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$L^2(\mathbb{R}^n)$ estimate of the solution to the Navier-Stokes equations with linearly growth initial data

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Abstract

In this article, we consider the incompressible Navier-Stokes equations with linearly growing initial data $U_0 := u_0(x) - Mx$. Here M is an $n \times n$ matrix, $\mathrm{tr}M = 0$, M^2 is symmetric and $u_0 \in L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$. Under these conditions, we consider $v(t) := u(t) - e^{-tA}u_0$, where $u(x) := U(x) - Mx$ and $U(x)$ is the mild solution of the incompressible Navier-Stokes equations with linearly growing initial data. We shall show that $D^\beta v(t)$ on the $L^2(\mathbb{R}^n)$ norm like $t^{-\frac{|\beta|-1}{2}-\frac{n}{4}}$ for all $|\beta| \geqslant 0$. ©2017 All rights reserved.

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1. Introduction

In this article, we investigate the Cauchy problems to the Navier-Stokes equations:

$$\begin{cases} U_t + U \cdot \nabla U - \Delta U + \nabla P = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ \nabla \cdot U = 0, & \text{in } \mathbb{R}^n \times (0,T), \\ U\,|_{t=0} = U_0, & \text{in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here U is the velocity field of the flow, and $P(\cdot,t) : \mathbb{R}^n \to \mathbb{R}$ represents the pressure function. $\nabla \cdot U = 0$ represents the incompressible condition, U_0 is a given initial data with $\nabla \cdot U_0 = 0$.

Let $U_0 := u_0(x) - Mx$ and $x \in \mathbb{R}^n$, where the initial disturbance u_0 is a function nondecaying at space infinity satisfying $\nabla \cdot u_0 = 0$ in the tempered distribution sense, $M = (m_{jk})_{1\leqslant k,j\leqslant n}$ is an $n \times n$ real-valued matrix fulfilling with $\mathrm{tr}M = 0$ and M^2 is symmetric. In the following, let

$$u := U + Mx, \quad \tilde{P} := P - (IIx, x)$$

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for all $x \in \mathbb{R}^n$ and $t > 0$,

$$\mathrm{II} := \frac{1}{2}(M_1^2 + M_2^2), \quad M_1 := \frac{1}{2}(M + M^T),$$

and

$$M_2 := \frac{1}{2}(M - M^T),$$

where M^T stands for the transposed matrix of M. If (U, P) is the classical solution of (1.1), then $(u, \tilde{p})$ should satisfy the following equations:

$$\begin{cases} u_t - Au + u \cdot \nabla u - 2Mu + \nabla \tilde{p} = 0, & \text{in } \mathbb{R}^n \times (0, T), \\ \nabla \cdot u = 0, & \text{in } \mathbb{R}^n \times (0, T), \\ u\,|_{t=0} = u_0, & \text{in } \mathbb{R}^n. \end{cases} \tag{1.2}$$

Let the operator A be

$$Au := -\Delta u - Mx \cdot \nabla u + Mu,$$

with domian

$$D(A)(\mathbb{R}^n) := \{u \in W^{2,p}(\mathbb{R}^n) \cap L^p_\sigma(\mathbb{R}^n)\}, \quad Mx \cdot \nabla u \in L^p(\mathbb{R}^n)\},$$

where $W^{2,p}$ is the Sobolev space whose norm is

$$\|u\|_{L^p(\mathbb{R}^n)} + \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

Thanks to the results of Ornstein-Uhlenbeck semigroup theory [5, 6], we know that $-A$ generates a C_0-semigroup in $L^p_\sigma(\mathbb{R}^n)$ for $p \in [1, \infty)$. Meanwhile, $-A$ generates a semigroup in $L^\infty_\sigma(\mathbb{R}^n)$. We also have a representation form of semigroup

$$e^{-tA}f(x) = \frac{1}{(4\pi)^{\frac{n}{2}}(\det Q_t)^{\frac{1}{2}}} e^{-tM} \int_{\mathbb{R}^n} f(e^{tM}x - y) e^{-\frac{1}{4}(Q_t y, y)} ds,$$

where

$$Q_t := \int_0^t e^{sM} e^{sM^T} ds$$

for all $t > 0$. According to the Duhamel principle, the mild solution (u, d) for system (1.2) can be represented as:

$$u = e^{-tA}u_0 - \int_0^t e^{-(t-s)A}\mathbb{P}\nabla \cdot (u \otimes u)(\cdot, s)ds + 2\int_0^t e^{-(t-s)A}\mathbb{P}Mu(s)ds. \tag{1.3}$$

Here $\mathbb{P}$ is the Leray projection operator, which can be expressed as an $n \times n$ matrix:

$$\mathbb{P} = \{\mathbb{P}_{j,k}\}_{1\leqslant i,k\leqslant n} = \{\delta_{j,k} + \mathbb{R}_j\mathbb{R}_k\}_{1\leqslant i,k\leqslant n},$$

with $\delta_{j,k}$ being the Kronecker symbol, $\mathbb{R}_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ being the Riesz transform [11, 12].

In the case $M = 0$, the existence of the mild solution with the initial data in L^n was proved by Kato [4] and it was proved that the unique mild solution u satisfies

$$t^{\frac{n}{2}(\frac{1}{n}) - \frac{1}{q}} u \in BC([0, T); L^q(\mathbb{R}^n))$$

for $n \leqslant q \leqslant \infty$ and

$$t^{\frac{n}{2}(\frac{1}{n}) - \frac{1}{q} + \frac{1}{2}} \nabla u \in BC([0, T); L^q(\mathbb{R}^n)),$$

when $n \leqslant q \leqslant \infty$ for some $T > 0$, where $BC((0, T), Z)$ stands for the class of bounded continuous functions from $(0, T)$ onto the Banach spaces Z. Recently, this result was extended by Giga and Sawada, they proved in [2] that if the initial dada $u_0 \in L^n(\mathbb{R}^n)$, there exist $T_0 > 0$ and a unique mild solution u satisfying

$$\|D^\beta u(t)\|_{L^q(\mathbb{R}^n)} \leqslant ct^{-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})-\frac{|\beta|}{2}}$$

for all $0 < t < T_0$ and fixed $\beta \in \mathbb{N}^n$, note that here $q \in [n, \infty]$. Moreover, it was proved in [4, 7] that if the initial data $u_0 \in L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \leqslant c, \quad \|\nabla u(t)\|_{L^2(\mathbb{R}^n)} \leqslant ct^{-\frac{1}{2}}$$

for all $t \in [0, T_1)$ with some $0 < T_1 \leqslant T_0$. In all these results we should assume that the initial data decay as $|x| \to \infty$.

In the case $M \neq 0$, when the initial data may grow as Mx, where $M = (m_{ij})_{1\leqslant i,j\leqslant n}$ is a revalued constant matrix satisfying $trM = 0$ and M^2 is symmetric. The authors [3] established the local-in-time solvability and show solution of system (1.1) is analytic in x. We also can refer the reader to [8, 9] about linearly growing initial data. Clearly, the authors obtained the following result about the systems (1.2) from [3, Proposition 4.1].

Theorem 1.1. *Let $n \geqslant 2$ and $\beta \in \mathbb{N}^n$ fixed. Assume that $\|e^{tM}\| \leqslant 1$ for all $t > 0$, and $r \in (n, \infty), q \in [n, \infty], u_0 \in L^n(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$. Let u be the local mild solution of (2.1) for some $T > 0$. Suppose that there exist constants $M_1, M_2 > 0$ such that*

$$\sup_{0<t<T} \|u(t)\|_{L^n(\mathbb{R}^n)} \leqslant M_1 < \infty, \quad \sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{r})} \|u(t)\|_{L^r(\mathbb{R}^n)} \leqslant M_2 < \infty. \tag{1.4}$$

Assume further that

$$\partial_x^\alpha u \in C((0, T]; L^q(\mathbb{R}^n))$$

for all $q \in [n, \infty]$ and all $\alpha \in \mathbb{N}_0^n$. Then, given $\delta \in (\frac{1}{2}, 1]$, there exist constants K_1 and K_2 (depending only on n, r, M, M_1, M_2, T and δ) such that,

$$\|\nabla^m u(t)\|_{L^q(\mathbb{R}^n)} \leqslant K_1(K_2 m)^{m-\delta} t^{-\frac{m}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} < \infty$$

for any $t \in (0, T], n \in \mathbb{N}_0$ and $q \in [n, \infty]$.

Note that q is restricted by $q \geqslant n$ in Theorem 1.1. The authors are not able to give any results for $q < n$ under the assumption $u_0 \in L^n$.

Based on the ideas introduced in [3, 4], we construct L^2 estimate of solution to the Navier-Stokes equations [1, 10] with linearly growth initial data of the form $U_0 := u_0(x) - Mx$, where $u_0 \in L^2(\mathbb{R}^n) \cap L^n(\mathbb{R}^n)$ for $n \geqslant 3$,

Proposition 1.2. *Let $n \geqslant 3$ and $u_0 \in L^n(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, where $1 < p < n$. The solution u of system (1.2) has the following properties:*

$$u \in BC((0, T_2]; L^n(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)), \quad t^{\frac{1}{2}}\nabla u \in BC((0, T_2]; L^n(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)),$$

with some $0 < T_2 \leqslant T$.

Proof. For the proof of Proposition 1.2, we refer the reader to see [3, Theorem 2.1] and [4, Theorem 3]. □

Our main result is the following.

Theorem 1.3. *Let $n \geqslant 3$ and $\beta \in \mathbb{N}_0^n$. Assume that $\| e^{tM} \| \leqslant 1$ for all $t > 0$, and $u_0 \in L^n(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with $\nabla \cdot u_0 = 0$. Let u be the local-in-time mild solution of the system (1.2) from Proposition 1.2 for some $T > 0$ and $\partial_x^{\tilde\beta} u \in C((0, T); L^2(\mathbb{R}^n))$ for all $\tilde\beta \in \mathbb{N}_0^n$ with $\tilde\beta \leqslant \beta$. If there exists constant K_1 such that*

$$\sup_{0<t<T} t^{\frac{n}{2}(\frac{1}{n}-\frac{1}{p})} \|u(t)\|_{L^p(\mathbb{R}^n)} \leqslant K_1 < \infty \tag{1.5}$$

for some $p \in (n, 2n]$, *then there exists a constant* $K > 0$ *dependingly only on* $K_1, n, p, |\beta|, \|u_0\|_{L^2(\mathbb{R}^n)}, \|u_0\|_{L^n(\mathbb{R}^n)}$ *and* T, *such that*

$$\left\|D^\beta \left(u(t) - e^{-tA}u_0\right)\right\|_{L^2(\mathbb{R}^n)} \leqslant K t^{-\frac{|\beta|-1}{2}-\frac{n}{4}} \tag{1.6}$$

for any $0 < t < T$.

Throughout this article, we use $c > 0$ to denote a constant independent of the main variables, which may be different from line to line. We will employ the notation $a \lesssim b$ to mean that $a \leqslant cb$ for a universal constant $c > 0$ that only depends on the parameters coming from the problems.

2. Preliminaries

In this section, we prepare the following lemmas.

Lemma 2.1. *If* $n \geqslant 1$, $1 \leqslant p \leqslant q \leqslant \infty$, *then there exist constants* $c > 0$ *and* $\omega \geqslant 0$ *such that for all* $f \in L^p$, *we have*

(I)
$$\left\|e^{-tA}f\right\|_{L^q(\mathbb{R}^n)} \leqslant c e^{\omega t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}, \quad \left\|\nabla e^{-tA}f\right\|_{L^p(\mathbb{R}^n)} \leqslant c e^{\omega t} t^{-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^n)}. \tag{2.1}$$

(II) *For all* $t > 0, m \in \mathbb{N}$, *and* $f \in W^{m,p}$, *we have*
$$\left\|\nabla^m e^{-tA}f\right\|_{L^p(\mathbb{R}^n)} \leqslant \tilde{c}_1 e^{(\omega_1+\omega_2 m)t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \|\nabla^m f\|_{L^p(\mathbb{R}^n)}.$$

(III) *For all* $t > 0, m \in \mathbb{N}$, *and* $f \in L^p$, *we have*
$$\left\|\nabla^m e^{-tA}f\right\|_{L^p(\mathbb{R}^n)} \leqslant \tilde{c}_2 (\tilde{c}_3 m)^{\frac{m}{2}} e^{(\omega_3+\omega_4 m)t} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})-\frac{m}{2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

(IV) *For all* $f \in L^p$ *and* $t \geqslant 0$, *we have*
$$\left\|\nabla e^{-tA}\mathbb{P}f\right\|_{L^p(\mathbb{R}^n)} \leqslant c_p e^{\omega_5 t} t^{-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. For the proof of Lemma 2.1, we refer the reader to [3]. □

To prove our result, we need the following estimates: For any $\alpha \in \mathbb{N}_0^n, |\alpha| = m$ and $f \in L^1(\mathbb{R}^n)$,
$$\left\|D^\alpha e^{-tA}\mathbb{P}f\right\|_{L^2(\mathbb{R}^n)} \leqslant c_p e^{\omega_5 t} t^{-\frac{|\alpha|}{2}-\frac{n}{4}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Actually, in the case of $|\alpha| = 0$, noting that A and $\mathbb{P}$ can commute, we have
$$\left\|e^{-tA}\mathbb{P}f\right\|_{L^2(\mathbb{R}^n)} = \left\|\mathbb{P}e^{-tA}f\right\|_{L^2(\mathbb{R}^n)} \leqslant c \left\|e^{-tA}f\right\|_{L^2(\mathbb{R}^n)} \leqslant c e^{\omega t} t^{-\frac{n}{4}} \|f\|_{L^1(\mathbb{R}^n)}. \tag{2.2}$$

And in the case of $|\alpha| \geqslant 1$, it follows from (2.1) and (2.2) that
$$\left\|\nabla e^{-tA}\mathbb{P}f\right\|_{L^1(\mathbb{R}^n)} \leqslant c_1 e^{\omega_5 t} t^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^n)}.$$

Then, for $1 \leqslant q \leqslant \infty$, since
$$\nabla e^{-tA} f = e^{tM} e^{-tA} \nabla e^{-tA} f,$$
and $\| e^{tM} \| \leqslant c e^{\omega_2 t}$ for all $t > 0$, we have

$$\begin{aligned}
\left\|D^\alpha e^{-tA}\mathbb{P}f\right\|_{L^q(\mathbb{R}^n)} &\lesssim e^{\frac{\omega_2 t}{2}} \tilde{c}_2(\tilde{c}_3(m-1))^{\frac{m-1}{2}} e^{(\omega_3+\omega_4 m-\omega_4)\frac{t}{2}} t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{m-1}{2}} \left\|\nabla e^{-\frac{tA}{2}}\mathbb{P}f\right\|_{L^1(\mathbb{R}^n)} \\
&\lesssim e^{\frac{\omega_2 t}{2}} \tilde{c}_2(\tilde{c}_3(m-1))^{\frac{m-1}{2}} e^{(\omega_3+\omega_5+\omega_4 m-\omega_4)\frac{t}{2}} t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{m-1}{2}-\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^n)} \\
&\leqslant C e^{\omega t} t^{-\frac{n}{2}(1-\frac{1}{q})-\frac{m-1}{2}-\frac{1}{2}} \|f\|_{L^1(\mathbb{R}^n)}.
\end{aligned} \tag{2.3}$$

3. Proof of Theorem 1.3

Proof of Theorem 1.3. For simplicity, we prove the theorem under the additional assumption that $T \leqslant 1$. Let $|\beta| = m$, we will prove Theorem 1.3 in the case of $m \geqslant 1$ by an induction in Step 1 and Step 2 below. Based on the result in Step 1, we will prove (1.6) in the case of $m = 0$ in Step 3. In this section, c denotes a positive constant depending only on $n, p, M_1, \| u_0 \|_{L^2}, \| u_0 \|_{L^n}$, while $C_j(\epsilon)(j = 1, 2, \cdots)$ depends only on $n, p, M_1, \| u_0 \|_{L^2(\mathbb{R}^n)}, \| u_0 \|_{L^n(\mathbb{R}^n)}, T$ and ϵ.

Step 1. We first consider the case $m = 1$. Differentiating (1.3) with respect to x, taking L^2 norm, and dividing the integral into four parts for some $\epsilon \in (0, 1)$, one has

$$
\begin{aligned}
\|\nabla v(t)\|_{L^2(\mathbb{R}^n)} \leqslant & \int_0^{t(1-\epsilon)} \left\|\nabla e^{-(t-s)A}\mathbb{P}u \cdot \nabla u(s)\right\|_{L^p(\mathbb{R}^n)} ds \\
& + \int_{t(1-\epsilon)}^{t} \left\|\nabla e^{-(t-s)A}\mathbb{P}u \cdot \nabla u(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
& + 2\int_0^{t(1-\epsilon)} \left\|\nabla e^{-(t-s)A}\mathbb{P}Mu(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
& + 2\int_{t(1-\epsilon)}^{t} \left\|\nabla e^{-(t-s)A}\mathbb{P}Mu(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
:= & A_1 + A_2 + B_1 + B_2.
\end{aligned}
$$

We shall estimate each term. First, by multiplying both sides of first term of system (1.2) by u, and integrating by parts, note that $\mathrm{tr}M = 0$, for all $t > 0$, the Gronwall Lemma furnishes that

$$
\|u(t)\|_{L^2(\mathbb{R}^n)} \leqslant \|u_0\|_{L^2(\mathbb{R}^n)} \exp(|M|t),
$$

we can refer the reader to see [8].

For A_1, by Proposition 1.2 and (2.3), we also make use of the estimate $\|\nabla u(s)\|_{L^2} \leqslant cs^{-\frac{1}{2}}$, it follows that

$$
\begin{aligned}
A_1 & \leqslant \int_0^{t(1-\epsilon)} (t-s)^{-\frac{n}{4}-\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} \|\nabla u(s)\|_{L^2(\mathbb{R}^n)} ds \\
& \leqslant \int_0^{t(1-\epsilon)} (t-s)^{-\frac{n}{4}-\frac{1}{2}} s^{-\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} ds \\
& \leqslant ct^{-\frac{n}{4}} \|u_0\|_{L^2(\mathbb{R}^n)} \int_\epsilon^1 \tilde{s}^{-\frac{n}{4}-\frac{1}{2}} (1-\tilde{s})^{-\frac{1}{2}} ds \\
& = C_1(\epsilon) t^{-\frac{n}{4}},
\end{aligned}
$$

where $\tilde{s} = \frac{t-s}{t}$ and

$$
C_1(\epsilon) = c\|u_0\|_{L^2(\mathbb{R}^n)} \int_\epsilon^1 \tilde{s}^{-\frac{n}{4}-\frac{1}{2}} (1-\tilde{s})^{-\frac{1}{2}} ds.
$$

Using (2.3) and Hölder's inequality, we have

$$
\begin{aligned}
A_2 & \leqslant \int_{t(1-\epsilon)}^1 (t-s)^{-\frac{n}{2}(\frac{p+2}{2p}-\frac{1}{2})-\frac{1}{2}} \|u \cdot \nabla u(s)\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)} ds \\
& \leqslant \int_{t(1-\epsilon)}^1 (t-s)^{-\frac{n}{2}(\frac{p+2}{2p}-\frac{1}{2})-\frac{1}{2}} \|u \cdot \nabla v(s)\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)} ds \\
& \quad + \int_{t(1-\epsilon)}^1 (t-s)^{-\frac{n}{2}(\frac{p+2}{2p}-\frac{1}{2})-\frac{1}{2}} \|u \cdot \nabla e^{-sA} u_0\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)} ds \\
& := A_{21} + A_{22}.
\end{aligned}
$$

It follows from (2.3) that

$$A_{22} \leqslant ct^{-\frac{n}{4}} \int_{(1-\epsilon)}^{1} (1-\tilde{s})^{-\frac{1}{2}-\frac{n}{2p}} \tilde{s}^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|u_0\|_{L^2(\mathbb{R}^n)} d\tilde{s} = C_2(\epsilon)t^{-\frac{n}{4}},$$

where $\tilde{s} = \frac{s}{t}$ and

$$C_2(\epsilon) = c\|u_0\|_{L^2(\mathbb{R}^n)} \int_{(1-\epsilon)}^{1} (1-\tilde{s})^{-\frac{1}{2}-\frac{n}{2p}} \tilde{s}^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} d\tilde{s}.$$

From (1.5), we obtain

$$A_{21} \leqslant cM_1 \int_{t(1-\epsilon)}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2p}} s^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)} ds.$$

For B_1, similarly as A_1, since $T \leqslant 1$, we thus have

$$B_1 \leqslant c\int_0^{t(1-\epsilon)} (t-s)^{-\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} ds \leqslant C_3(\epsilon)t^{-\frac{n}{4}},$$

where

$$C_3(\epsilon) = c\|u_0\|_{L^2(\mathbb{R}^n)} \int_0^{1-\epsilon} (1-\tilde{s})^{-\frac{1}{2}} d\tilde{s}.$$

For B_2, since $T \leqslant 1$, we thus have

$$\begin{aligned}
B_2 &\leqslant c\int_{t(1-\epsilon)}^{t} \left\| e^{-(t-s)A}\mathbb{P}M \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \|\nabla u(s)\|_{L^2(\mathbb{R}^n)} ds \\
&\leqslant c\int_{t(1-\epsilon)}^{t} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)} ds + c\int_{t(1-\epsilon)}^{t} s^{-\frac{1}{2}} \|u_0\|_{L^2(\mathbb{R}^n)} ds \\
&\leqslant c\int_{t(1-\epsilon)}^{t} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)} ds + C_4(\epsilon)t^{-\frac{n}{4}} \\
&\leqslant c\int_{t(1-\epsilon)}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2p}} s^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)} ds + C_4(\epsilon)t^{-\frac{n}{4}},
\end{aligned}$$

where

$$c_4(\epsilon) = c\|u_0\|_{L^2(\mathbb{R}^n)} \int_{1-\epsilon}^{1} s^{-\frac{1}{2}} ds.$$

Combining these estimates A_1, A_{21}, A_{22}, B_1 and B_2, we finally get

$$\|\nabla v(t)\|_{L^2(\mathbb{R}^n)} \leqslant \sum_{i=1}^{4} C_i(\epsilon)t^{-\frac{n}{4}} + cM_1 \int_{t(1-\epsilon)}^{t} (t-s)^{-\frac{1}{2}-\frac{n}{2p}} s^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)} ds. \tag{3.1}$$

Let $\eta > 0$ and

$$\phi_0(t) = \sup_{\eta\leqslant s\leqslant t} s^{\frac{n}{4}} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)}.$$

From (3.1), we have

$$\phi_0(t) \leqslant \sum_{i=1}^{4} C_i(\epsilon)t^{-\frac{n}{4}} + cM_1\phi_0(t) \int_{(1-\epsilon)}^{1} (1-\tilde{s})^{-\frac{1}{2}-\frac{n}{2p}} \tilde{s}^{-\frac{1}{2}+\frac{n}{2p}-\frac{n}{4}} d\tilde{s}$$

for all $t \in (\frac{\eta}{1-\epsilon}, T)$, where $\tilde{s} = \frac{s}{t}$. Let $\epsilon_0 \in (0,1)$, we thus obtain

$$\int_{(1-\epsilon)}^{1} (1-\tilde{s})^{-\frac{1}{2}-\frac{n}{2p}} \tilde{s}^{-\frac{1}{2}+\frac{n}{2p}-\frac{n}{4}} d\tilde{s} \leqslant \frac{1}{2CM_1}.$$

Thus we have

$$\phi_0(t) \leqslant 2\sum_{i=1}^{4} C_i(\epsilon)t^{-\frac{n}{4}}$$

for all $t \in (\frac{\eta}{1-\epsilon}, T)$, that is, there exists a constant

$$K = K(M_1, n, p, \|u_0\|_{L^2(\mathbb{R}^n)}, \|u_0\|_{L^n(\mathbb{R}^n)}) > 0,$$

such that for all $t \in (0, T)$, we have

$$\left\|\nabla(u(t) - e^{-tA}u_0)\right\|_{L^2(\mathbb{R}^n)} \leqslant Kt^{-\frac{n}{4}}.$$

Step 2. We shall prove (1.6) for $m \geqslant 2$. We argue by an induction, thus we assume that $m \geqslant 1$ and (1.6) holds for $|\ \beta\ | \leqslant m-1$, we proceed to prove that (1.6) also holds for $|\beta| = m$. We see that for some $\epsilon \in (0, 1)$,

$$\begin{aligned}
\|\nabla^m v(t)\|_{L^2(\mathbb{R}^n)} &\leqslant \int_0^{t(1-\epsilon)} \left\|\nabla^m e^{-(t-s)A}\mathbb{P}u\cdot\nabla u(s)\right\|_{L^p(\mathbb{R}^n)} ds \\
&\quad + \int_{t(1-\epsilon)}^{t} \left\|\nabla^m e^{-(t-s)A}\mathbb{P}u\cdot\nabla u(s)\right\|_{L^p(\mathbb{R}^n)} ds \\
&\quad + 2\int_0^{t(1-\epsilon)} \left\|\nabla^m e^{-(t-s)A}\mathbb{P}Mu(s)\right\|_{L^p(\mathbb{R}^n)} ds \\
&\quad + 2\int_{t(1-\epsilon)}^{t} \left\|\nabla^m e^{-(t-s)A}\mathbb{P}Mu(s)\right\|_{L^p(\mathbb{R}^n)} ds \\
&=: \tilde{A}_1 + \tilde{A}_2 + \tilde{B}_1 + \tilde{B}_2.
\end{aligned}$$

For $\tilde{A}_1$, by (2.3), we obtain

$$\begin{aligned}
\tilde{A}_1 &\leqslant \int_0^{t(1-\epsilon)} \left\|\nabla^m e^{-(t-s)A}\mathbb{P}\right\|_{L^1(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \|u\cdot\nabla u(s)\|_{L^1(\mathbb{R}^n)} ds \\
&\leqslant \int_0^{t(1-\epsilon)} (t-s)^{-\frac{n}{4}-\frac{m}{2}} \|u\|_{L^2(\mathbb{R}^n)} \|\nabla u(s)\|_{L^2(\mathbb{R}^n)} ds \\
&\leqslant \int_0^{t(1-\epsilon)} (t-s)^{-\frac{n}{4}-\frac{m}{2}} s^{-\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)} ds \\
&= C_5(\epsilon)t^{-\frac{n}{4}},
\end{aligned}$$

where

$$\tilde{s} = \frac{t-s}{t}, \quad C_5(\epsilon) = c\,\|u_0\|_{L^2(\mathbb{R}^n)} \int_\epsilon^1 \tilde{s}^{-\frac{n}{4}-\frac{m}{2}}(1-\tilde{s})^{-\frac{1}{2}} ds.$$

Applying (2.3) and the Leibniz's rule, one obtains

$$\begin{aligned}
\tilde{A}_2 &\leqslant \int_{t(1-\epsilon)}^{1} (t-s)^{-\frac{n}{2p}-\frac{1}{2}} \left\|(D^\beta u)\otimes u\right\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)} ds \\
&\quad + \int_{t(1-\epsilon)}^{1} (t-s)^{-\frac{n}{2p}-\frac{1}{2}} \left\|(D^\alpha u)\otimes (D^{\beta-\alpha}u)\right\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)} ds \\
&\leqslant \int_{t(1-\epsilon)}^{1} (t-s)^{-\frac{n}{2p}-\frac{1}{2}} \|u\|_{L^p(\mathbb{R}^n)} \left\|D^\beta v(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
&\quad + \int_{t(1-\epsilon)}^{1} (t-s)^{-\frac{n}{2p}-\frac{1}{2}} \|u(s)\|_{L^2(\mathbb{R}^n)} \left\|D^\beta e^{-sA}u_0\right\|_{L^2(\mathbb{R}^n)}) ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\left\|(D^{\alpha}v)\otimes D^{\beta-\alpha}u)\right\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)}ds\\
&+\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\left\|(D^{\alpha}e^{-tA}u_0)\otimes D^{\beta-\alpha}u)\right\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)}ds\\
&:=\tilde{A}_{21}+\tilde{A}_{22}+\tilde{A}_{23}+\tilde{A}_{24}.
\end{aligned}
$$

For $\tilde{A}_{21}$, we have

$$
\tilde{A}_{21}\leqslant cM_1\int_{t(1-\epsilon)}^{t}(t-s)^{-\frac{1}{2}-\frac{n}{2p}}s^{-\frac{1}{2}+\frac{n}{2p}}\left\|D^{\beta}v(s)\right\|_{L^2(\mathbb{R}^n)}ds.
$$

It follows from (2.3) that

$$
\tilde{A}_{22}\leqslant ct^{-\frac{m-1}{2}-\frac{n}{4}}\int_{(1-\epsilon)}^{1}(1-\tilde{s})^{-\frac{1}{2}-\frac{n}{2p}}\tilde{s}^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{2}-\frac{1}{p})}\|u_0\|_{L^2(\mathbb{R}^n)}d\tilde{s}=C_6(\epsilon)t^{-\frac{m-1}{2}-\frac{n}{4}}.
$$

By our induction assumption, we have

$$
\begin{aligned}
\tilde{A}_{23}&\leqslant c\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\|D^{\alpha}v\|_{L^2(\mathbb{R}^n)}\left\|D^{\beta-\alpha}u\right\|_{L^p(\mathbb{R}^n)}ds\\
&\leqslant c\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}s^{-\frac{|\alpha|-1}{2}-\frac{n}{4}}s^{-\frac{|\beta-\alpha|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}ds\\
&=C_7(\epsilon)t^{-\frac{m-1}{2}-\frac{n}{4}},
\end{aligned}
$$

where $\tilde{s}=\frac{s}{t}$ and

$$
C_7(\epsilon)=c\int_{(1-\epsilon)}^{1}(1-\tilde{s})^{-\frac{n}{2p}-\frac{1}{2}}\tilde{s}^{-\frac{m-1}{2}-\frac{n}{4}-\frac{1}{2}+\frac{n}{2p}}d\tilde{s}.
$$

Applying (2.3) and the induction assumption, it follows that

$$
\begin{aligned}
\tilde{A}_{24}\leqslant{}& c\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\left\|D^{\beta-\alpha}v\right\|_{L^2(\mathbb{R}^n)}\left\|D^{\alpha}e^{-tA}u_0\right\|_{L^p(\mathbb{R}^n)}ds\\
&+c\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\left\|D^{\beta-\alpha}e^{-tA}u_0\right\|_{L^2(\mathbb{R}^n)}\left\|D^{\alpha}e^{-tA}u_0\right\|_{L^p(\mathbb{R}^n)}ds\\
\leqslant{}& c\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\|u_0\|_{L^n(\mathbb{R}^n)}s^{-\frac{|\alpha|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}s^{-\frac{|\beta-\alpha|-1}{2}-\frac{n}{4}}ds\\
&+c\int_{t(1-\epsilon)}^{1}(t-s)^{-\frac{n}{2p}-\frac{1}{2}}\|u_0\|^2_{L^2(\mathbb{R}^n)}s^{-\frac{|\alpha|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{p})}s^{-\frac{|\beta-\alpha|}{2}}ds\\
={}&C_8(\epsilon)t^{-\frac{m-1}{2}-\frac{n}{4}},
\end{aligned}
$$

where $\tilde{s}=\frac{s}{t}$ and

$$
C_8(\epsilon)=c\int_{(1-\epsilon)}^{1}(1-\tilde{s})^{-\frac{n}{2p}-\frac{1}{2}}\tilde{s}^{-\frac{m-1}{2}-\frac{n}{4}-\frac{1}{2}+\frac{n}{2p}}d\tilde{s}.
$$

For $\tilde{B}_1$, we have

$$
\begin{aligned}
\tilde{B}_1&\leqslant c\int_0^{t(1-\epsilon)}(t-s)^{-\frac{m}{2}}\|u\|_{L^2}ds\\
&\leqslant c\|u_0\|_{L^2}t^{-\frac{m}{2}}\int_0^{t(1-\epsilon)}(t-s)^{-\frac{m}{2}}ds\\
&\leqslant c\|u_0\|_{L^2}t^{-\frac{m}{2}}t^{\frac{1}{2}-\frac{n}{4}}\int_0^{(1-\epsilon)}(1-s)^{-\frac{m}{2}}ds\\
&=C_9(\epsilon)t^{-\frac{m-1}{2}-\frac{n}{4}}.
\end{aligned}
$$

For $\tilde{B}_2$, since $T \leqslant 1$, we have

$$\begin{aligned}
\tilde{B}_2 &\leqslant c\int_{t(1-\epsilon)}^{t} \left\| e^{-(t-s)A}\mathbb{P}M \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \|\nabla^m u(s)\|_{L^2(\mathbb{R}^n)}\, ds \\
&\leqslant c\int_{t(1-\epsilon)}^{t} \|\nabla^m v(s)\|_{L^2(\mathbb{R}^n)}\, ds + c\int_{t(1-\epsilon)}^{t} \left\|\nabla^m e^{-sA}u_0(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
&\leqslant c\int_{t(1-\epsilon)}^{t} \|\nabla^m v(s)\|_{L^2(\mathbb{R}^n)}\, ds + c\int_{t(1-\epsilon)}^{t} s^{-\frac{m}{2}} \|u_0\|_{L^2(\mathbb{R}^n)}\, ds \\
&\leqslant c\int_{t(1-\epsilon)}^{t} (t-s)^{-\frac12-\frac{n}{2p}} s^{-\frac12+\frac{n}{2p}} \left\|D^\beta v(s)\right\|_{L^2(\mathbb{R}^n)} ds + ct^{-\frac{m-1}{2}-\frac{n}{4}} \|u_0\|_{L^2(\mathbb{R}^n)} \int_{1-\epsilon}^{1} s^{-\frac{m}{2}}\, ds \\
&= \int_{t(1-\epsilon)}^{t} (t-s)^{-\frac12-\frac{n}{2p}} s^{-\frac12+\frac{n}{2p}} \left\|D^\beta v(s)\right\|_{L^2(\mathbb{R}^n)} ds + C_{10}(\epsilon) t^{-\frac{m-1}{2}-\frac{n}{4}}.
\end{aligned}$$

Combining above estimates, we finally obtain

$$\left\|\nabla^m \left(u(t) - e^{-tA}u_0\right)\right\|_{L^2(\mathbb{R}^n)} \leqslant Kt^{-\frac{m-1}{2}-\frac{n}{4}}.$$

Step 3. We shall prove (1.6) for $m = 0$. Similarly as Step 1,

$$\begin{aligned}
\|v(t)\|_{L^2(\mathbb{R}^n)} \leqslant& \int_0^{\frac{t}{2}} \left\|e^{-(t-s)A}\mathbb{P}u\cdot\nabla u(s)\right\|_{L^2(\mathbb{R}^n)} ds + \int_{\frac{t}{2}}^{t} \left\|e^{-(t-s)A}\mathbb{P}u_j\cdot\nabla u(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
&+ 2\int_0^t \left\|e^{-(t-s)A}\mathbb{P}Mu(s)\right\|_{L^2(\mathbb{R}^n)} ds \\
:=& \bar{A}_1 + \bar{A}_2 + \bar{B}_1.
\end{aligned}$$

By (2.3), we obtain

$$\bar{A}_1 \leqslant \int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{4}} \|u\|_{L^2(\mathbb{R}^n)} \|\nabla u(s)\|_{L^2}\, ds \leqslant \int_0^{\frac{t}{2}} (t-s)^{-\frac{n}{4}} s^{-\frac12} \|u\|_{L^2(\mathbb{R}^n)}\, ds \leqslant Kt^{-\frac{n}{4}+\frac12}.$$

It follows from (2.3) and Hölder's inequality that

$$\begin{aligned}
\bar{A}_2 &\leqslant \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}(\frac{p+2}{2p}-\frac12)} \left\|u\cdot\nabla\left(v(s)+e^{-sA}u_0\right)\right\|_{L^{\frac{2p}{p+2}}(\mathbb{R}^n)} ds \\
&\leqslant \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2}(\frac{p+2}{2p}-\frac12)} \left(\|u\|_{L^p} \|\nabla v(s)\|_{L^2(\mathbb{R}^n)} + \|u(s)\|_{L^2} \left\|\nabla e^{-sA}u_0\right\|_{L^2}\right) ds \\
&\leqslant \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2p}} \left(s^{-\frac{n}{2}} s^{-\frac{n}{2}(\frac1n-\frac1p)} + s^{-\frac12-\frac{n}{2}(\frac12-\frac1p)}\right) ds \\
&\leqslant \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{n}{2p}} \left(s^{-\frac{n}{2}} s^{-\frac{n}{2}(\frac1n-\frac1p)} + s^{-\frac12-\frac{n}{2}(\frac12-\frac1p)}\right) ds \\
&\leqslant Kt^{-\frac{n}{4}+\frac12}.
\end{aligned}$$

We can estimate $\bar{B}_1$ as

$$\bar{B}_1 \leqslant c\int_0^t \|u\|_{L^2(\mathbb{R}^n)}\, ds \leqslant ct\,\|u_0\|_{L^2(\mathbb{R}^n)} \leqslant Kt^{-\frac{n}{4}+\frac12}.$$

Thus, we completely prove (1.6) for $m = 0$. Thus the proof of the theorem is now completed. □

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