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Inequalities on asymmetric L_p-harmonic radial bodies

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Abstract

Lutwak introduced the L_p-harmonic radial body of a star body. In this paper, we define the notion of asymmetric L_p-harmonic radial bodies and study their properties. In particular, we obtain the extremum values of dual quermassintegrals and the volume of the polars of the asymmetric L_p-harmonic radial bodies, respectively.

Keywords: Star body, L_p-harmonic radial body, asymmetric L_p-harmonic radial body, dual quermassintegrals, polar.
2010 MSC: 52A20, 52A39, 52A40.

1. Introduction and main results

Lutwak in 1996 extended the classical Brunn-Minkowski theory to L_p-Brunn-Minkowski theory (see [15]), which had an enormous impact, providing stronger affine isoperimetric inequalities than the classical counterparts (see [22, 30]). Several years later, Ludwig, Haberl and Schuster extended the L_p-Brunn-Minkowski theory to asymmetric L_p Brunn-Minkowski theory by introducing a continuous parameter (see [6–9, 11, 12]). The asymmetric L_p-Brunn-Minkowski theory is far more general since the continuous parameter makes the L_p-asymmetric geometric bodies can be studied by analytical methods. In the past ten years, the investigations of L_p-asymmetric geometric bodies have received great interest from many articles (see [2, 10, 16, 17, 19, 20, 22, 24–29]).

The L_p-harmonic radial combination of convex bodies was first investigated by Firey (see [3, 4]) and was extended to star bodies by Lutwak (see [15]), which played an important role in L_p Brunn-Minkowski theory. For the further researches of L_p-harmonic radial combination, also see ([1, 21, 30]).

We will use $\mathcal{K}^n$ to denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. $\mathcal{K}^n_o$ denotes the set of convex bodies containing the origin in their interiors. The n-dimensional volume of a body K is written as $V(K)$ and $\widetilde{W}_i(K)$ denotes the dual quermassintegral of K. The unit sphere in $\mathbb{R}^n$ is denoted as S^{n-1} and B denotes the standard unit ball in $\mathbb{R}^n$.

If K is a compact star shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \to [0,\infty)$, is defined by (see [5, 18])

$$\rho(K,u) = \max\{\lambda \geqslant 0,\ \lambda u \in K\},\ \ u \in S^{n-1}.$$

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If ρ_K is positive and continuous, then K is called a star body (about the origin) and S^n denotes the set of star bodies in $\mathbb{R}^n$. We will use S^n_o and S^n_{os} to denote the subset of S^n containing the origin in their interiors and the set of origin-symmetric star body respectively. Two star bodies K and L are said to be dilated of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

For $K, L \in S^n_o$, $p \geqslant 1$ and $\lambda, \mu \geqslant 0$ (not both zero), the Firey L_p-harmonic radial combination, $\lambda \star K \tilde{+}_{-p} \mu \star L \in S^n_o$ of K and L, was defined by

$$\rho^{-p}(\lambda \star K \tilde{+}_{-p} \mu \star L, \cdot) = \lambda \rho^{-p}(K, \cdot) + \mu \rho^{-p}(L, \cdot). \tag{1.1}$$

In (1.1), let $\lambda = \mu = \frac{1}{2}$ and $L = -K$, the L_p-harmonic radial body was defined as follows,

$$\widetilde{\Delta}_p K = \frac{1}{2} \star K \tilde{+}_{-p} \frac{1}{2} \star (-K). \tag{1.2}$$

Based on the definitions of (1.1) and (1.2), Wang and Leng [23] gave the following results.

Theorem 1.1. *If $K, L \in S^n_0$, $p \geqslant 1$ and $\lambda, \mu \geqslant 0$ (not both zero), $i \neq n$, then for $i < n$ or $n < i < n+p$,*

$$\widetilde{W}_i(\lambda \star K \tilde{+}_{-p} \mu \star L)^{-\frac{p}{n-i}} \geqslant \lambda \widetilde{W}_i(K)^{-\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{-\frac{p}{n-i}}, \tag{1.3}$$

for $i > n+p$,

$$\widetilde{W}_i(\lambda \star K \tilde{+}_{-p} \mu \star L)^{-\frac{p}{n-i}} \leqslant \lambda \widetilde{W}_i(K)^{-\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{-\frac{p}{n-i}},$$

with equality in each inequality if and only if K and L are dilates.

Theorem 1.2. *If $K \in S^n_0$, $p \geqslant 1$, then for $i < n$ or $n < i < n+p$,*

$$\widetilde{W}_i(\widetilde{\Delta}_p K) \geqslant \widetilde{W}_i(K),$$

for $i > n+p$,

$$\widetilde{W}_i(\widetilde{\Delta}_p K) \leqslant \widetilde{W}_i(K),$$

with equality in each inequality if and only if K is origin-symmetric.

Here $\widetilde{W}_i(K)$ denotes the dual quermassintegrals of $K \in S^n_o$ which be defined by (see [13])

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n-i} du,$$

for any real i.

Motivated by the idea of Wang and Ma (see [26]), in this paper we extend the notion of L_p-harmonic radial bodies to asymmetric L_p-harmonic radial bodies as follows. For $K \in S^n_0$, $p \geqslant 1$ and $\tau \in [-1, 1]$, the asymmetric L_p-harmonic radial body of K is defined by

$$\widetilde{\Delta}^\tau_p K = f_1(\tau) \star K \tilde{+}_{-p} f_2(\tau) \star (-K), \tag{1.4}$$

here the functions $f_1(\tau)$ and $f_2(\tau)$ are defined as

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \quad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.5}$$

The function $f_1(\tau)$ and $f_2(\tau)$ were first defined in [9, 11]. From (1.5), we easily see that $f_1(\tau)$ and $f_2(\tau)$ satisfy

$$f_1(\tau) + f_2(\tau) = 1,$$
$$f_1(-\tau) = f_2(\tau), \quad f_2(-\tau) = f_1(\tau).$$

Together with (1.2), (1.4) and (1.5), we get if $\tau = 0$, then $\widetilde{\Delta}^\tau_p K = \widetilde{\Delta}_p K$; if $\tau = \pm 1$, then $\widetilde{\Delta}^{+1}_p K = K$ and $\widetilde{\Delta}^{-1}_p K = -K$.

The main goal of this paper is to give the extremum values of the dual quermassintegrals of asymmetric L_p-harmonic radial bodies and their polars.

Theorem 1.3. *If $K \in S_0^n$, $p \geqslant 1$, and $\tau \in [-1, 1]$, then for $i < n$ or $n < i < n+p$,*

$$\widetilde{W}_i(\widetilde{\Delta}_p K) \geqslant \widetilde{W}_i(\widetilde{\Delta}_p^\tau K) \geqslant \widetilde{W}_i(K). \tag{1.6}$$

For $i > n+p$,

$$\widetilde{W}_i(\widetilde{\Delta}_p K) \leqslant \widetilde{W}_i(\widetilde{\Delta}_p^\tau K) \leqslant \widetilde{W}_i(K),$$

if K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

When $i = 0$, we know $\widetilde{W}_i(K) = V(K)$ and we have the following result.

Corollary 1.4. *If $K \in S_0^n$, $p \geqslant 1$, and $\tau \in [-1, 1]$, then,*

$$V(K) \leqslant V(\widetilde{\Delta}_p^\tau K) \leqslant V(\widetilde{\Delta}_p K). \tag{1.7}$$

If K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$.

If we denote by $\widetilde{\Delta}_p^{\tau,*} K$ the polar body of the asymmetric L_p-harmonic radial body $\widetilde{\Delta}_p^\tau K$ of K, we can have the following result.

Theorem 1.5. *If $K \in S_0^n$, $p \geqslant 1$, and $\tau \in [-1, 1]$, then,*

$$V(K^*) \leqslant V(\widetilde{\Delta}_p^{\tau,*} K) \leqslant V(\widetilde{\Delta}_p^* K).$$

If K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = \pm 1$ and equality holds in the right inequality if and only if $\tau = 0$.

Since $\widetilde{\Delta}_p K \in S_{os}^n$, using Theorem 1.1 and extensive Blaschke-Santaló inequality ([15]) for symmetric star bodies, we deduce the following.

Theorem 1.6. *If $K \in S_0^n$, $p \geqslant 1$, and $\tau \in [-1, 1]$, then,*

$$V(K)V(\widetilde{\Delta}_p^{\tau,c} K) \leqslant \omega_n^2, \tag{1.8}$$

with equality if and only if K is an ellipsoid. Here $\widetilde{\Delta}_p^{\tau,c} K = (\widetilde{\Delta}_p^\tau K)^c$.

As a direct result, we have the following corollary.

Corollary 1.7. *If $K \in S_0^n$, $p \geqslant 1$, then,*

$$V(K)V(\widetilde{\Delta}_p^* K) \leqslant \omega_n^2, \tag{1.9}$$

with equality if and only if K is an ellipsoid centered at the origin. Here $\omega_n = V(B)$.

The proofs of Theorem 1.3-1.6 will be given in Section 3 of this paper. In the next section, we will give the associated background material and derive some properties of the asymmetric L_p-harmonic radial bodies needed in the proof of the main results.

2. Preliminaries

2.1. Background material

The notion of L_p-dual mixed volume was introduced by Lutwak ([14]). For $K, L \in S_o^n$, $p \geqslant 1$ and $\varepsilon > 0$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K, L)$, of the K and L is defined by

$$\frac{n}{-p}\widetilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_{-p} \varepsilon \star L) - V(K)}{\varepsilon}.$$

Lutwak ([14]) has proved the L_p-dual mixed volume has the following integral representation

$$\widetilde{V}_{-p}(K, L) = \frac{1}{n}\int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} du. \tag{2.1}$$

Obviously, by (2.1),

$$\widetilde{V}_{-p}(K,\ K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n du.$$

The L_p-dual Brunn-Minkowski inequality can be stated as follows ([14]): If $K,\ L \in S_o^n$, and $p \geqslant 1$, $\lambda, \mu \geqslant 0$ (not both zero), then

$$V(\lambda \star K \tilde{+}_{-p} \mu \star L)^{\frac{-p}{n}} \geqslant \lambda V(K)^{\frac{-p}{n}} + \mu V(L)^{\frac{-p}{n}}, \tag{2.2}$$

with equality if and only if K and L are dilates.

Associated with the L_p-Minkowski sum, Firey has proved the following interesting result:

$$(\lambda K +_p \mu L)^* = \lambda \star K^* \tilde{+}_{-p} \mu \star L^*. \tag{2.3}$$

Here E^* denotes the polar set of a non-empty set E which is defined by (see [5])

$$E^* = \{x \in R^n :\ x \cdot y \leqslant 1,\ \text{any } y \in E\}.$$

From definition (2.3), it follows that if $K \in K_o^n$, then

$$h_{K^*} = \frac{1}{\rho_K}, \quad \rho_{K^*} = \frac{1}{h_K}. \tag{2.4}$$

For $K \in S_{os}^n$, an extension of the well-known Blaschke-Santaló inequality takes the following form ([14]):

$$V(K)V(K^*) \leqslant \omega_n^2, \tag{2.5}$$

with equality if and only if K is an ellipsoid.

For a general star body $K \in S_o^n$, the Blaschke-Santaló inequality means

$$V(K)V(K^c) \leqslant \omega_n^2,$$

with equality if and only if K is an ellipsoid. Here K^c denotes the polar body of K with respect to the centroid of K, i.e., $K^c = (K - \text{cent}(K))^*$.

2.2. *Properties of asymmetric L_p-harmonic radial bodies*

In this subsection, we establish several properties of the asymmetric L_p-radial bodies needed in Section 3.

Theorem 2.1. *If $K \in S_o^n$, $p \geqslant 1$ and $\tau \in [-1,\ 1]$, then*

$$\widetilde{\Delta}_p^{-\tau} K = \widetilde{\Delta}_p^{\tau}(-K) = -\widetilde{\Delta}_p^{\tau} K. \tag{2.6}$$

Proof. From the definitions (1.4) and (1.5), we have

$$\widetilde{\Delta}_p^{-\tau} K = f_1(-\tau) \star K \tilde{+}_{-p} f_2(-\tau) \star (-K) = f_1(\tau) \star (-K) \tilde{+}_{-p} f_2(\tau) \star (-(-K)) = \widetilde{\Delta}_p^{\tau}(-K).$$

Further, together with definitions (1.1), (1.4) and (1.5),

$$\begin{aligned}
\rho^{-p}\big(-\widetilde{\Delta}_p^{\tau} K, u\big) &= \rho^{-p}\big(\widetilde{\Delta}_p^{\tau} K, -u\big) \\
&= f_1(\tau)\rho^{-p}(K, -u) + f_2(\tau)\rho^{-p}(-K, -u) \\
&= f_1(\tau)\rho^{-p}(-K, u) + f_2(\tau)\rho^{-p}((-(-K)), u) \\
&= \rho^{-p}\big(\widetilde{\Delta}_p^{\tau}(-K), u\big).
\end{aligned}$$

This yields the right equality of (2.6). □

Theorem 2.2. *If $K \in S_o^n$, $p \geqslant 1$ and $\tau \in [-1, 1]$, $\tau \neq 0$, then $\widetilde{\Delta}_p^{\tau} K = \widetilde{\Delta}_p^{-\tau} K$ if and only if K is an origin-symmetric star body.*

Proof. From the definitions (1.1), (1.4) and (1.5), we have that for all $u \in S_o^n$,

$$\rho^{-p}\left(\widetilde{\Delta}_p^{\tau} K, u\right) = f_1(\tau)\rho^{-p}(K, u) + f_2(\tau)\rho^{-p}(-K, u), \tag{2.7}$$

and

$$\rho^{-p}\left(\widetilde{\Delta}_p^{-\tau} K, u\right) = f_2(\tau)\rho^{-p}(K, u) + f_1(\tau)\rho^{-p}(-K, u). \tag{2.8}$$

Hence, if $\widetilde{\Delta}_p^{\tau} K = \widetilde{\Delta}_p^{-\tau} K$, then we get

$$[f_1(\tau) - f_2(\tau)]\rho^{-p}(K, u) = [f_1(\tau) - f_2(\tau)]\rho^{-p}(-K, u).$$

Since $\tau \neq 0$, we have $f_1(\tau) - f_2(\tau) \neq 0$, thus $\rho^{-p}(K, u) = \rho^{-p}(-K, u)$, for all $u \in S_o^n$, i.e., K is an origin-symmetric star body.

Contrarily, if K is an origin-symmetric star body, i.e., $K = -K$, then (2.7) and (2.8) yield $\rho^{-p}\left(\widetilde{\Delta}_p^{\tau} K, u\right) = \rho^{-p}\left(\widetilde{\Delta}_p^{-\tau} K, u\right)$ for all $u \in S_o^n$ which means $\widetilde{\Delta}_p^{\tau} K = \widetilde{\Delta}_p^{-\tau} K$. □

From Theorem 2.2, we deduce immediately:

Corollary 2.3. *For $K \in S_o^n$, $p \geqslant 1$ and $\tau \in [-1, 1]$, if K is not an origin-symmetric star body, then $\widetilde{\Delta}_p^{\tau} K = \widetilde{\Delta}_p^{-\tau} K$ if and only if $\tau = 0$.*

Theorem 2.4. *If $K \in S_{os}^n$, $p \geqslant 1$ and $\tau \in [-1, 1]$, then $\widetilde{\Delta}_p^{\tau} K = K$.*

Proof. Since $K \in S_{os}^n$, that is $K = -K$, by (1.1) and (1.4) we have

$$\rho^{-p}\left(\widetilde{\Delta}_p^{\tau} K, u\right) = f_1(\tau)\rho^{-p}(K, u) + f_2(\tau)\rho^{-p}(K, u) = \rho^{-p}(K, u),$$

for all $u \in S_o^n$. This gives $\widetilde{\Delta}_p^{\tau} K = K$. □

Theorem 2.5. *If $K \in S_o^n$, $p \geqslant 1$ and $\tau \in [-1, 1]$, then,*

$$\widetilde{\Delta}_p K = \frac{1}{2} \star \widetilde{\Delta}_p^{\tau} K \tilde{+}_{-p} \frac{1}{2} \star \widetilde{\Delta}_p^{-\tau} K. \tag{2.9}$$

Proof. From (1.1) and (1.4), it follows that for all $u \in S_o^n$,

$$\begin{aligned}
\rho^{-p}\left(\frac{1}{2} \star \widetilde{\Delta}_p^{\tau} K \tilde{+}_{-p} \frac{1}{2} \star \widetilde{\Delta}_p^{-\tau} K, u\right) &= \frac{1}{2}\rho^{-p}(\widetilde{\Delta}_p^{\tau} K, u) + \frac{1}{2}\rho^{-p}(\widetilde{\Delta}_p^{-\tau} K, u) \\
&= \frac{1}{2} f_1(\tau)\rho^{-p}(K, u) + \frac{1}{2} f_2(\tau)\rho^{-p}(-K, u) \\
&\quad + \frac{1}{2} f_1(-\tau)\rho^{-p}(K, u) + \frac{1}{2} f_2(-\tau)\rho^{-p}(-K, u) \\
&= \frac{1}{2}\rho^{-p}(K, u) + \frac{1}{2}\rho^{-p}(-K, u) \\
&= \rho^{-p}(\widetilde{\Delta}_p K, u),
\end{aligned}$$

i.e., $\widetilde{\Delta}_p K = \frac{1}{2} \star \widetilde{\Delta}_p^{\tau} K \tilde{+}_{-p} \frac{1}{2} \star \widetilde{\Delta}_p^{-\tau} K$. □

3. Proofs of the main results

In this section, we prove Theorem 1.3-1.6. First we prove Theorem 1.3.

Proof of Theorem 1.3. For $i < n$ or $n < i < n + p$, in inequality (1.3), let $\lambda = f_1(\tau)$, $\mu = f_2(\tau)$, notice (1.4), we have

$$\widetilde{W}_i(\widetilde{\Delta}_p^{\tau} K)^{-\frac{p}{n-i}} \geqslant f_1(\tau)\widetilde{W}_i(K)^{-\frac{p}{n-i}} + f_2(\tau)\widetilde{W}_i(-K)^{-\frac{p}{n-i}} = \widetilde{W}_i(K)^{-\frac{p}{n-i}},$$

which yields

$$\widetilde{W}_i(\widetilde{\Delta}_p^{\tau} K) \geqslant \widetilde{W}_i(K). \tag{3.1}$$

Clearly, equality holds in (3.1) when $\tau = \pm 1$. If $\tau = \pm 1$, by the equality condition of (1.3), we see that equality holds in (3.1), if and only if K and $-K$ are dilates which means $K = -K$, i.e., K is an origin-symmetric star body. Hence, if K is not origin-symmetric, equality holds in the right inequality if and only if $\tau = \pm 1$.

Now we prove the left hand inequality of (1.6). Notice (2.9), similarly to the right inequality of (1.6) and (2.6), we have

$$\begin{aligned}\widetilde{W}_i(\widetilde{\Delta}_p K)^{-\frac{p}{n-i}} &= \widetilde{W}_i(\frac{1}{2} \star \widetilde{\Delta}_p^{\tau} K \tilde{+}_{-p} \frac{1}{2} \star \widetilde{\Delta}_p^{-\tau} K)^{-\frac{p}{n-i}}\\ &\geqslant \frac{1}{2}\widetilde{W}_i(\widetilde{\Delta}_p^{\tau} K)^{-\frac{p}{n-i}} + \frac{1}{2}\widetilde{W}_i(\widetilde{\Delta}_p^{-\tau} K)^{-\frac{p}{n-i}}\\ &= \widetilde{W}_i(\widetilde{\Delta}_p^{\tau} K)^{-\frac{p}{n-i}},\end{aligned}$$

i.e., $\widetilde{W}_i(\widetilde{\Delta}_p K) \geqslant \widetilde{W}_i(\widetilde{\Delta}_p^{\tau} K)$, which is just the left hand inequality of (1.6).

Similarly, from the condition of equality in inequality (1.3), we know that equality holds in the left hand inequality of (1.6) if and only if $\widetilde{\Delta}_p^{\tau} K$ and $\widetilde{\Delta}_p^{-\tau} K$ are dilates, that is $\widetilde{\Delta}_p^{\tau} K = \widetilde{\Delta}_p^{-\tau} K$, thus using Corollary 2.3, we deduce that if K is not origin-symmetric, then equality holds in the left hand inequality of (1.6) if and only if $\tau = 0$.

For the case $i > n + p$, the proof is similar. □

The proof of Theorem 1.5 needs the following lemma ([26]).

Lemma 3.1. *If* $K \in K_o^n$, $p \geqslant 1$, *and* $\tau \in [-1, 1]$, *then*

$$V(\triangle_p K) \geqslant V(\triangle_p^{\tau} K) \geqslant V(K).$$

If K *is not origin-symmetric and* $p > 1$ *(or* K *is not central if* $p = 1$*), there is equality in the left if and only if* $\tau = 0$ *and equality in the right inequality if and only if* $\tau = \pm 1$.

Here $\triangle_p^{\tau} K$ denotes the asymmetric L_p-difference body and is defined as follows.

$$\triangle_p^{\tau} K = f_1(\tau)K +_p f_2(\tau)(-K). \tag{3.2}$$

Proof of Theorem 1.5. Clearly, by (2.2) and (3.2), we have

$$\begin{aligned}\widetilde{\Delta}_p^{\tau,*} K &= (\widetilde{\Delta}_p^{\tau} K)^*\\ &= (f_1(\tau) \star K \tilde{+}_{-p} f_2(\tau) \star (-K))^*\\ &= f_1(\tau)K^* +_p f_2(\tau)(-K)^*\\ &= \triangle_p^{\tau} K^*.\end{aligned}$$

Hence, we have $\widetilde{\Delta}_p^{*} K = \triangle_p K^*$ and $\widetilde{\Delta}_p^{+1,*} K = K^*$, $\widetilde{\Delta}_p^{-1,*} K = (-K)^*$.

Thus by Lemma 3.1, we have

$$V(K^*) \leqslant V(\widetilde{\Delta}_p^{\tau,*} K) \leqslant V(\widetilde{\Delta}_p^{*} K).$$

The equality conditions of Lemma 3.1 lead to that if K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = \pm 1$ and equality holds in the right inequality if and only if $\tau = 0$. □

Proof of Theorem 1.6. Since $K \in S_0^n$, using (1.7) and the general Blaschke-Santaló inequality (2.5). We have

$$V(K)V(\widetilde{\Delta}_p^{\tau,c} K) \leqslant V(\widetilde{\Delta}_p^{\tau} K)V(\widetilde{\Delta}_p^{\tau,c} K) \leqslant \omega_n^2.$$

The equality conditions of inequalities (1.7) and (2.5) mean that equality holds in (1.8) if and only if K is an ellipsoid. □

Proof of Corollary 1.7. Since $\widetilde{\Delta}_p K \in S^n_{os}$, combining with (1.7) and the extensive Blaschke-Santaló inequality (2.4), we have

$$V(K)V(\widetilde{\Delta}^*_p K) \leqslant V(\widetilde{\Delta}_p K)V(\widetilde{\Delta}^*_p K) \leqslant \omega_n^2.$$

i.e., (1.9) holds.

The equality conditions of inequalities (1.7) and (2.4) tell ue that equality holds in (1.9) if and only if $\widetilde{\Delta}_p K \in S^n_{os}$ and $\widetilde{\Delta}_p K$ is an ellipsoid. This and Theorem 2.4 indicate that equality holds in (1.9) if and only if K is an ellipsoid centered at the origin. □

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References

[1] Y. D. Chai, Y. S. Lee, *Harmonic radial combinations and dual mixed volumes*, Asian J. Math., **5**(2001), 493–498. 1
[2] Y.-B. Feng, W.-D. Wang, *Some inequalities for L_p-dual affine surface area*, Math. Inequal. Appl., **17** (2014), 431–441. 1
[3] W. J. Firey, *Mean cross-section measures of harmonic means of convex bodies*, Pacific J. Math., **11** (1961), 1263–1266. 1
[4] W. J. Firey, *Polar means of convex bodies and a dual to the Brunn-Minkowski theorem*, Canad. J. Math., **13** (1961), 444–453. 1
[5] R. J. Gardner, *Geometric tomography*, Second edition, Encyclopedia of Mathematics and its Applications, Cambridge University Press, New York, (2006). 1, 2.1
[6] C. Haberl, *L_p intersection bodies*, Adv. Math., **217** (2008), 2599–2624. 1
[7] C. Haberl, M. Ludwig, *A characterization of L_p intersection bodies*, Int. Math. Res. Not., **2006** (2006), 29 pages.
[8] C. Haberl, F. E. Schuster, *Asymmetric affine L_p Sobolev inequalities*, J. Funct. Anal., **257** (2009), 641–658.
[9] C. Haberl, F. E. Schuster, *General L_p affine isoperimetric inequalities*, J. Differential Geom., **83** (2009), 1–26. 1, 1
[10] C. Haberl, F. E. Schuster, J. Xiao, *An asymmetric affine Pólya-Szegö principle*, Math. Ann., **352** (2012), 517–542. 1
[11] M. Ludwig, *Minkowski valuations*, Trans. Amer. Math. Soc., **357** (2005), 4191–4213. 1, 1
[12] M. Ludwig, *Intersection bodies and valuations*, Amer. J. Math., **128** (2006), 1409–1428. 1
[13] M. Ludwig, *Valuations in the affine geometry of convex bodies*, Integral geometry and convexity, World Sci. Publ., Hackensack, NJ, (2006), 49–65. 1
[14] E. Lutwak, *Extended affine surface area*, Adv. Math., **85** (1991), 39–68. 2.1, 2.1, 2.1
[15] E. Lutwak, *The Brunn-Minkowski-Firey theory*, II, Affine and geominimal surface areas, Adv. Math., **118** (1996), 244–294. 1, 1
[16] L. Parapatits, *SL(n)-contravariant L_p-Minkowski valuations*, Trans. Amer. Math. Soc., **366** (2014), 1195–1211. 1
[17] L. Parapatits, *SL(n)-covariant L_p-Minkowski valuations*, J. Lond. Math. Soc., **89** (2014), 397–414. 1
[18] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, (1993). 1
[19] F. E. Schuster, T. Wannerer, GL(n) *contravariant Minkowski valuations*, Trans. Amer. Math. Soc., **364** (2012), 815–826. 1
[20] F. E. Schuster, M. Weberndorfer, *Volume inequalities for asymmetric Wulff shapes*, J. Differential Geom., **92** (2012), 263–283. 1
[21] W.-D. Wang, Q. Chen, *L_p-dual geominimal surface area*, J. Inequal. Appl., **2011** (2011), 10 pages. 1
[22] W.-D. Wang, Y.-B. Feng, *A general L_p-version of Petty's affine projection inequality*, Taiwanese J. Math., **17** (2013), 517–528. 1
[23] W.-D. Wang, G.-S. Leng, *L_p-dual mixed quermassintegrals*, Indian J. Pure Appl. Math., **36** (2005), 177–188. 1
[24] W.-D. Wang, Y.-N. Li, *Busemann-Petty problems for general L_p-intersection bodies*, Acta Math. Sin. (Engl. Ser.), **31** (2015), 777–786. 1
[25] W.-D. Wang, Y.-N. Li, *General L_p-intersection bodies*, Taiwanese J. Math., **19** (2015), 1247–1259.
[26] W.-D. Wang, T.-Y. Ma, *Asymmetric L_p-difference bodies*, Proc. Amer. Math. Soc., **142** (2014), 2517–2527. 1, 3
[27] T. Wannerer, GL(n) *equivariant Minkowski valuations*, Indiana Univ. Math. J., **60** (2011), 1655–1672.
[28] M. Weberndorfer, *Shadow systems of asymmetric L_p zonotopes*, Adv. Math., **240** (2013), 613–635.
[29] F. Yibin, W. Weidong, L. Fenghong, *Some inequalities on general L_p-centroid bodies*, Math. Inequal. Appl., **18** (2015), 39–49. 1
[30] B.-C. Zhu, N. Li, J.-Z. Zhou, *Isoperimetric inequalities for L_p geominimal surface area*, Glasg. Math. J., **53** (2011), 717–726. 1