Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 4451–4458
Research Article

ISSN : 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Singular left-definite Hamiltonian systems in the Sobolev space

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Communicated by A. Atangana

Abstract

This paper is devoted to construct Weyl's theory for the singular left-definite even-order Hamiltonian systems in the corresponding Sobolev space. In particular, it is proved that there exist at least n-linearly independent solutions in the Sobolev space for the $2n$-dimensional Hamiltonian system.

Keywords: Hamiltonian system, left-definite problems, Weyl theory.
2010 MSC: 34B20, 46C05.

1. Introduction

In 1964, Atkinson studied the following first-order differential equation [1]

$$JY' = [\lambda A(x) + B(x)]\, Y, \quad x \in (a, b) \subseteq (-\infty, \infty), \tag{1.1}$$

where J, A and B are square matrices of order k, Y is a $k \times 1$ column matrix, A and B are integrable over (a, b), J is a constant matrix and

$$J^* = -J, \quad A^*(x) = A(x) \geqslant 0, \quad B^*(x) = B(x). \tag{1.2}$$

Equation (1.1) is called the Hamiltonian system and contains kth order formally selfadjoint differential equations [23] as well as more interesting differential equations.

Equation (1.1) with conditions (1.2), especially with $A(x) \geqslant 0$, has been studied for right-definite equations in the Hilbert space $L^2_A(a, b)$ which is equipped with the inner product

$$(Y, Z) = \int_a^b Z^* A Y dx.$$

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doi:10.22436/jnsa.010.08.37

Received 2017-06-22

In particular, in 1910, Weyl proved with his extraordinary way the second-order equation

$$-(p(x)y')' + q(x)y = \lambda y, \quad x \in [0, \infty), \tag{1.3}$$

has at least one solution $\chi(x,\lambda) = \varphi(x,\lambda) + m_\infty(\lambda)\psi(x,\lambda)$ satisfying

$$\int_a^\infty |\chi|^2 \, dx < \infty,$$

where p and q are real-valued functions on the given interval, φ and ψ are linearly independent solutions of (1.3) and m_∞ is a point on the limiting-point or limiting-circle [25]. Atkinson generalized this result for the linear $2n$-dimensional Hamiltonian system (1.1) satisfying (1.2) and he proved that at least n-linearly independent solutions of (1.1), (1.2) lie in $L_A^2(a,b)$.

A different approach was given by Niessen [17–19]. Niessen examined the matrix

$$\mathcal{A}(x) = (1/2 \operatorname{Im}\lambda)\mathcal{Y}^*(x,\lambda)(J/i)\mathcal{Y}(x,\lambda), \quad \operatorname{Im}\lambda \neq 0,$$

where $\mathcal{Y}(x,\lambda)$ is the fundamental solution of (1.1), (1.2). However, more efficient method was introduced by Hinton and Shaw [4–6]. In this method, Hinton and Shaw used matrix function $M(\lambda)$ which is similar with Weyl's function to construct the circle or ellipsoid equations. Then they proved that (1.1), (1.2) have at least n-linearly independent solutions belonging to $L_A^2(a,b)$. Similar approach was given by Krall [8, 12].

However, all these results were introduced for the right-definite Hamiltonian systems. Right-definite case is related with the right-hand side of (1.1). Positiveness condition in the right-hand side of the equation generates a weighted Hilbert space. On the other side, in real-world problems there exist functions in the right-hand side of the equations changing the sign on the interval. As a famous application we can give the Camassa-Holm equation [2]

$$-y'' + \frac{1}{4}y = \lambda w(x)y.$$

To introduce the motivation of the left-definite equations consider the equation

$$y'' + y = 0. \tag{1.4}$$

Multiplying (1.4) with y', it is found that

$$(y')^2 + y^2 = c^2, \tag{1.5}$$

where c is a constant. Solving for y' we have

$$y' = \sqrt{c^2 - y^2}.$$

Choosing $y = c_1 \sin\gamma$ it is obtained

$$y = c_1 \sin(x + c_2),$$

where c_1, c_2 are constants. The left-side of (1.5) may arise in the standart Sturm-Liouville equations. In fact, for sufficiently nice functions one obtains

$$\int_c^d \left[-(py')' + qy\right] \overline{y} dx = \int_c^d \left[p\left|y'\right|^2 + q\left|y\right|^2\right] dx - (py')\overline{y}\, |_c^d,$$

where $-\infty \leqslant c < d \leqslant \infty$. Therefore imposing positiveness condition on p and q one can construct the Sobolev space $H^1(c,d;p,q)$ with the inner product

$$\langle y, z\rangle = \int_c^d \left[py'\overline{z}' + qy\overline{z}\right] dx.$$

In recent years, the authors have studied some spectral properties of the regular and singular left-definite Sturm-Liouville differential and difference equations [2, 3, 7, 14–16, 20, 24].

In 1995, Krall and Race [13] studied the singular left-definite second-order Sturm-Liouville equation

$$-(py')' + qy = \lambda w y, \quad (a,b) \subseteq (-\infty, \infty), \tag{1.6}$$

where p, q, w are real-valued, positive functions over (a,b) such that p^{-1} is locally integrable on (a,b), $\epsilon_1 w \leqslant q \leqslant \epsilon_2 w$, q and w are in $L^1(a,b)$ (also see [9, 10]). They proved that there is a solution $\chi(x,\lambda) = \varphi(x,\lambda) + m_b(\lambda)\psi(x,\lambda)$ of (1.6) belonging to $H^1(a,b;p,q)$. However, for a singular left-definite fourth/sixth/... order or a linear singular left-definite Hamiltonian system has not been studied yet. Beside this, Weyl's theory for the singular Dirac system has been investigated in [21]. In this paper, our main aim is to develop Weyl's theory for the singular left-definite linear even-dimensional Hamiltonian system. It should be noted that regular left-definite Hamiltonian system has been studied in [11] and some properties of the regular fractional operator in the Sobolev space has been investigated in [22].

2. Preliminaries

In this section we shall remind some known results on singular right-definite Hamiltonian system in $L^2_A(a,b)$.

Let us assume that a is the regular point and b is the singular point for the $2n$-dimensional Hamiltonian system (1.1), (1.2). Let $\mathcal{Y}$ be a fundamental matrix of size $2n \times 2n$ of (1.1), (1.2) satisfying

$$\mathcal{Y}(a,\lambda) = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix},$$

where α_1, α_2 are $n \times n$ real-matrices such that $\operatorname{rank}(\alpha_1, \alpha_2) = n$,

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_n, \qquad \alpha_1\alpha_2^* - \alpha_2\alpha_1^* = 0,$$

and I_n is the $n \times n$ identity matrix. If $\mathcal{Y}$ is partitioned into

$$Y = \begin{pmatrix} \theta & \phi \end{pmatrix} = \begin{pmatrix} \theta_1 & \phi_1 \\ \theta_2 & \phi_2 \end{pmatrix},$$

we may assume that

$$\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \theta(a) = I_n, \quad \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \phi(a) = 0.$$

Now consider the following boundary condition at b', $b' < b$,

$$\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} Y(b') = 0, \tag{2.1}$$

where β_1, β_2 are $n \times n$ real-matrices such that $\operatorname{rank}(\beta_1, \beta_2) = n$ and

$$\beta_1\beta_1^* + \beta_2\beta_2^* = I_n, \qquad \beta_1\beta_2^* - \beta_2\beta_1^* = 0.$$

We set the solution χ of (1.1), (1.2) as

$$\chi = \mathcal{Y}\begin{pmatrix} I_n \\ M(b') \end{pmatrix}.$$

Then χ satisfies the boundary condition (2.1) at b' if

$$M(b') = -\left(\beta_1\phi_1(b',\lambda) + \beta_2\phi_2(b',\lambda)\right)^{-1}\left(\beta_1\theta_1(b',\lambda) + \beta_2\theta_2(b',\lambda)\right), \tag{2.2}$$

and $\chi^*(b',\lambda)J\chi(b',\lambda) = 0$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Circle equation can be introduced as

$$\pm \begin{pmatrix} I_n & M^* \end{pmatrix} \mathcal{Y}^*(b')\,(J/i)\,\mathcal{Y}(b') \begin{pmatrix} I_n \\ M \end{pmatrix} = 0,$$

where "+" holds when $\operatorname{Im}\lambda > 0$ and "−" holds when $\operatorname{Im}\lambda < 0$.

Now let

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & \mathcal{D} \end{pmatrix} = \begin{cases} \mathcal{Y}^*(b')\,(J/i)\,\mathcal{Y}(b'), & \operatorname{Im}\lambda > 0, \\ -\mathcal{Y}^*(b')\,(J/i)\,\mathcal{Y}(b'), & \operatorname{Im}\lambda < 0, \end{cases}$$

and

$$\mathbb{E}(M) = \begin{pmatrix} I_n & M^* \end{pmatrix} \begin{pmatrix} \mathcal{A} & \mathcal{B}^* \\ \mathcal{B} & \mathcal{D} \end{pmatrix} \begin{pmatrix} I_n \\ M \end{pmatrix}.$$

The circle equation $\mathbb{E}(M) = 0$ can be expressed as

$$\mathbb{E}(M) = (M - C)^* R_1^{-2} (M - C) - R_2^2 = 0,$$

where $C = -\mathcal{D}^{-1}\mathcal{B}$, $R_1 = \mathcal{D}^{-1/2}$, $R_2 = (\mathcal{B}^*\mathcal{D}^{-1}\mathcal{B} - \mathcal{A})^{1/2}$.

Then following theorem is valid [8, 12].

Theorem 2.1.

(i) $\mathcal{D} > 0$;

(ii) $\mathcal{B}^*\mathcal{D}^{-1}\mathcal{B} - \mathcal{A} = \mathcal{D}^{-1}(\overline{\lambda})$;

(iii) $R_2 = \overline{R}_1$;

(iv) *as b' increases, $\mathcal{D}$ increases, R_1 decreases and R_2 decreases;*

(v) $\lim_{b' \to b} R_1(b', \lambda) = R_0(\lambda) = R_0$, $\lim_{b' \to b} R_2(b', \lambda) = R_0(\overline{\lambda}) = \widetilde{R}_0$, $R_0 \geqslant 0$, $\widetilde{R}_0 \geqslant 0$;

(vi) *as b' approaches b, the circles $\mathbb{E}(M) = 0$ are nested and $\lim_{b' \to b} C(b', \lambda) = C_0$ exists;*

(vii) $M = C_0 + R_0 U \overline{R}_0$, $U = R_1^{-1}(M - C)\overline{R}_1^{-1}$ *is well-defined. As U varies over the unit-circle in the $n \times n$ sphere the limit-circle or -point $\mathbb{E}_0(M)$ is covered.*

3. Dirichlet formula

To construct the Sobolev space we let [11]

$$A = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} -B_{11} & B_{12} \\ B_{12}^* & B_{22} \end{pmatrix}, \tag{3.1}$$

and $-B_{11} \leqslant 0 \leqslant B_{22}$, $\rho E \leqslant B_{11}$. Therefore classical $L_A^2(a, b)$ space implies

$$(Y, Z) = \int_a^b Z^* A Y dx = \int_a^b Z_1^* E Y_1 dx.$$

On the other hand, Sobolev space $H^1(a, b; B_{11}, B_{22})$ is equipped with the inner product

$$\langle Y, Z \rangle = \int_a^b Z^* \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} Y dx = \int_a^b [Z_1^* B_{11} Y_1 + Z_2^* B_{22} Y_2]\, dx.$$

Now consider the equations

$$JY' - BY = AF, \quad LY := F,$$

where $Y, F \in L^2_A(a, b)$. Then

$$\begin{aligned} -Y_2' + B_{11}Y_1 - B_{12}Y_2 &= EF_1, \\ Y_1' - B_{12}^*Y_1 - B_{22}Y_2 &= 0. \end{aligned}$$

Therefore

$$\begin{aligned} (LY, Z) &= \int_a^b Z_1^* \left[-Y_2' + B_{11}Y_1 - B_{12}Y_2\right] dx \\ &= -Z_1^*Y_2 \mid_a^b + \int_a^b \left[Z_1^{*'}Y_2 + Z_1^*B_{11}Y_1 - Z_1^*B_{12}Y_2\right] dx \\ &= -Z_1^*Y_2 \mid_a^b + \int_a^b \left[(B_{12}^*Z_1 + B_{22}Z_2)^* Y_2 + Z_1^*B_{11}Y_1 - Z_1^*B_{12}Y_2\right] dx \\ &= -Z_1^*Y_2 \mid_a^b + \int_a^b [Z_1^*B_{11}Y_1 + Z_2^*B_{22}Y_2] \, dx, \end{aligned}$$

provided that

$$Z_1' - B_{12}^*Z_1 = B_{22}Z_2.$$

Hence we have the Dirichlet formula

$$(LY, Z) = \langle Y, Z\rangle - Z_1^*Y_2 \mid_a^b . \tag{3.2}$$

4. Singular left-definite Hamiltonian system

In this section we introduce the main results.

Equation (3.2) implies that

$$\lambda \int_a^{b'} Y_1^*E_1Y_1 dx = \int_a^{b'} Y_1^*B_{11}Y_1 dx + \int_a^{b'} Y_2^*B_{22}Y_2 dx - Y_1^*Y_2 \mid_a^{b'} . \tag{4.1}$$

Now consider the boundary condition (2.1) at b'. Then the solution

$$\chi = \mathcal{Y}\begin{pmatrix} I_n \\ M(b') \end{pmatrix},$$

satisfies (2.1) if $M(b')$ is of the form (2.2). Equation (4.1) implies that

$$\begin{aligned} \lambda \int_a^{b'} \chi_1^*E\chi_1 dx &= \int_a^{b'} \chi_1^*B_{11}\chi_1 dx + \int_a^{b'} \chi_2^*B_{22}\chi_2 dx - \chi_1^*\chi_2 \mid_a^{b'} \\ &= \int_a^{b'} \chi_1^*B_{11}\chi_1 dx + \int_a^{b'} \chi_2^*B_{22}\chi_2 dx - \chi_1^*(b')\chi_2(b') + \chi_1^*(a)\chi_2(a). \end{aligned} \tag{4.2}$$

Note that

$$\begin{pmatrix} \chi_1(a) \\ \chi_2(a) \end{pmatrix} = \begin{pmatrix} \alpha_1^* - \alpha_2^*M(b') \\ \alpha_2^* + \alpha_1^*M(b') \end{pmatrix}, \quad \begin{pmatrix} \chi_1(b') \\ \chi_2(b') \end{pmatrix} = \begin{pmatrix} \beta_2^* \\ -\beta_1^* \end{pmatrix}. \tag{4.3}$$

Substitution (4.3) into (4.2) we find

$$\lambda\int_a^{b'}\chi_1^*E\chi_1\,dx=\int_a^{b'}\chi_1^*B_{11}\chi_1\,dx+\int_a^{b'}\chi_2^*B_{22}\chi_2\,dx+\beta_2\beta_1^*-M^*\alpha_2\alpha_2^* \\ -M^*\alpha_2\alpha_1^*M+\alpha_1\alpha_2^*+\alpha_1\alpha_1^*M. \tag{4.4}$$

Now, let $\beta_1=0$ such that $\operatorname{rank}\beta_2=n$. This case corresponds to

$$\chi_2(b')=0,$$

and (2.2) coincides with

$$M(b')=-\phi_2^{-1}(b')\theta_2(b').$$

Then (4.4) gives

$$\int_a^{b'}\chi_1^*B_{11}\chi_1\,dx+\int_a^{b'}\chi_2^*B_{22}\chi_2\,dx=\lambda\int_a^{b'}\chi_1^*E\chi_1\,dx+M^*\alpha_2\alpha_2^* \\ +M^*\alpha_2\alpha_1^*M-\alpha_1\alpha_2^*-\alpha_1\alpha_1^*M.$$

Fixing b' in the upper limit in the integral we find

$$\int_a^{b'}\chi_1^*B_{11}\chi_1\,dx+\int_a^{b'}\chi_2^*B_{22}\chi_2dx\leqslant \operatorname{Re}\lambda\int_a^{b}\chi_1^*E\chi_1\,dx-\alpha_1\alpha_2^*+\operatorname{Re}\{M^*(b)\alpha_2\alpha_2^* \\ +M^*(b)\alpha_2\alpha_1^*M(b)-\alpha_1\alpha_1^*M(b)\}.$$

This implies the following theorem.

Theorem 4.1. *There exists a solution*

$$\chi=\mathcal{Y}\begin{pmatrix} I_n \\ M(b)\end{pmatrix}$$

of (1.1), (3.1) *such that for all* λ *with* $\operatorname{Im}\lambda\neq 0$ χ *lies in* $H^1(a,b;B_{11},B_{22})$.

Remark 4.2. It seems that there is no need to consider $\beta_1=0$ to introduce Theorem 4.1. However, in [13] Krall and Race showed that there is needed to restrict the boundary conditions for further calculation in their work. So we take $\beta_1=0$ to coincide the further results in [13].

Theorem 4.3. *Let* $\operatorname{rank}R_0=r_1$, $\operatorname{rank}\overline{R}_0=r_2$, $\nu=n+\min(r_1,r_2)$. *Then for* $\operatorname{Im}\lambda\neq 0$, *there exist* ν *solutions of* (1.1) *satisfying*

$$\int_a^{b'}Y_1^*B_{11}Y_1dx+\int_a^{b'}Y_2^*B_{22}Y_2dx<\infty.$$

Proof. Consider the solution

$$\chi(x,\lambda)=\mathcal{Y}(x,\lambda)\begin{pmatrix} I_n \\ C_0\end{pmatrix}=\begin{pmatrix} Y_1 & \cdots & Y_n\end{pmatrix}(x,\lambda).$$

Then

$$\int_a^{b'}Y_{j,1}^*B_{11}Y_{j,1}\,dx+\int_a^{b'}Y_{j,2}^*B_{22}Y_{j,2}\,dx<\infty,$$

where Y_j, $1 \leqslant j \leqslant n$, are n-linearly independent solutions.

Now let

$$\widetilde{\chi}(x,\lambda) = \mathcal{Y}(x,\lambda)\begin{pmatrix} I_n \\ M(b) \end{pmatrix} = \begin{pmatrix} Z_1 & \cdots & Z_n \end{pmatrix}(x,\lambda),$$

where $M(b) = C_0 + R_0 U \overline{R}_0$ and $U^* U \leqslant I_n$. Therefore

$$\int_a^{b'} Z_{k,1}^* B_{11} Z_{k,1} dx + \int_a^{b'} Z_{k,2}^* B_{22} Z_{k,2} dx < \infty,$$

where Z_j, $n+1 \leqslant k \leqslant 2n$, are n-linearly independent solutions. Therefore

$$\begin{pmatrix} Y_1 & \cdots & Y_n & Z_1 & \cdots & Z_n \end{pmatrix}(x,\lambda) = \mathcal{Y}(x,\lambda)\begin{pmatrix} I_n & I_n \\ C_0 & M(b) \end{pmatrix}.$$

One can write

$$\begin{pmatrix} I_n & I_n \\ C_0 & M(b) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ C_0 & R_0 U \overline{R}_0 \end{pmatrix}\begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}. \tag{4.5}$$

Equation (4.5) shows that

$$\operatorname{rank}\begin{pmatrix} I_n & I_n \\ C_0 & M(b) \end{pmatrix} = n + \min(r_1, r_2) = \nu.$$

Since the right matrix on the right side in (4.5) and $\mathcal{Y}(x,\lambda)$ are invertible,

$$\operatorname{rank}\begin{pmatrix} Y_1 & \cdots & Y_n & Z_1 & \cdots & Z_n \end{pmatrix} = \nu,$$

and this completes the proof. □

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