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Some Hermite-Hadamard type inequalities for harmonically extended s-convex functions

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Abstract

In this paper, we establish some inequalities of Hermite-Hadamard type for functions whose derivatives absolute values are harmonically extended s-convex functions. ©2017 All rights reserved.

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1. Introduction

The following definitions are well-known in the literature.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex function if

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [1, 4] the concept of s-convex functions below was given.

Definition 1.2 ([1, 4])**.** Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R} \to \mathbb{R}_0 = [0, \infty)$ is said to be s-convex (in the second sense) if

$$f(tx + (1-t)y) \leqslant t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

In [16] the definition of extended s-convex functions was put forward, i.e.,

Definition 1.3 ([16])**.** A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if

$$f(tx + (1-t)y) \leqslant t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in (0, 1)$ and for some fixed $s \in [-1, 1]$.

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Definition 1.4 ([6]). Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant tf(y)+(1-t)f(x) \tag{1.1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.1) is reversed, then f is said to be harmonically concave.

A formal definition for harmonically s-convex functions is stated as follows (see [7, 9]):

Definition 1.5 ([7, 9]). A function $f : I \subseteq \mathbb{R}_+ = (0, +\infty) \to \mathbb{R}$ is said to be harmonically s-convex function of second kind, where $s \in (0,1]$, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant t^s f(y)+(1-t)^s f(x), \quad x, y \in I, \quad t \in [0,1].$$

Next, let us recall the concepts of harmonically P-convex and harmonically s-Godunova-Levin convex functions.

Definition 1.6 ([9]). A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically P-convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant f(x)+f(y), \quad x, y \in I, \quad t \in [0,1].$$

Definition 1.7 ([9]). A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically Godunova-Levin convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant \frac{f(y)}{t}+\frac{f(x)}{1-t}, \quad x, y \in I, \quad t \in (0,1).$$

Definition 1.8 ([9]). A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically s-Godunova-Levin convex function of second kind, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant \frac{f(y)}{t^s}+\frac{f(x)}{(1-t)^s}, \quad x, y \in I, \quad t \in (0,1), \quad s \in (0,1].$$

In order to unify the above concepts, we focus on the definition of harmonically extended s-convex functions below.

Definition 1.9 ([9]). A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically extended s-convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant t^s f(y)+(1-t)^s f(x), \quad x, y \in I, \quad t \in (0,1), \quad s \in [-1,1].$$

It is obvious that the harmonically extended 0-convex function and harmonically extended -1-convex function are just the harmonically P-convex functions (Definition 1.6) and harmonically Godunova-Levin convex functions (Definition 1.7), respectively.

In [3, 8, 10] and [5], the following Hermite-Hadamard type inequalities for functions whose derivatives absolute values are convex (or s-convex) functions were established.

Theorem 1.10 ([3]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.11 ([10]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a,b]$ and $q \geqslant 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q},$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Theorem 1.12 ([8]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q \geqslant 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leqslant \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-1/q}\left[\frac{2+1/2^s}{(s+1)(s+2)}\right]^{1/q}\left[|f'(a)|^q + |f'(b)|^q\right]^{1/q}.$$

Theorem 1.13 ([5]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I°, $a, b \in I$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$, then*

$$\begin{aligned}\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leqslant \frac{b-a}{4}\left[\frac{1}{(s+1)(s+2)}\right]^{1/q}\left(\frac{1}{2}\right)^{1/p}\\ &\times\left\{\left[|f'(a)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[|f'(b)|^q + (s+1)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right\},\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In recent years, some other kinds of Hermite-Hadamard type inequalities were generated in, for example, [2, 11–15] and [17].

In this paper we shall establish some new Hermite-Hadamard type inequalities for harmonically extended s-convex functions.

2. Two lemmas

To establish some new Hermite-Hadamard type inequalities for harmonically extended s-convex functions, we need the following integral identities:

Lemma 2.1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1([a, b])$, then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2} + \frac{ab[f'(b)-f'(a)]}{2(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx\\ &\quad= \frac{1}{2(ab)^2}\int_0^1 \left[(1-t)a^2 + tb^2\right]\left(\frac{t}{a} + \frac{1-t}{b}\right)^{-4} f''\left(\left(\frac{t}{a} + \frac{1-t}{b}\right)^{-1}\right)dt.\end{aligned}$$

Proof. Letting $x = \left(\frac{t}{a} + \frac{1-t}{b}\right)^{-1}$ for $t \in [0, 1]$, then

$$\begin{aligned}&\frac{1}{2(ab)^2}\int_0^1 \left[(1-t)a^2 + tb^2\right]\left(\frac{t}{a} + \frac{1-t}{b}\right)^{-4} f''\left(\left(\frac{t}{a} + \frac{1-t}{b}\right)^{-1}\right)dt\\ &\quad= \frac{1}{2(b-a)^2}\int_a^b \left[a(x-a)x + b(b-x)x\right] f''(x)dx\\ &\quad= \frac{ab[f'(b)-f'(a)]}{2(b-a)} - \frac{1}{2(b-a)}\int_a^b \left[a(2x-a) + b(b-2x)\right] f'(x)dx\\ &\quad= \frac{f(a)+f(b)}{2} + \frac{ab[f'(b)-f'(a)]}{2(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx.\end{aligned}$$

Lemma 2.1 is proved. □

Lemma 2.2. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I^\circ$ with $a < b$. If $f'' \in L_1([a, b])$, then*

$$\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a) + \frac{1}{b^2}f(b)\right]}{2} - \frac{1}{b-a}\int_a^b \left(\frac{3}{x^4} - \frac{2}{H(a,b)x^3}\right) f(x)dx$$

$$= \frac{(b-a)^2}{2(ab)^3}\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2} f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt,$$

where $H(a,b)=\frac{2ab}{a+b}$.

Proof. Letting $x=\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}$ for $t\in[0,1]$, then

$$\begin{aligned}
&\frac{(b-a)^2}{2(ab)^3}\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2} f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)dt\\
&\quad=\frac{1}{2(b-a)}\int_a^b\left(\frac{1}{x}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{x}\right)f''(x)dx\\
&\quad=-\frac{1}{2(b-a)}\int_a^b\left[\frac{2}{x^3}-\frac{1}{x^2}\left(\frac{1}{a}+\frac{1}{b}\right)\right]f'(x)dx\\
&\quad=\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}+\frac{1}{2(b-a)}\int_a^b\left[-\frac{6}{x^4}+\frac{2}{x^3}\left(\frac{1}{a}+\frac{1}{b}\right)\right]f(x)dx\\
&\quad=\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}-\frac{1}{b-a}\int_a^b\left(\frac{3}{x^4}-\frac{2}{H(a,b)x^3}\right)f(x)dx.
\end{aligned}$$

The proof of Lemma 2.2 is completed. □

3. Main results

Our main results are given in the following theorems.

Theorem 3.1. *Let* $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ *be a twice differentiable function on* I°, $a,b\in I$ *with* $a<b$, *and* $f''\in L_1([a,b])$. *If* $|f''|^q$ *for* $q\geqslant 1$ *is harmonically extended* s*-convex on* I° *for some fixed* $s\in(-1,1]$, *then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{[H(a,b,0)]^{1-1/q}}{2(ab)^2}\left[H(a,b,s)|f''(a)|^q+H(b,a,s)\left|f''(b)\right|^q\right]^{1/q},
\end{aligned}$$

where

$$H(a,b,s)=\frac{(s+1)[a^2+(s+2)b^2]a^4+[2a^2+(s+1)b^2]b^4}{(s+3)(s+2)(s+1)}.$$

Proof. Using Lemma 2.1, and the Hölder's integral inequality and the harmonically extended s-convexity of function $|f''|^q$, we obtain

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{1}{2(ab)^2}\int_0^1\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4}\left|f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right|dt\\
&\qquad\leqslant\frac{1}{2(ab)^2}\left[\int_0^1\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4}dt\right]^{1-1/q}\\
&\qquad\quad\times\left[\int_0^1\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4}\left|f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right|^q dt\right]^{1/q}
\end{aligned}$$

$$\leqslant \frac{1}{2(ab)^2}\left[\int_0^1 \left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4} dt\right]^{1-1/q}$$
$$\times \left[\int_0^1 \left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4}\left[t^s|f''(a)|^q+(1-t)^s\left|f''(b)\right|^q\right] dt\right]^{1/q}.$$

Since $\left(ta^{-1}+(1-t)b^{-1}\right)^{-p} \leqslant ta^p+(1-t)b^p$ for $t\in[0,1]$ and $p>1$, we have used the facts

$$\int_0^1 \left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4} dt$$
$$\leqslant \int_0^1 \left[(1-t)a^2+tb^2\right][ta^4+(1-t)b^4]dt = H(a,b,0),$$
$$\times \int_0^1 t^s\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4} dt$$
$$\leqslant \int_0^1 t^s\left[(1-t)a^2+tb^2\right][ta^4+(1-t)b^4]dt = H(a,b,s),$$
$$\times \int_0^1 (1-t)^s\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4} dt$$
$$\leqslant \int_0^1 (1-t)^s\left[(1-t)a^2+tb^2\right][ta^4+(1-t)b^4]dt = H(b,a,s).$$

The proof of Theorem 3.1 is completed. □

Corollary 3.2. *Under the conditions of Theorem 3.1, when* $q=1$*, then*

$$\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{1}{2(ab)^2}\left[H(a,b,s)|f''(a)|+H(b,a,s)\left|f''(b)\right|\right].$$

Theorem 3.3. *Let* $f: I\subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a twice differentiable function on* I°, $a,b\in I$ *with* $a<b$, *and* $f''\in L_1([a,b])$. *If* $|f''|^q$ *for* $q\geqslant 1$ *is harmonically extended* s*-convex on* I *for* $s=-1$*, then*

$$\left|\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leqslant \frac{b-a}{2(ab)^2L(a,b)}\left[\frac{ab\left[(a+b)-2L(a,b)\right]}{b-a}\right]^{1-1/q}$$
$$\times \left[b\left[L(a,b)-a\right]|f''(a)|^q+a\left[b-L(a,b)\right]\left|f''(b)\right|^q\right]^{1/q},$$

where $L(u,v)$ *is the logarithmic mean defined by*

$$L(u,v)=\begin{cases}\dfrac{v-u}{\ln v-\ln u}, & u\neq v,\\ u, & u=v,\end{cases} \quad u,v>0.$$

Proof. Since $|f''|^q$ is harmonically extended -1-convex function on I°, by Lemma 2.2 and the Hölder's integral inequality, we have

$$\left|\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$
\begin{aligned}
&\leqslant \frac{(b-a)^2}{2(ab)^3}\left[\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}dt\right]^{1-1/q}\\
&\quad\times\left[\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\left|f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right|^q dt\right]^{1/q}\\
&\leqslant \frac{(b-a)^2}{2(ab)^3}\left[\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}dt\right]^{1-1/q}\\
&\quad\times\left[\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\left[t^{-1}|f''(a)|^q+(1-t)^{-1}\left|f''(b)\right|^q\right]dt\right]^{1/q}\\
&= \frac{b-a}{2(ab)^2L(a,b)}\left[\frac{ab\left[(a+b)-2L(a,b)\right]}{b-a}\right]^{1-1/q}\\
&\quad\times\left[b\left[L(a,b)-a\right]|f''(a)|^q+a\left[b-L(a,b)\right]\left|f''(b)\right|^q\right]^{1/q}.
\end{aligned}
$$

The proof of Theorem 3.3 is completed. □

Corollary 3.4. *Under the conditions of Theorem 3.3, when $q=1$, then*

$$
\begin{aligned}
&\left|\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{(b-a)^2}{2(ab)^2}\left[\frac{b\left[1-aL(a,b)\right]}{b-a}|f''(a)|+\frac{a\left[bL(a,b)-1\right]}{b-a}\left|f''(b)\right|\right].
\end{aligned}
$$

Theorem 3.5. *Let $f: I\subseteq\mathbb{R}_+\to\mathbb{R}$ be a twice differentiable function on I°, $a,b\in I$ with $a<b$, and $f''\in L_1([a,b])$. If $|f''|^q$ is harmonically extended s-convex on I for some fixed $s\in(-1,1]$ and $q>1$, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{1}{2(ab)^{2/q}}\\
&\qquad\times\left[\frac{(q-1)\left(b^{\frac{2(q+1)}{q-1}}(a+b(q-1)+3aq)-a^{\frac{2(q+1)}{q-1}}(b+a(q-1)+3bq)\right)}{2(b-a)(3q+1)(q+1)}\right]^{1-1/q}\\
&\qquad\times\left[\frac{a^2+(s+1)b^2}{(s+2)(s+1)}|f''(a)|^q+\frac{(s+1)a^2+b^2}{(s+2)(s+1)}\left|f''(b)\right|^q\right]^{1/q}.
\end{aligned}
$$

Proof. By Lemma 2.1, and the Hölder's integral inequality and the harmonically extended s-convexity of function $|f''|^q$, we deduce

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{1}{2(ab)^2}\left[\int_0^1\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4q/(q-1)}dt\right]^{1-1/q}\\
&\qquad\quad\times\left[\int_0^1\left[(1-t)a^2+tb^2\right]\left|f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right|^q dt\right]^{1/q}\\
&\qquad\leqslant\frac{1}{2(ab)^2}\left[\int_0^1\left[(1-t)a^2+tb^2\right]\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4q/(q-1)}dt\right]^{1-1/q}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\int_0^1\left[(1-t)a^2+tb^2\right]\left[t^s|f''(a)|^q+(1-t)^s\left|f''(b)\right|^q\right]dt\right]^{1/q}\\
&\leqslant\frac{1}{2(ab)^{2/q}}\left[\frac{(q-1)\left[b^{\frac{2(q+1)}{q-1}}(a+b(q-1)+3aq)-a^{\frac{2(q+1)}{q-1}}(b+a(q-1)+3bq)\right]}{2(b-a)(3q+1)(q+1)}\right]^{1-1/q}\\
&\times\left[\frac{a^2+(s+1)b^2}{(s+2)(s+1)}|f''(a)|^q+\frac{(s+1)a^2+b^2}{(s+2)(s+1)}\left|f''(b)\right|^q\right]^{1/q}.
\end{aligned}
$$

This completes the proof of Theorem 3.5. ☐

Theorem 3.6. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$, and $f'' \in L_1([a, b])$. If $|f''|^q$ is harmonically extended -1-convex on I for $q > 1$, then*

$$
\begin{aligned}
&\left|\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{(b-a)^2}{4(ab)^3}\left[\frac{a^{2q/(q-1)}+b^{2q/(q-1)}}{6}\right]^{1-1/q}\left[|f''(a)|^q+\left|f''(b)\right|^q\right]^{1/q}.
\end{aligned}
$$

Proof. By Lemma 2.2, and the Hölder's integral inequality and the harmonically extended -1-convexity of function $|f''|^q$, we deduce

$$
\begin{aligned}
&\left|\frac{\frac{1}{ab}\left[\frac{1}{a^2}f(a)+\frac{1}{b^2}f(b)\right]}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{(b-a)^2}{2(ab)^3}\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2}\left|f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right|dt\\
&\qquad\leqslant\frac{(b-a)^2}{2(ab)^3}\left[\int_0^1 t(1-t)\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-2q/(q-1)}dt\right]^{1-1/q}\\
&\qquad\quad\times\left[\int_0^1 t(1-t)\left[t^{-1}|f''(a)|^q+(1-t)^{-1}\left|f''(b)\right|^q\right]dt\right]^{1/q}\\
&\qquad\leqslant\frac{(b-a)^2}{4(ab)^3}\left[\frac{a^{2q/(q-1)}+b^{2q/(q-1)}}{6}\right]^{1-1/q}\left[|f''(a)|^q+\left|f''(b)\right|^q\right]^{1/q}.
\end{aligned}
$$

Theorem 3.6 is proved. ☐

Theorem 3.7. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a twice differentiable function on I°, $a, b \in I$ with $a < b$, and $f'' \in L_1([a, b])$. If $|f''|^q$ for $q > 1$ is harmonically extended s-convex on I for some fixed $s \in (-1, 1]$, then*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|\\
&\qquad\leqslant\frac{1}{2(ab)^2}\left\{\frac{q-1}{(2q-1)(3q-2)(b^2-a^2)^2}\right.\\
&\qquad\quad\times\left[b^{\frac{2(2q-1)}{q-1}}\left(b^{\frac{2(3q-1)}{q-1}}(q-1)+a^{\frac{4q}{q-1}}b^2(2q-1)-a^{\frac{2(3q-1)}{q-1}}(3q-2)\right)\right.\\
&\qquad\quad\left.\left.+a^{\frac{2(2q-1)}{q-1}}\left(a^{\frac{2(3q-1)}{q-1}}(q-1)+a^2b^{\frac{4q}{q-1}}(2q-1)-b^{\frac{2(3q-1)}{q-1}}(3q-2)\right)\right]\right\}^{1-1/q}\\
&\qquad\quad\times\left[\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right]^{1/q}.
\end{aligned}
$$

Proof. From Lemma 2.1, and the Hölder's integral inequality and the harmonically extended s-convexity of function $|f''|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}+\frac{ab[f'(b)-f'(a)]}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\leqslant \frac{1}{2(ab)^2}\left[\int_0^1 \left[(1-t)a^2+tb^2\right]^{q/(q-1)}\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4q/(q-1)} dt\right]^{1-1/q} \\
&\quad\times\left[\int_0^1 \left|f''\left(\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-1}\right)\right|^q dt\right]^{1/q} \\
&\leqslant \frac{1}{2(ab)^2}\left[\int_0^1 \left[(1-t)a^2+tb^2\right]^{q/(q-1)}\left(\frac{t}{a}+\frac{1-t}{b}\right)^{-4q/(q-1)} dt\right]^{1-1/q} \\
&\quad\times\left[\int_0^1 \left[t^s|f''(a)|^q+(1-t)^s\left|f''(b)\right|^q\right] dt\right]^{1/q} \\
&\leqslant \frac{1}{2(ab)^2}\left\{\frac{q-1}{(2q-1)(3q-2)(b^2-a^2)^2}\right. \\
&\quad\times\left[b^{\frac{2(2q-1)}{q-1}}\left(b^{\frac{2(3q-1)}{q-1}}(q-1)+a^{\frac{4q}{q-1}}b^2(2q-1)-a^{\frac{2(3q-1)}{q-1}}(3q-2)\right)\right. \\
&\quad\left.\left.+a^{\frac{2(2q-1)}{q-1}}\left(a^{\frac{2(3q-1)}{q-1}}(q-1)+a^2b^{\frac{4q}{q-1}}(2q-1)-b^{\frac{2(3q-1)}{q-1}}(3q-2)\right)\right]\right\}^{1-1/q} \\
&\quad\times\left[\frac{|f''(a)|^q+|f''(b)|^q}{s+1}\right]^{1/q}.
\end{aligned}
$$

This completes the proof of Theorem 3.7. □

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