) + d(y, Ty)d(y, Tx) &= d(2, 1)d(2, 0) + d(3, 0)d(3, 2) = 5, \\ \max\{d(x, Ty), d(Tx, y)\} &= \max\{d(2, 0), d(1, 3)\} = 2. \end{aligned}$$

For $\tau \in (0, \ln \frac{3}{2}]$, we have $3e^{-\tau} \geqslant 2$ and $2e^{-\tau} \geqslant 1$. Therefore, T satisfies in condition (2.2).
For $\tau \in (0, \frac{1}{6}]$, we have $\frac{2}{2\tau+1} \geqslant 1$ and $\frac{3}{3\tau+1} \geqslant 2$. Therefore, T satisfies in condition (2.3).
For $\tau \in (0, 2]$, T satisfies in condition (2.4).

3. Main results

Our main theorem is essentially inspired by Khan [6], Fisher [4], Wardowski [9], and Piri et al. [8]. More precisely, we state and prove the following result.

Theorem 3.1. *Let (X, d) be a complete metric space and $T\colon X \to X$ be an F-Khan-contraction. Then, T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof. Let $x_0 = x \in X$. Put $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n = 0, 1, 2, \cdots$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n-1}$, then x_{n-1} is a fixed point of T. This completes the proof. Therefore, we suppose $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$. We shall divide the proof into two cases.

Cases 1. Assume that $d(x_{n-1}, Tx_n) \neq 0$, for all $n \in \mathbb{N}$. Then, from (2.1) we have

$$\begin{aligned} F(d(x_n, Tx_n)) &< \tau + F(d(Tx_{n-1}, Tx_n)) \\ &\leqslant F\left(\frac{d(x_{n-1}, Tx_{n-1})d(x_{n-1}, Tx_n) + d(x_n, Tx_n)d(x_n, Tx_{n-1})}{\max\{d(x_{n-1}, Tx_n), d(Tx_{n-1}, x_n)\}}\right) \\ &= F(d(x_{n-1}, Tx_{n-1})). \end{aligned} \tag{3.1}$$

Since $F \in F_K$, so from (3.1) we have

$$d(x_n, Tx_n) < d(x_{n-1}, Tx_{n-1}), \quad \forall n \in \mathbb{N}.$$

Therefore $\{d(x_n, Tx_n)\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence of nonnegative real numbers, and hence

$$\lim_{n\to\infty} d(x_n, Tx_n) = \gamma \geqslant 0.$$

Since $\{d(x_n, Tx_n)\}_{n\in\mathbb{N}}$ is a nonnegative strictly decreasing sequence, so for every $n \in \mathbb{N}$, we have

$$d(x_n, Tx_n) \geqslant \gamma. \tag{3.2}$$

Now, we claim that $\gamma = 0$. Arguing by contradiction, we assume that $\gamma > 0$. From (3.2) and $F \in \mathcal{F}_K$, we get

$$F(\gamma) \leqslant F(d(x_n, Tx_n)) \leqslant F(d(x_{n-1}, Tx_{n-1})) - \tau \leqslant F(d(x_{n-2}, Tx_{n-2})) - 2\tau \leqslant \cdots \leqslant F(d(x_0, Tx_0)) - n\tau \tag{3.3}$$

for all $n \in \mathbb{N}$. Since $F(\gamma) \in \mathbb{R}$ and $\lim_{n\to\infty}[F(d(x_0, Tx_0)) - n\tau] = -\infty$, so there exists $n_1 \in \mathbb{N}$ such that

$$F(d(x_0, Tx_0)) - n\tau < F(\gamma), \quad \forall n > n_1. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$F(\gamma) \leqslant F(d(x_0, Tx_0)) - n\tau < F(\gamma), \quad \forall n > n_1.$$

It is a contradiction. Therefore, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.5}$$

Now, we claim that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Arguing by contradiction, we assume that there exist $\epsilon > 0$, the sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n, \;\; d(x_{p(n)}, x_{q(n)}) \geqslant \epsilon, \;\; d(x_{p(n)-1}, x_{q(n)}) < \epsilon, \quad \forall n \in \mathbb{N}. \tag{3.6}$$

By triangular inequality, we have

$$d(x_{p(n)}, x_{q(n)}) \leqslant d(x_{p(n)}, Tx_{q(n)}) + d(Tx_{q(n)}, x_{q(n)}).$$

It follows from (3.5) and (3.6) that

$$\epsilon \leqslant \liminf_{n\to\infty} d(x_{p(n)}, Tx_{q(n)}).$$

So, there exists $n_2 \in \mathbb{N}$ such that for all $n \geqslant n_2$, $d(x_{p(n)}, Tx_{q(n)}) > \frac{\epsilon}{2}$. Therefore

$$\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\} > \frac{\epsilon}{2}, \qquad \forall n \geqslant n_2. \tag{3.7}$$

Again by triangular inequality, we have

$$d(x_{p(n)}, x_{q(n)}) \leqslant d(x_{p(n)}, Tx_{p(n)}) + d(Tx_{p(n)}, Tx_{q(n)}) + d(Tx_{q(n)}, x_{q(n)}).$$

It follows from (3.5) and (3.6) that

$$\epsilon \leqslant \liminf_{n\to\infty} d(Tx_{p(n)}, Tx_{q(n)}).$$

So, there exists $n_3 \in \mathbb{N}$ such that for all $n \geqslant n_3$,

$$d(Tx_{p(n)}, Tx_{q(n)}) > \frac{\epsilon}{2}. \tag{3.8}$$

Since $F \in F_K$, so from (2.1), (3.7), and (3.8), for all $n \geqslant \max\{n_2, n_3\}$, we have

$$\begin{aligned} \tau + F(\frac{\epsilon}{2}) &\leqslant \tau + F(d(Tx_{p(n)}, Tx_{q(n)})) \\ &\leqslant F\left(\frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}}\right). \end{aligned} \tag{3.9}$$

From (3.7), for $n \geqslant n_2$, we have

$$\begin{aligned} 0 &\leqslant \frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}} \\ &= \frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}} + \frac{d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}} \\ &\leqslant \frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)})}{d(x_{p(n)}, Tx_{q(n)})} + \frac{d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{d(Tx_{p(n)}, x_{q(n)})} \\ &= d(x_{p(n)}, Tx_{p(n)}) + d(x_{q(n)}, Tx_{q(n)}). \end{aligned} \tag{3.10}$$

It follows from (3.5), (3.10), and sandwich theorem that

$$\lim_{n\to\infty} \frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}} = 0.$$

So there exists $n_4 \in \mathbb{N}$ such that for all $n > n_4$,

$$\frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}} < \frac{\epsilon}{2}.$$

Since $F \in F_K$, so for all $n > n_4$, we have

$$F\left(\frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}}\right) \leqslant F(\frac{\epsilon}{2}). \tag{3.11}$$

From (3.9) and (3.11), for all $n \geqslant \max\{n_2, n_3, n_4\}$, we obtain that

$$\tau + F(\frac{\epsilon}{2}) \leqslant F\left(\frac{d(x_{p(n)}, Tx_{p(n)})d(x_{p(n)}, Tx_{q(n)}) + d(x_{q(n)}, Tx_{q(n)})d(x_{q(n)}, Tx_{p(n)})}{\max\{d(x_{p(n)}, Tx_{q(n)}), d(Tx_{p(n)}, x_{q(n)})\}}\right) \leqslant F(\frac{\epsilon}{2}).$$

This contradiction shows that $\{x_n\}$ is a Cauchy sequence. By Completeness of (X, d), $\{x_n\}$ converges to some point x^* in X. Therefore

$$\lim_{n\to\infty} d(x_n, x^*) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Tx_n, Tx^*) = d(x^*, Tx^*). \tag{3.12}$$

Now, we claim that $d(x^*, Tx^*) = 0$. By contradiction, we assume that $d(x^*, Tx^*) > 0$. We only have the following two cases

(I) $\forall n \in \mathbb{N},\ \exists i_n \in \mathbb{N},\ i_n > i_{n-1},\ i_0 = 1$ and $x_{i_n+1} = Tx^*$;

(II) $\exists N \in \mathbb{N},\ \ \forall n \geqslant N,\ d(x_n, Tx^*) > 0$.

In the first case, from (3.12) we have

$$x^* = \lim_{n\to\infty} x_{i_{n+1}} = Tx^*.$$

In the second case, for all $n \geqslant N$, we have

$$\max\{d(x_n, Tx^*), d(Tx_n, x^*)\} > 0.$$

So from (2.1), we get

$$\tau + F(d(Tx_n, Tx^*)) \leqslant F\left(\frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}}\right). \tag{3.13}$$

On the other hand, from (3.5) and (3.12) we have

$$\lim_{n\to\infty} \frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}} = 0.$$

Since $d(x^*, Tx^*) > 0$, so there exists $n_5 \in \mathbb{N}$ such that for all $n \geqslant n_5$,

$$\frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}} < \frac{1}{2}d(x^*, Tx^*).$$

Therefore

$$F\left(\frac{d(x_n, Tx_n)d(x_n, Tx^*) + d(x^*, Tx^*)d(x^*, Tx_n)}{\max\{d(x_n, Tx^*), d(Tx_n, x^*)\}}\right) \leqslant F\left(\frac{1}{2}d(x^*, Tx^*)\right), \qquad \forall n \geqslant n_5. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\tau + F(d(Tx_n, Tx^*)) \leqslant F\left(\frac{1}{2}d(x^*, Tx^*)\right), \quad \forall n \geqslant n_5.$$

So, we get

$$d(Tx_n, Tx^*) \leqslant \frac{1}{2}d(x^*, Tx^*), \quad \forall n \geqslant n_5,$$

and from (3.12), we obtain

$$d(x^*, Tx^*) \leqslant \frac{1}{2}d(x^*, Tx^*).$$

This is contradiction. So, we have $x^* = Tx^*$. Now, we show that T has a unique fixed point. For this, we assume that y^* is another fixed point of T in X such that $d(x^*, y^*) > 0$. Therefore

$$\max\{d(x^*, Ty^*), d(Tx^*, y^*)\} > 0.$$

So from (2.1), we get

$$F(d(x^*, y^*)) = F(d(Tx^*, Ty^*)) < \tau + F(d(Tx^*, Ty^*)) \leqslant F\left(\frac{d(x^*, Tx^*)d(x^*, Ty^*) + d(y^*, Ty^*)d(y^*, Tx^*)}{\max\{d(x^*, Ty^*), d(Tx^*, y^*)\}}\right).$$

Since

$$\frac{d(x^*, Tx^*)d(x^*, Ty^*) + d(y^*, Ty^*)d(y^*, Tx^*)}{\max\{d(x^*, Ty^*), d(Tx^*, y^*)\}} = 0,$$

this leads to a contradiction and hence $x^* = y^*$. This completes the proof.

Cases 2. Assume that there exists $m \in \mathbb{N}$ such that

$$d(x_{m-1}, Tx_m) = 0.$$

By assumption of theorem, we have $d(Tx_{m-1}, Tx_m) = 0$ and hence $x_m = Tx_m$. This completes the proof of the existence of a fixed point of T. The uniqueness follows as in Case 1. □

Remark 3.2. Theorem 3.1 is the generalization of Theorem 1.2, since for the mapping F of the form in Example 2.6, an F-Khan contraction mapping becomes the contraction explained in Theorem 1.2.

Theorem 3.3. *Let (X, d) be a complete metric space and $T^n : X \to X$ be an F-Khan-contraction for some $n \in \mathbb{N}$. Then, T has a unique fixed point $x^* \in X$ and for every $x \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.*

Proof. By Theorem 3.1, T^n has a unique fixed point $x^* \in X$. Then, we have $T^n(Tx^*) = T(T^n x^*) = Tx^*$ and so Tx^* is a fixed point of T^n. Therefore, by uniqueness of the fixed point of T^n it must be that $Tx^* = x^*$. □

Example 3.4. Consider the sequence $\{S_n\}_{n\in\mathbb{N}}$ as follows:

$$S_1 = 1\times 2,\ S_2 = 1\times 2+2\times 3,\cdots,\ S_n = 1\times 2+2\times 3+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3},\cdots.$$

Let $X = \{S_n : n\in\mathbb{N}\}$ and $d : X\times X \to [0,\infty)$ be defined by $d(x,y) = \max\{x,y\}$, if $x\neq y$ and $d(x,y) = 0$, if $x = y$. Then (X,d) is complete metric space. Define the mapping $T\colon X\to X$ by $T(S_1) = S_1$ and $T(S_n) = S_{n-1}$ for every $n > 1$. Since

$$\begin{aligned}\lim_{n\to\infty}\frac{d(TS_n,TS_1)}{\frac{d(S_n,TS_n)d(S_n,TS_1)+d(S_1,TS_1)d(S_1,TS_n)}{\max\{d(S_n,TS_1),d(S_1,TS_n)\}}} &= \lim_{n\to\infty}\frac{d(S_{n-1},S_1)}{\frac{d(S_n,S_{n-1})d(S_n,S_1)+d(S_1,S_1)d(S_1,S_{n-1})}{\max\{d(S_n,S_1),d(S_1,S_{n-1})\}}}\\ &= \lim_{n\to\infty}\frac{S_{n-1}}{\frac{S_n\times S_n+0}{S_n}} = \lim_{n\to\infty}\frac{n-1}{n+2} = 1.\end{aligned}$$

Therefore T is not satisfies in assumption of Theorem 1.2 and since (1.1) implies (1.2), so T is not satisfies in assumption of Theorem 1.1. On the other hand, taking

$$F_2 : (0,\infty)\to\mathbb{R},\qquad F_2(\alpha) = \ln(\alpha)+\alpha,$$

we obtain that T is an F_2-Khan-contraction with $\tau = \ln 3$. To see this, let us consider the following calculation. First observe that

$$\max\{d(x_n,Tx_m),d(Tx_n,x_m)\}\neq 0 \Leftrightarrow [(m>2\wedge n=1)\vee(m>n>1)].$$

For every $m\in\mathbb{N}$, $m>2$, we have

$$\begin{aligned}&\frac{d(TS_m,TS_1)}{\frac{d(S_m,TS_m)d(S_m,TS_1)+d(S_1,TS_1)d(S_1,TS_m)}{\max\{d(S_m,TS_1),d(S_1,TS_m)\}}}e^{d(TS_m,TS_1)-\frac{d(S_m,TS_m)d(S_m,TS_1)+d(S_1,TS_1)d(S_1,TS_m)}{\max\{d(S_m,TS_1),d(S_1,TS_m)\}}}\\ &= \frac{S_{m-1}}{S_m}e^{S_{m-1}-S_m} = \frac{m-1}{m+2}e^{\frac{-1}{3}(3m^2+3m)} < e^{-3}.\end{aligned}$$

For every $m,n\in\mathbb{N}$, $m>n>1$, we have

$$\begin{aligned}&\frac{d(TS_m,TS_n)}{\frac{d(S_m,TS_m)d(S_m,TS_n)+d(S_n,TS_n)d(S_n,TS_m)}{\max\{d(S_m,TS_n),d(S_n,TS_m)\}}}e^{d(T(S_m)),T(S_n))-\frac{d(S_m,TS_m)d(S_m,TS_n)+d(S_n,TS_n)d(S_n,TS_m)}{\max\{d(S_m,TS_n),d(S_n,TS_m)\}}}\\ &= \frac{S_mS_{m-1}}{S_nS_{m-1}+S_mS_m}e^{S_{m-1}-\frac{S_nS_{m-1}+S_mS_m}{S_m}}\\ &\leqslant \frac{S_mS_{m-1}}{(S_n+S_m)S_{m-1}}e^{S_m-\frac{S_nS_{m-1}+S_mS_m}{S_m}} \leqslant \frac{S_m}{S_n+S_m}e^{-\frac{S_nS_{m-1}}{S_m}} < e^{-3}.\end{aligned}$$

4. Application

In this section, we present an application where Theorem 3.1 can be applied. This application is inspired by [3].

Let $X = C([0,1])$ be the set of all real continuous functions on $[0,1]$. Clearly, (X,d) is a complete metric space, where d is defined by

$$d(f,g) = \|f-g\| = \max_{t\in[0,1]}|f(t)-g(t)|,\quad f,g\in X.$$

Let

$$Y = \left\{f\in X : 0\leqslant f(t)\leqslant\frac{1}{8},\ \ t\in[0,1]\ \text{ or }\ f(t)=1,\ t\in[0,1]\right\},$$

and $G : [0,1] \times [0,1] \times Y \to X$ be defined by

$$G(t,s,f(r)) = \begin{cases} \frac{1}{2}, & 0 \leqslant f(r) \leqslant \frac{1}{8}, \\ \frac{1}{4}, & f(r) = 1 \end{cases}$$

for all $r, s, t \in [0,1]$ and $f \in Y$. Obviously Y is complete metric space and $G(s,r,f(r))$ is integrable with respect to r on $[0,1]$ for all $s \in [0,1]$.

Let T be defined on Y by $Tf(s) = \int_0^1 G(s,r,f(r))dr$ for all $s \in [0,1]$. We have

$$Tf(s) = \begin{cases} \int_0^1 G(s,r,f(r))dr = \int_0^1 \frac{1}{2} dr = \frac{1}{2}, & 0 \leqslant f(r) \leqslant \frac{1}{8}, \\ \int_0^1 G(s,r,f(r))dr = \int_0^1 \frac{1}{4} dr = \frac{1}{4}, & f(r) = 1. \end{cases}$$

This proves that $Tf \in Y$ for all $f \in Y$. For all $r, s \in [0,1]$ and $f, g \in Y$, we have

$$|G(s,r,f(r)) - G(s,r,g(r))| = \begin{cases} 0, & f(r) = g(r) = 1 \text{ or } 0 \leqslant f(r), g(r) \leqslant \frac{1}{8}, \\ \frac{1}{4}, & \text{otherwise}, \end{cases} \tag{4.1}$$

and

$$|f(r) - Tf(r)| = \begin{cases} |f(r) - \frac{1}{2}|, & 0 \leqslant f(r) \leqslant \frac{1}{8}, \\ |1 - \frac{1}{4}| = \frac{3}{4}, & f(r) = 1, \end{cases}$$

and

$$|f(r) - Tg(r)| = \begin{cases} |f(r) - \frac{1}{2}|, & 0 \leqslant f(r), g(r) \leqslant \frac{1}{8}, \\ |f(r) - \frac{1}{4}|, & 0 \leqslant f(r) \leqslant \frac{1}{8}, g(r) = 1 \\ |1 - \frac{1}{2}| = \frac{1}{2}, & f(r) = 1, 0 \leqslant g(r) \leqslant \frac{1}{8} \\ |1 - \frac{1}{4}| = \frac{3}{4}, & f(r) = g(r) = 1. \end{cases}$$

According symmetry the above relations are established for $|g(r) - Tg(r)|$ and $|g(r) - Tf(r)|$. Obviously $0 \leqslant f(r) \leqslant \frac{1}{8}$ implies that $\frac{3}{8} \leqslant |f(r) - \frac{1}{2}| \leqslant \frac{1}{2}$ and $\frac{1}{8} \leqslant |f(r) - \frac{1}{4}| \leqslant \frac{1}{4}$. Therefore

$$\frac{1}{8} \leqslant \max\{\|f - Tg\|, \|g - Tf\|\} \leqslant \frac{1}{4}$$

Observe that

$$\begin{aligned} &\frac{|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|}{\max\{\|f - Tg\|, \|g - Tf\|\}} \\ &\geqslant 4[|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|] \geqslant 4[\frac{3}{8}\frac{1}{8} + \frac{3}{8}\frac{1}{8}] = \frac{3}{8}. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2) for all $k \in [\frac{2}{3}, 1)$, we have

$$|G(s,r,f(r)) - G(s,r,g(r))| \leqslant k \frac{|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|}{\max\{\|f - Tg\|, \|g - Tf\|\}}. \tag{4.3}$$

Now, we prove that the integral equation

$$f(s) = \int_0^1 G(s,r,f(r))dr \tag{4.4}$$

has a unique solution $f^* \in Y$.

For all $f, g \in Y$ and $s \in [0, 1]$ from (4.3), we have

$$\begin{aligned}
|Tf(s) - Tg(s)| &= \left| \int_0^1 G(s, r, f(r))dr - \int_0^1 G(s, r, g(r))dr \right| \\
&\leqslant \int_0^1 |G(s, r, f(r))dr - G(s, r, g(r))| \, dr \\
&\leqslant \int_0^1 k \frac{|f(r) - Tf(r)||f(r) - Tg(r)| + |g(r) - Tg(r)||g(r) - Tf(r)|}{\max\{\|f - Tg\|, \|g - Tf\|\}} dr \\
&\leqslant \int_0^1 k \frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}} dr \\
&= k \frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}}.
\end{aligned}$$

So, for all $f, g \in Y$, we have

$$\|Tf - Tg\| \leqslant k \frac{\|f - Tf\|\|f - Tg\| + \|g - Tg\|\|g - Tf\|}{\max\{\|f - Tg\|, \|g - Tf\|\}}.$$

Consequently, by passing to logarithms, one can obtain

$$-\ln k + \ln d(Tf, Tg) \leqslant \ln \frac{d(f, Tf)d(f, Tg) + d(g, Tg)d(g, Tf)}{\max\{d(f, Tg), d(g, Tf)\}}.$$

Consequently, all the conditions of Theorem 3.1 are satisfied by operator T with the function F_1 defined as in Example 2.6 and $\tau = -\ln k$. Therefore T has a fixed point which is the solution of the integral equation (4.4).

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