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Split equality problem with equilibrium problem, variational inequality problem, and fixed point problem of nonexpansive semigroups

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Abstract

In this paper, we present a new algorithm for the split equality problem for finding a common element of solution of equilibrium problem, solution of variational inequality problem for monotone and Lipschitz continuous operators, and common fixed point of nonexpansive semigroups. We establish strong convergence of the algorithm in an infinite dimensional Hilbert spaces. Our results improve and generalize some recent results in the literature.

Keywords: Split equality problem, equilibrium problem, variational inequality, nonexpansive semigroups, fixed point.
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1. Introduction

Let C be a nonempty closed convex subset of $\mathcal{H}$. Let $\Phi : C \times C \to \mathbb{R}$ be a bifunction. In [6] it was shown that a broad class of problems in optimization, such as variational inequality, convex minimization, fixed point, and Nash equilibrium problems can be formulated as the equilibrium problem associated to the bifunction Φ and the set C: find $x \in C$ such that

$$\Phi(x, y) \geqslant 0, \qquad \forall y \in C.$$

A point $x \in C$ solving this problem is said to be an equilibrium point. The set of solutions is denoted by $EP(\Phi)$. Numerous problems in physics, optimization, and economics are reduced to find a solution of the equilibrium problem; see [36]. Many techniques and algorithms have been devised to analyze the existence and approximation of a solution to equilibrium problems; see [16].

Let $F : \mathcal{H} \to \mathcal{H}$ be a nonlinear operator. It is well-known that the variational inequality problem is to find $u \in C$ such that

$$\langle Fu, v - u\rangle \geqslant 0, \qquad \forall v \in C. \tag{1.1}$$

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We denote by $VI(C, F)$ the solution set of (1.1). The theory of variational inequalities has played an important role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, partial differential equation, operations research and engineering sciences. During the last decades this problem has been studied by many authors, (see [5, 23, 25, 37, 38]).

We recall the following definition on $F : \mathcal{H} \to \mathcal{H}$. The operator F is called

- Lipschitz continuous on $C \subset \mathcal{H}$ with constant $L > 0$ if
$$\|F(x) - F(y)\| \leqslant L\|x - y\|, \qquad \forall x, y \in C;$$
- nonexpansive on C if
$$\|F(x) - F(y)\| \leqslant \|x - y\|, \qquad \forall x, y \in C;$$
- monotone on C if
$$\langle F(x) - F(y), x - y\rangle \geqslant 0, \quad \forall x, y \in C;$$
- inverse strongly monotone with constant $\beta > 0$, (β-ism) if
$$\langle F(x) - F(y), x - y\rangle \geqslant \beta\|F(x) - F(y)\|^2, \quad \forall x, y \in C.$$

We note that every β-inverse strongly monotone operator is monotone and Lipschitz continuous. It is known that if F is β-inverse strongly monotone, and $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda F)$ is nonexpansive. It is worth noting that there exists a monotone Lipschitz continuous operator F such that $P_C(I - \lambda F)$ fails to be nonexpansive [15].

A family $\mathcal{T} := \{T(s) : 0 \leqslant s < \infty\}$ of mappings on C is called a nonexpansive semigroup if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;
(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geqslant 0$;
(iii) $\|T(s)x - T(s)y\| \leqslant \|x - y\|$ for all $x, y \in C$ and $s \geqslant 0$;
(iv) for all $x \in C$, $s \to T(s)x$ is continuous.

We use $F(\mathcal{T})$ to denote the common fixed point set of the semigroup $\mathcal{T}$, i.e., $F(\mathcal{T}) = \{x \in C : T(s)x = x, \forall s \geqslant 0\}$. It is well-known that $F(\mathcal{T})$ is closed and convex [7]. A nonexpansive semigroup $\mathcal{T}$ on C is said to be uniformly asymptotically regular (in short, u.a.r.) on C if for all $h \geqslant 0$ and any bounded subset E of C,
$$\lim_{t\to\infty} \sup_{x\in E} \|T(h)(T(t)x) - T(t)x\| = 0.$$
For each $h \geqslant 0$, define $\sigma_t(x) = \frac{1}{t}\int_0^t T(s)x\, ds$. Then
$$\lim_{t\to\infty} \sup_{x\in E} \|T(h)(\sigma_t(x)) - \sigma_t(x)\| = 0,$$
provides E be a closed bounded convex subset of C. It is known that the set $\{\sigma_t(x) : t > 0\}$ is a u.a.r. nonexpansive semigroup; see [14]. The other examples of u.a.r. operator semigroup can be found in [1].

It is well-known that the semigroup result has implications in partial differential equation theory, evolutionary equation theory, and fixed point theory. The nonexpansive semigroup is directly linked to solutions of differential equations and has been studied by several authors (see, for example, [29, 39] and the references therein).

In the last years, many authors studied the problems of finding a common element of the set of fixed points of nonlinear operator and the set of solutions of an equilibrium problem (and the set of solutions of variational inequality problem) in the framework of Hilbert spaces, see, for instance, [2, 11, 18, 20, 24, 31] and the references therein. The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or equilibrium problem. This happens, in particular, in the practical problems as signal processing, network resource allocation, and image recovery; see, for instance, [24, 31].

Let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is formulated as:

$$\text{to find} \quad x^* \in C \quad \text{and} \quad \mathcal{A}x^* \in Q,$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator. In 1994, Censor and Elfving [11] first introduced the SFP in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [8]. The split feasibility problem has received much attention due to its applications in signal processing and image reconstruction [9], with particular progress in intensity-modulated therapy [10].

Recently, Moudafi [33] introduced the following split equality problem. Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let C and Q be nonempty closed convex subsets of real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split equality problem (SEP) is to find

$$x \in C, \quad y \in Q \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y,$$

which allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, for instance in decomposition methods for PDEs, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows consideration of agents that interplay only via some components of their decision variables (see [3, 4]).

Since every nonempty closed convex subset of a Hilbert space can be regarded as a set of fixed points of a projection, Moudafi [34] introduced the following split equality fixed point problem (SEFP). Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$, $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be two bounded linear operators, let $S : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be two nonlinear operators such that $\mathrm{Fix}(S) \neq \emptyset$ and $\mathrm{Fix}(T) \neq \emptyset$. The split equality fixed point problem (SEFP) is to find

$$x \in \mathrm{Fix}(S), \quad y \in \mathrm{Fix}(T) \qquad \text{such that} \quad \mathcal{A}x = \mathcal{B}y. \tag{1.2}$$

If $\mathcal{H}_2 = \mathcal{H}_3$ and $\mathcal{B} = I$, then the split equality fixed point problem (1.1) reduces to the split common fixed point problem (SCFP) originally introduced in Censor and Segal [13] which is to find $x \in \mathrm{Fix}(S)$ with $\mathcal{A}x \in \mathrm{Fix}(T)$. Algorithms for solving the SEP and SCFP received great attention (see [17, 19, 21, 22, 26–28, 30, 32, 35, 40, 42] and references therein).

In [35], Moudafi and Al-Shemas introduced the following simultaneous iterative method to solve SEFP (1.2)

$$\begin{cases} x_{n+1} = S(x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)), \\ y_{n+1} = T(y_n + \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)), \end{cases} \qquad \forall n \geqslant 0, \tag{1.3}$$

for firmly quasi-nonexpansive operators S and T, where $\gamma_n \in (\epsilon, \frac{2}{\lambda_{\mathcal{A}} + \lambda_{\mathcal{B}}} - \epsilon)$ where $\lambda_{\mathcal{A}}$ and $\lambda_{\mathcal{B}}$ stand for the spectral radius of $\mathcal{A}^\star\mathcal{A}$ and $\mathcal{B}^\star\mathcal{B}$, respectively. We note that in the above algorithm the determination of the step-size $\{\gamma_n\}$ depends on the operator (matrix) norms $\|\mathcal{A}\|$ and $\|\mathcal{B}\|$ (or the largest eigenvalues of $\mathcal{A}^\star\mathcal{A}$ and $\mathcal{B}^\star\mathcal{B}$). In order to implement the alternating algorithm (1.3) for solving SEFP (1.2), the computation (or, at least, estimation) of the operator norms of $\mathcal{A}$ and $\mathcal{B}$, which is in general not an easy task in practice, will be required. To overcome this difficulty, Lopez et al. [30] introduced a method for estimating the step-sizes which does not require any knowledge of the operator norms for solving the split feasibility problems. Inspired by them, Zhao [42] and Dong et al. [17] introduced some new choice of the step-size sequence $\{\gamma_n\}$ for the simultaneous Mann iterative algorithm to solve SEFP.

Now, we consider a new type of split equality problem with equilibrium problem, variational inequality problem and fixed point problem as follows:

Let $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let

$\mathcal{T} := \{T(t) : t \geqslant 0\}$ and $\mathcal{S} := \{S(t) : t \geqslant 0\}$ be two u.a.r. nonexpansive semigroups on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let, $F : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and L-Lipschitz continuous operator on C and $G : \mathcal{H}_2 \to \mathcal{H}_2$ be a monotone and K-Lipschitz continuous operator on Q and that $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ be functions. We find a point $x \in Fix(\mathcal{T}) \bigcap EP(\Phi) \bigcap VI(C, F)$ and $y \in Fix(\mathcal{S}) \bigcap EP(\Phi) \bigcap VI(Q, G)$ such that $\mathcal{A}x = \mathcal{B}y$.

In this paper, using the extragradient method introduced by Korpelevich [26], we present a new algorithm for the split equality problem for finding a common element of solution of equilibrium problem, solution of variational inequality problem for monotone and Lipschitz continuous operators, and common fixed point of nonexpansive semigroups. Our algorithm does not require any knowledge of the operator norms. We establish strong convergence of the algorithm in an infinite dimensional Hilbert space. Our results improve and generalize the result of Moudafi [34, 35], Censor et al. [12], Zhao [42], and many others.

2. Preliminaries

In the sequel, we use the notions $\rightharpoonup$ and $\to$ for weak convergence and strong convergence, respectively. We need the following lemmas to prove our main result.

Lemma 2.1 ([40])**.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1 - \vartheta_n)a_n + \vartheta_n \delta_n, \qquad n \geqslant 0,$$

where $\{\vartheta_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \vartheta_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leqslant 0$ *or* $\sum_{n=1}^{\infty} |\vartheta_n \delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([32])**.** *Let $\{t_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} < t_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\tau(n) \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$t_{\tau(n)} \leqslant t_{\tau(n)+1}, \qquad t_n \leqslant t_{\tau(n)+1}.$$

In fact

$$\tau(n) = \max\{k \leqslant n : t_k < t_{k+1}\}.$$

Lemma 2.3 ([41])**.** *For each $x_1, \cdots, x_m \in H$ and $\alpha_1, \cdots, \alpha_m \in [0, 1]$ with $\sum_{i=1}^{m} \alpha_i = 1$, the following equality holds*

$$\|\alpha_1 x_1 + \cdots + \alpha_m x_m\|^2 = \sum_{i=1}^{m} \alpha_i \|x_i\|^2 - \sum_{1\leqslant i<j\leqslant m} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Given a nonempty closed convex set $C \subset \mathcal{H}$, the mapping that assigns every point $x \in \mathcal{H}$ to its unique nearest point in C is called the metric projection onto C and is denoted by P_C, i.e., $P_C \in C$ and $\|x - P_C x\| = \inf_{y\in C} \|x - y\|$. The metric projection P_C is characterized by the fact that $P_C(x) \in C$ and

$$\langle y - P_C(x), x - P_C(x)\rangle \leqslant 0, \qquad \forall x \in H, y \in C.$$

Lemma 2.4 ([2])**.** *Let $F : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz operator on C and λ be a positive number. Let $u_n = P_C(x_n - F(x_n))$ and $v_n = P_C(x_n - \lambda F(u_n))$. Then for all $x^\star \in VI(C, F)$ we have*

$$\|v_n - x^\star\|^2 \leqslant \|x_n - x^\star\|^2 - (1 - \lambda L)\|u_n - x_n\|^2 - (1 - \lambda L)\|u_n - v_n\|^2.$$

For solving the equilibrium problem, let us assume that the bifunction Φ satisfies the following conditions:

(A1) $\Phi(x,x)=0$ for all $x\in C$;
(A2) Φ is monotone, i.e., $\Phi(x,y)+\Phi(y,x)\leqslant 0$ for any $x,y\in C$;
(A3) for each $x,y,z\in C$
$$\limsup_{t\to 0^+}\Phi(tz+(1-t)x,y)\leqslant\Phi(x,y);$$
(A4) for each $x\in C$, $y\to\Phi(x,y)$ is convex and lower semi-continuous.

We know the following lemma which appeared implicitly in Blum et al. [6] and Combettes et al. [16].

Lemma 2.5 ([6, 16]). *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let Φ be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r>0$ and $x\in\mathcal{H}$. Then, there exists $z\in C$ such that*
$$\Phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geqslant 0,\qquad\forall y\in C.$$
Further, if
$$U_r^{\Phi}x=\{z\in C:\Phi(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geqslant 0,\quad\forall y\in C\},$$
then, the followings hold

(i) U_r^{Φ} *is single-valued;*
(ii) U_r^{Φ} *is firmly nonexpansive, i.e., for any $x,y\in\mathcal{H}$,*
$$\|U_r^{\Phi}x-U_r^{\Phi}y\|^2\leqslant\langle U_r^{\Phi}x-U_r^{\Phi}y,x-y\rangle;$$
(iii) $\mathrm{Fix}(U_r^{\Phi})=\mathrm{EP}(\Phi)$;
(iv) $\mathrm{EP}(\Phi)$ *is closed and convex.*

3. Split equality problem

In this section, by combining Halpern's iterative method and the extragradient method [26], we present our algorithm for solving the split equality problem with equilibrium problem, variational inequality problem, and fixed point problem in Hilbert spaces.

Theorem 3.1. *Let $\mathcal{H}_1,\mathcal{H}_2$, and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A}:\mathcal{H}_1\to\mathcal{H}_3$ and $\mathcal{B}:\mathcal{H}_2\to\mathcal{H}_3$ be bounded linear operators and let C and Q be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{T}:=\{T(t):t\geqslant 0\}$ and $\mathcal{S}:=\{S(t):t\geqslant 0\}$ be two u.a.r. nonexpansive semigroups on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let, $F:\mathcal{H}_1\to\mathcal{H}_1$ be a monotone and L- Lipschitz continuous operator on C and $G:\mathcal{H}_2\to\mathcal{H}_2$ be a monotone and K- Lipschitz continuous operator on Q and that $\Phi:C\times C\to\mathbb{R}$ and $\Psi:Q\times Q\to\mathbb{R}$ be functions satisfying conditions (A1)-(A4). Suppose $\Omega=\{x\in\mathrm{Fix}(\mathcal{T})\bigcap\mathrm{VI}(C,F)\bigcap\mathrm{EP}(\Phi),\quad y\in\mathrm{Fix}(\mathcal{S})\bigcap\mathrm{VI}(Q,G)\bigcap\mathrm{EP}(\Psi):\mathcal{A}x=\mathcal{B}y\}\neq\emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0,\vartheta\in\mathcal{H}_1,y_0,\zeta\in\mathcal{H}_2$, and by*
$$\begin{cases}z_n=x_n-\gamma_n\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n),\\ \eta_n=U^{\Phi}_{\kappa_{n,1}}z_n,\\ u_n=P_C(\eta_n-\lambda_nF(\eta_n)),\\ v_n=P_C(\eta_n-\lambda_nF(u_n)),\\ x_{n+1}=\alpha_n\vartheta+\beta_nv_n+\delta_nT(r_n)v_n\\ w_n=y_n+\gamma_n\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n),\\ \sigma_n=U^{\Psi}_{\kappa_{n,2}}w_n,\\ s_n=P_Q(\sigma_n-\rho_nG(\sigma_n)),\\ t_n=P_Q(\sigma_n-\rho_nG(s_n)),\\ y_{n+1}=\alpha_n\zeta+\beta_nt_n+\delta_nS(\iota_n)t_n,\quad\forall n\geqslant 0,\end{cases}\tag{3.1}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{r_n\}$, $\{\iota_n\}$, $\{\kappa_{n,1}\}$, $\{\kappa_{n,2}\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\lambda_n\}$, and $\{\rho_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \delta_n = 1$, *and* $\liminf_n \beta_n\delta_n > 0$;
(ii) $\lim_{n\to\infty} r_n = \infty$ *and* $\lim_{n\to\infty} \iota_n = \infty$;
(iii) $\lambda_n \subset [a, b] \subset (0, \frac{1}{L})$ *and* $\rho_n \subset [c, d] \subset (0, \frac{1}{K})$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(v) $\liminf_n \kappa_{n,1} > 0$ *and* $\liminf_n \kappa_{n,2} > 0$.

Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

Proof. Take $(x^\star, y^\star) \in \Omega$. Since the operator $U^{\Phi}_{\kappa_{n,1}}$ is firmly nonexpansive (see Lemma 2.5), we have

$$\|\eta_n - x^\star\|^2 = \|U^{\Phi}_{\kappa_{n,1}} z_n - x^\star\|^2 \leqslant \|z_n - x^\star\|^2 - \|z_n - \eta_n\|^2. \tag{3.2}$$

Also, we have

$$\|\sigma_n - y^\star\|^2 = \|U^{\Psi}_{\kappa_{n,2}} w_n - y^\star\|^2 \leqslant \|w_n - y^\star\|^2 - \|w_n - \sigma_n\|^2. \tag{3.3}$$

From Lemma 2.4 we deduce

$$\|v_n - x^\star\|^2 \leqslant \|\eta_n - x^\star\|^2 - (1 - \lambda_n L)\|\eta_n - u_n\|^2 - (1 - \lambda_n L)\|u_n - v_n\|^2, \tag{3.4}$$

and

$$\|t_n - y^\star\|^2 \leqslant \|\sigma_n - y^\star\|^2 - (1 - \rho_n K)\|\sigma_n - s_n\|^2 - (1 - \rho_n K)\|t_n - s_n\|^2. \tag{3.5}$$

Applying Lemma 2.3 and inequalities (3.2) and (3.4) we have

$$\begin{aligned}
\|x_{n+1} - x^\star\|^2 &= \|\alpha_n \vartheta + \beta_n v_n + \delta_n T(r_n)v_n - x^\star\|^2 \\
&\leqslant \alpha_n\|\vartheta - x^\star\|^2 + \beta_n\|v_n - x^\star\|^2 + \delta_n\|T(r_n)v_n - x^\star\|^2 - \beta_n\delta_n\|T(r_n)v_n - v_n\|^2 \\
&\leqslant \alpha_n\|\vartheta - x^\star\|^2 + \beta_n\|v_n - x^\star\|^2 + \delta_n\|v_n - x^\star\|^2 - \beta_n\delta_n\|T(r_n)v_n - v_n\|^2 \\
&\leqslant \alpha_n\|\vartheta - x^\star\|^2 + (1 - \alpha_n)\|z_n - x^\star\|^2 - \beta_n\delta_n\|T(r_n)v_n - v_n\|^2 \\
&\quad - (1 - \alpha_n)\|z_n - \eta_n\|^2 - (1 - \alpha_n)(1 - \lambda_n L)\|\eta_n - u_n\|^2 \\
&\quad - (1 - \alpha_n)(1 - \lambda_n L)\|u_n - v_n\|^2.
\end{aligned} \tag{3.6}$$

Similarly, from inequalities (3.3) and (3.5) we have

$$\begin{aligned}
\|y_{n+1} - y^\star\|^2 &= \|\alpha_n \zeta + \beta_n t_n + \delta_n S(\iota_n)t_n - y^\star\|^2 \\
&\leqslant \alpha_n\|\zeta - y^\star\|^2 + \beta_n\|t_n - y^\star\|^2 + \delta_n\|S(\iota_n)t_n - y^\star\|^2 - \beta_n\delta_n\|S(\iota_n)t_n - t_n\|^2 \\
&\leqslant \alpha_n\|\zeta - y^\star\|^2 + (1 - \alpha_n)\|w_n - y^\star\|^2 - \beta_n\delta_n\|S(\iota_n)t_n - t_n\|^2 \\
&\quad - (1 - \alpha_n)\|w_n - \sigma_n\|^2 - (1 - \alpha_n)(1 - \rho_n K)\|\sigma_n - s_n\|^2 \\
&\quad - (1 - \alpha_n)(1 - \rho_n K)\|t_n - s_n\|^2.
\end{aligned} \tag{3.7}$$

From algorithm (3.1) we have that

$$\begin{aligned}
\|z_n - x^\star\|^2 &= \|x_n - \gamma_n\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n) - x^\star\|^2 \\
&= \|x_n - x^\star\|^2 + \gamma_n^2\|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - 2\gamma_n\langle x_n - x^\star, \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2\|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - 2\gamma_n\langle \mathcal{A}x_n - \mathcal{A}x^\star, (\mathcal{A}x_n - \mathcal{B}y_n)\rangle \\
&= \|x_n - x^\star\|^2 + \gamma_n^2\|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n\|\mathcal{A}x_n - \mathcal{A}x^\star\|^2 \\
&\quad - \gamma_n\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n\|\mathcal{B}y_n - \mathcal{A}x^\star\|^2.
\end{aligned}$$

By similar way we obtain that

$$\begin{aligned}\|w_n - y^\star\|^2 &= \|y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n) - y^\star\|^2\\ &= \|y_n - y^\star\|^2 + \gamma_n^2\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 - \gamma_n\|\mathcal{B}y_n - \mathcal{B}y^\star\|^2\\ &\quad - \gamma_n\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 + \gamma_n\|\mathcal{A}x_n - \mathcal{B}y^\star\|^2.\end{aligned}$$

By adding the two last inequalities and by taking into account the fact that $\mathcal{A}x^\star = \mathcal{B}y^\star$ we obtain

$$\begin{aligned}\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2 &= \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2\\ &\quad - \gamma_n[2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 - \gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)]\\ &\leqslant \|x_n - x^\star\|^2 + \|y_n - y^\star\|^2.\end{aligned} \tag{3.8}$$

This implies that

$$\begin{aligned}\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2 &\leqslant (1-\alpha_n)(\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2)\\ &\leqslant (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2)\\ &\leqslant \max\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\}\\ &\;\;\vdots\\ &\leqslant \max\{\|x_0 - x^\star\|^2 + \|y_0 - y^\star\|^2, \|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2\}.\end{aligned}$$

Thus $\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2$ is bounded. Therefore $\{x_n\}$ and $\{y_n\}$ are bounded. Consequently $\{z_n\}, \{w_n\}, \{v_n\}$, and $\{t_n\}$ are all bounded. From (3.6), (3.7), and (3.8) we have that

$$\begin{aligned}&\|x_{n+1} - x^\star\|^2 + \|y_{n+1} - y^\star\|^2\\ &\leqslant (1-\alpha_n)(\|z_n - x^\star\|^2 + \|w_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2)\\ &\quad - \beta_n\delta_n\|T(r_n)v_n - v_n\|^2 - \beta_n\delta_n\|S(\iota_n)t_n - t_n\|^2\\ &\quad - (1-\alpha_n)\|z_n - \eta_n\|^2 - (1-\alpha_n)\|w_n - \sigma_n\|^2\\ &\quad - (1-\alpha_n)(1-\lambda_n L)\|\eta_n - u_n\|^2 - (1-\alpha_n)(1-\lambda_n L)\|u_n - v_n\|^2,\\ &\quad - (1-\alpha_n)(1-\rho_n K)\|\sigma_n - s_n\|^2 - (1-\alpha_n)(1-\rho_n K)\|t_n - s_n\|^2\\ &\leqslant (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2) + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2)\\ &\quad - (1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2 - \gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2)]\\ &\quad - \beta_n\delta_n\|T(r_n)v_n - v_n\|^2 - \beta_n\delta_n\|S(\iota_n)t_n - t_n\|^2\\ &\quad - (1-\alpha_n)\|z_n - \eta_n\|^2 - (1-\alpha_n)\|w_n - \sigma_n\|^2\\ &\quad - (1-\alpha_n)(1-\lambda_n L)\|\eta_n - u_n\|^2 - (1-\alpha_n)(1-\lambda_n L)\|u_n - v_n\|^2,\\ &\quad - (1-\alpha_n)(1-\rho_n K)\|\sigma_n - s_n\|^2 - (1-\alpha_n)(1-\rho_n K)\|t_n - s_n\|^2.\end{aligned} \tag{3.9}$$

From above inequality we have that

$$\begin{aligned}(1-\alpha_n)(1-\lambda_n L)\|\eta_n - u_n\|^2 &\leqslant (1-\alpha_n)(\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2)\\ &\quad - \|x_{n+1} - x^\star\|^2 - \|y_{n+1} - y^\star\|^2 + \alpha_n(\|\vartheta - x^\star\|^2 + \|\zeta - y^\star\|^2).\end{aligned} \tag{3.10}$$

By our assumption that

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon),$$

we have that

$$(\gamma_n+\epsilon)\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2\leqslant 2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2.$$

From above inequality and inequality (3.9) we have that

$$\begin{aligned}(1-\alpha_n)\gamma_n^2(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)&\\ \leqslant (1-\alpha_n)\gamma_n[2\|\mathcal{A}x_n-\mathcal{B}y_n\|^2-\gamma_n(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)]&\\ \leqslant (1-\alpha_n)(\|x_n-x^\star\|^2+\|y_n-y^\star\|^2)-\|x_{n+1}-x^\star\|^2-\|y_{n+1}-y^\star\|^2&\\ +\alpha_n(\|\vartheta-x^\star\|^2+\|\zeta-y^\star\|^2).&\end{aligned}\tag{3.11}$$

We finally analyze the inequalities (3.10) and (3.11) by considering the following two cases.

Case A. Put $\Gamma_n=\|x_n-x^\star\|^2+\|y_n-y^\star\|^2$ for all $n\in\mathbb{N}$. Suppose that $\Gamma_{n+1}\leqslant\Gamma_n$ for all $n\geqslant n_0$ (for n_0 large enough). In this case, since Γ_n is bounded, the limit $\lim_{n\to\infty}\Gamma_n$ exists. Since $\lim_{n\to\infty}\alpha_n=0$, from (3.11) and by our assumption on $\{\gamma_n\}$, we have

$$\lim_{n\to\infty}(\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2+\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|^2)=0.$$

So we obtain that $\lim_{n\to\infty}\|\mathcal{B}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|=0$ and $\lim_{n\to\infty}\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|=0$. This implies that $\lim_{n\to\infty}\|\mathcal{A}x_n-\mathcal{B}y_n\|=0$. Also from (3.10) we deduce

$$\lim_{n\to\infty}(1-\alpha_n)(1-\lambda_n L)\|\eta_n-u_n\|^2=0.$$

By our assumption that $\lambda_n\subset[a,b]\subset(0,\frac{1}{L})$, we obtain that

$$\lim_{n\to\infty}\|\eta_n-u_n\|=0.\tag{3.12}$$

By similar argument we get that

$$\lim_{n\to\infty}\|u_n-v_n\|=\|\sigma_n-s_n\|=\|t_n-s_n\|=0,\tag{3.13}$$

and

$$\lim_{n\to\infty}\|S(\iota_n)t_n-t_n\|=\lim_{n\to\infty}\|T(r_n)v_n-v_n\|=\|z_n-\eta_n\|=\|w_n-\sigma_n\|=0.$$

Further, for all $h\geqslant 0$ and $n\geqslant 0$, we see that

$$\begin{aligned}\|t_n-S(h)t_n\|&\leqslant\|t_n-S(\iota_n)t_n\|+\|S(\iota_n)t_n-S(h)S(\iota_n)t_n\|+\|S(h)S(\iota_n)t_n-S(h)t_n\|\\ &\leqslant 2\|t_n-S(\iota_n)t_n\|+\sup_{x\in\{t_n\}}\|S(\iota_n)t_n-S(h)S(\iota_n)t_n\|\end{aligned}$$

Since $\{S(h)\}$ is u.a.r. nonexpansive semigroup and $\lim_{n\to\infty}\iota_n=\infty$, so we have

$$\lim_{n\to\infty}\|t_n-S(h)t_n\|=0.$$

By similar argument for all $h\geqslant 0$, we can obtain that $\lim_{n\to\infty}\|T(h)v_n-v_n\|=0$. Since $\|z_n-x_n\|=\gamma_n\|\mathcal{A}^*(\mathcal{A}x_n-\mathcal{B}y_n)\|$ and $\{\gamma_n\}$ is bounded, we have

$$\lim_{n\to\infty}\|z_n-x_n\|=0.\tag{3.14}$$

From (3.12), (3.13), and (3.14) we have

$$\|x_n-v_n\|\leqslant\|x_n-z_n\|+\|z_n-\eta_n\|+\|\eta_n-u_n\|+\|u_n-v_n\|\to 0,\quad\text{as}\quad n\to\infty.$$

Therefore

$$\|x_{n+1}-x_n\| \leqslant \alpha_n\|\vartheta - x_n\| + \beta_n\|v_n - x_n\| + \delta_n\|T(r_n)v_n - x_n\| \to 0, \qquad \text{as} \quad n \to \infty.$$

Similarly we have that $\lim_{n\to\infty}\|y_{n+1}-y_n\| = 0$.

Now we claim that $(\omega_w(x_n), \omega_w(y_n)) \subset \Omega$, where

$$\omega_w(x_n) = \{x \in \mathcal{H}_1 : x_{n_i} \rightharpoonup x \text{ for some subsequences } \{x_{n_i}\} \text{of} \{x_n\}\}.$$

Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded we have $\omega_w(x_n)$ and $\omega_w(y_n)$ are nonempty. Now, take $\widehat{x} \in \omega_w(x_n)$ and $\widehat{y} \in \omega_w(y_n)$. Thus, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to $\widehat{x}$. Without loss of generality, we can assume that $x_n \rightharpoonup \widehat{x}$. Since $\lim_{n\to\infty}\|\eta_n - x_n\| = 0$, we have $\eta_n \rightharpoonup \widehat{x}$. From $u_n = P_C(\eta_n - \lambda_n F(\eta_n))$, for each $x \in C$ we have that

$$\langle x - u_n, \eta_n - \lambda_n F(\eta_n) - u_n\rangle \leqslant 0. \tag{3.15}$$

Since, F is monotone, for each $x \in C$ we have

$$\langle \lambda_n F(x), \eta_n - x\rangle \leqslant \langle \lambda_n F(\eta_n), \eta_n - x\rangle. \tag{3.16}$$

Utilizing the inequalities (3.15) and (3.16) we have

$$\begin{aligned}\langle \lambda_n F(x), \eta_n - x\rangle &\leqslant \langle \lambda_n F(\eta_n), \eta_n - x\rangle \\ &= \langle \lambda_n F(\eta_n), \eta_n - u_n\rangle + \langle \lambda_n F(\eta_n), u_n - x\rangle \\ &= \langle \lambda_n F(\eta_n), \eta_n - u_n\rangle + \langle \lambda_n F(\eta_n) - \eta_n + u_n, u_n - x\rangle + \langle \eta_n - u_n, u_n - x\rangle \\ &\leqslant \lambda_n\langle F(\eta_n), \eta_n - u_n\rangle + \langle \eta_n - u_n, u_n - x\rangle \\ &\leqslant \lambda_n\|F(\eta_n)\|\|\eta_n - u_n\| + \|\eta_n - u_n\|\|u_n - x\|.\end{aligned}$$

Hence

$$\langle Fx, \eta_n - x\rangle \leqslant \|F(\eta_n)\|\|\eta_n - u_n\| + \frac{1}{\lambda_n}\|\eta_n - u_n\|\|u_n - x\|.$$

Since $\{F(\eta_n)\}$ is bounded, $\eta_n - u_n \to 0$ and $\eta_n \rightharpoonup \widehat{x}$, we have

$$\langle F(x), \widehat{x} - x\rangle = \lim_{n\to\infty}\langle F(x), \eta_n - x\rangle \leqslant 0, \qquad \forall x \in C.$$

This implies that $\widehat{x} \in VI(C, F)$. By similar argument we can obtain that $\widehat{y} \in VI(Q, G)$. Next we show that $\widehat{x} \in Fix(\mathcal{T})$ and $\widehat{y} \in Fix(\mathcal{S})$. Since $\lim_{n\to\infty}\|v_n - x_n\| = 0$, we have $v_n \rightharpoonup \widehat{x}$. Now, for all $r \geqslant 0$ we have

$$\|v_n - T(r)\widehat{x}\| \leqslant \|v_n - T(r)v_n\| + \|T(r)v_n - T(r)\widehat{x}\| \leqslant \|v_n - T(r)v_n\| + \|v_n - \widehat{x}\|.$$

This implies that

$$\liminf_{n\to\infty}\|v_n - T(r)\widehat{x}\| \leqslant \liminf_{n\to\infty}\|v_n - \widehat{x}\|.$$

By the Opial property of the Hilbert space $\mathcal{H}_1$ we obtain that $T(r)(\widehat{x}) = \widehat{x}$ for all $r \geqslant 0$, which implies that $\widehat{x} \in Fix(\mathcal{T})$. By similar argument we obtain that $\widehat{y} \in Fix(\mathcal{S})$. Since $\lim_{n\to\infty}\|\eta_n - z_n\| = \lim_{n\to\infty}\|U^{\Phi}_{\kappa_{n,1}} z_n - z_n\| = 0$, and $z_n \rightharpoonup \widehat{x}$, we have $\widehat{x} \in EP(\Phi)$, (see [18] for details). Similarly we have $\widehat{y} \in EP(\Psi)$. On the other hand, $\mathcal{A}\widehat{x} - \mathcal{B}\widehat{y} \in \omega_w(\mathcal{A}x_n - \mathcal{B}y_n)$ and weakly lower semi continuity of the norm imply that

$$\|\mathcal{A}\widehat{x} - \mathcal{B}\widehat{y}\| \leqslant \liminf_{n\to\infty}\|\mathcal{A}x_n - \mathcal{B}y_n\| = 0.$$

Thus $(\widehat{x}, \widehat{y}) \in \Omega$. We also have that the uniqueness of the weak cluster point of $\{x_n\}$ are $\{y_n\}$, (see [42] for details) which implies that the whole sequence $\{(x_n, y_n)\}$ weakly converges to a point $(\widehat{x}, \widehat{y}) \in \Omega$. Next we prove that the sequence $\{(x_n, y_n)\}$ converges strongly to $(\vartheta^\star, \zeta^\star)$ where $\vartheta^\star = P_\Omega\,\vartheta$ and $\zeta^\star = P_\Omega\,\zeta$. First we show that

$$\limsup_{n\to\infty}\langle \vartheta - \vartheta^\star, x_n - \vartheta^\star\rangle \leqslant 0.$$

To show this inequality, we choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty}\langle \vartheta - \vartheta^\star, x_{n_k} - \vartheta^\star\rangle = \limsup_{n\to\infty}\langle \vartheta - \vartheta^\star, x_n - \vartheta^\star\rangle.$$

Since $\{x_{n_k}\}$ converges weakly to $\widehat{x}$, it follows that

$$\limsup_{n\to\infty}\langle \vartheta - \vartheta^\star, x_n - \vartheta^\star\rangle = \lim_{k\to\infty}\langle \vartheta - \vartheta^\star, x_{n_k} - \vartheta^\star\rangle = \langle \vartheta - \vartheta^\star, \widehat{x} - \vartheta^\star\rangle \leqslant 0.$$

By similar argument we obtain that

$$\limsup_{n\to\infty}\langle \zeta - \zeta^\star, y_n - \zeta^\star\rangle \leqslant 0.$$

From the inequality $\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, x+y\rangle$, $(\forall x, y \in \mathcal{H}_1)$, we find that

$$\begin{aligned}
\|x_{n+1} - \vartheta^\star\|^2 &\leqslant \|\beta_n v_n + \delta_n T(r_n)v_n - (1-\alpha_n)\vartheta^\star\|^2 + 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle \\
&= (1-\alpha_n)^2\|\frac{\beta_n}{(1-\alpha_n)}v_n + \frac{\delta_n}{(1-\alpha_n)}T(r_n)v_n - \vartheta^\star\|^2 + 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle \\
&\leqslant \beta_n(1-\alpha_n)\|v_n - \vartheta^\star)\|^2 + \delta_n(1-\alpha_n)\|T(r_n)v_n - \vartheta^\star\|^2 + 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle \\
&\leqslant (1-\alpha_n)^2\|v_n - \vartheta^\star\|^2 + 2\alpha_n\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle.
\end{aligned}$$

Similarly we obtain that

$$\|y_{n+1} - \zeta^\star\|^2 \leqslant (1-\alpha_n)^2\|w_n - \zeta^\star\|^2 + 2\alpha_n\langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle.$$

By adding the two last inequalities we have that

$$\begin{aligned}
\|x_{n+1} - \vartheta^\star\|^2 + \|y_{n+1} - \zeta^\star\|^2 &\leqslant (1-\alpha_n)^2(\|x_n - \vartheta^\star\|^2 + \|y_n - \zeta^\star\|^2) \\
&\quad + 2\alpha_n(\langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle + \langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle).
\end{aligned}$$

It immediately follows that

$$\begin{aligned}
\Gamma_{n+1} &\leqslant (1-\alpha_n)^2\Gamma_n + 2\alpha_n\eta_n \\
&= (1-2\alpha_n)\Gamma_n + \alpha_n^2\Gamma_n + 2\alpha_n\eta_n \\
&\leqslant (1-2\alpha_n)\Gamma_n + 2\alpha_n\{\frac{\alpha_n N}{2} + \eta_n) \\
&\leqslant (1-\zeta_n)\|x_n - x^\star\|^2 + \zeta_n\delta_n,
\end{aligned}$$

where $\eta_n = \langle \vartheta - \vartheta^\star, x_{n+1} - \vartheta^\star\rangle + \langle \zeta - \zeta^\star, y_{n+1} - \zeta^\star\rangle$, $N = \sup\{\|x_n - x^\star\|^2 + \|y_n - y^\star\|^2 : n \geqslant 0\}$, $\zeta_n = 2\alpha_n$ and $\delta_n = \frac{\alpha_n N}{2} + \eta_n$. It is easy to see that $\zeta_n \to 0$, $\sum_{n=1}^\infty \zeta_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leqslant 0$. Hence, all conditions of Lemma 2.1 are satisfied. Therefore, we immediately deduce that $\lim_{n\to\infty}\Gamma_n = 0$. Consequently, $\lim_{n\to\infty}\|x_n - \vartheta^\star\| = \lim_{n\to\infty}\|y_n - \zeta^\star\| = 0$, that is $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$.

Case B. Assume that $\{\Gamma_n\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geqslant n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \leqslant n : \Gamma_k < \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geqslant n_0$, $\Gamma_{\tau(n)} < \Gamma_{\tau(n)+1}$. Following an argument similar to that in Case A we have

$$\Gamma_{\tau(n)+1} \leqslant (1-\zeta_{\tau(n)})\Gamma_{\tau(n)} + \zeta_{\tau(n)}\eta_{\tau(n)},$$

where $\zeta_{\tau(n)} \to 0, \sum_{n=1}^{\infty} \zeta_{\tau(n)} = \infty$, and $\limsup_{n\to\infty} \zeta_{\tau(n)} \leqslant 0$. Hence, by Lemma 2.1, we obtain $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$. Since $\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$, we also have $\lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$. Now Lemma 2.2 implies

$$0 \leqslant \Gamma_n \leqslant \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leqslant \Gamma_{\tau(n)+1}.$$

Therefore $(x_n, y_n) \to (\vartheta^\star, \zeta^\star)$. This completes the proof. □

From Theorem 3.1, we obtain the following result for split equality equilibrium problem and fixed point problem.

Theorem 3.2. *Let $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$ be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\mathcal{T} := \{T(t) : t \geqslant 0\}$ and $\mathcal{S} := \{S(t) : t \geqslant 0\}$ be two u.a.r. nonexpansive semigroups on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $\Phi : C \times C \to \mathbb{R}$ and $\Psi : Q \times Q \to \mathbb{R}$ be functions satisfying conditions (A1)-(A4). Suppose $\Omega = \{x \in \mathrm{Fix}(\mathcal{T}) \bigcap \mathrm{EP}(\Phi),\quad y \in \mathrm{Fix}(\mathcal{S}) \bigcap \mathrm{EP}(\Psi) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_0, \vartheta \in \mathcal{H}_1$, $y_0, \zeta \in \mathcal{H}_2$, and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ v_n = U^{\Phi}_{\kappa_{n,1}} z_n, \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \delta_n T(r_n) v_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ t_n = U^{\Psi}_{\kappa_{n,2}} w_n, \\ y_{n+1} = \alpha_n \zeta + \beta_n t_n + \delta_n S(\iota_n) t_n, \quad \forall n \geqslant 0, \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{r_n\}$, $\{\iota_n\}$, $\{\kappa_{n,1}\}$, $\{\kappa_{n,2}\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \delta_n = 1$ *and* $\liminf_n \beta_n \delta_n > 0$;
(ii) $\lim_{n\to\infty} r_n = \infty$ *and* $\lim_{n\to\infty} \iota_n = \infty$;
(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iv) $\liminf_n \kappa_{n,1} > 0$ *and* $\liminf_n \kappa_{n,2} > 0$.

Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

As a corollary we obtain the following result for split equality fixed point problem of nonexpansive semigroups.

Corollary 3.3. *Let $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators. Let $\mathcal{T} := \{T(t) : t \geqslant 0\}$ and $\mathcal{S} := \{S(t) : t \geqslant 0\}$ be two u.a.r. nonexpansive semigroups on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Suppose $\Omega = \{x \in \mathrm{Fix}(\mathcal{T}),\quad y \in \mathrm{Fix}(\mathcal{S}) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by x_0, $\vartheta \in \mathcal{H}_1$, $y_0, \zeta \in \mathcal{H}_2$, and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ x_{n+1} = \alpha_n \vartheta + \beta_n z_n + \delta_n T(r_n) z_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ y_{n+1} = \alpha_n \zeta + \beta_n w_n + \delta_n S(\iota_n) w_n, \quad \forall n \geqslant 0, \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{r_n\}$, $\{\iota_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \gamma_n = 1$ *and* $\liminf_n \beta_n\delta_n > 0$;
(ii) $\lim_{n\to\infty} r_n = \infty$ *and* $\lim_{n\to\infty} \iota_n = \infty$;
(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

As another corollary of our main result we obtain the following result for split equality variational inequality problem for monotone and Lipschitz continuous operators.

Theorem 3.4. *Let $\mathcal{H}_1, \mathcal{H}_2$, and $\mathcal{H}_3$, be real Hilbert spaces, $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_3$ and $\mathcal{B} : \mathcal{H}_2 \to \mathcal{H}_3$ be bounded linear operators, and let C and Q be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $F : \mathcal{H}_1 \to \mathcal{H}_1$ be a monotone and L-Lipschitz continuous operator on C and $G : \mathcal{H}_2 \to \mathcal{H}_2$ be a monotone and K-Lipschitz continuous operator on Q. Suppose $\Omega = \{x \in VI(C, F), \quad y \in VI(Q, G) : \mathcal{A}x = \mathcal{B}y\} \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by x_0, $\vartheta \in \mathcal{H}_1$, $y_0, \zeta \in \mathcal{H}_2$, and by*

$$\begin{cases} z_n = x_n - \gamma_n \mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ u_n = P_C(z_n - \lambda_n F(z_n)), \\ v_n = P_C(z_n - \lambda_n F(u_n)), \\ x_{n+1} = \alpha_n \vartheta + (1-\alpha_n)v_n, \\ w_n = y_n + \gamma_n \mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n), \\ s_n = P_Q(w_n - \eta_n G(w_n)), \\ t_n = P_Q(w_n - \eta_n G(s_n)), \\ y_{n+1} = \alpha_n \zeta + (1-\alpha_n)t_n, \quad \forall n \geqslant 0, \end{cases}$$

where the step-size γ_n is chosen in such a way that

$$\gamma_n \in (\epsilon, \frac{2\|\mathcal{A}x_n - \mathcal{B}y_n\|^2}{\|\mathcal{B}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2 + \|\mathcal{A}^*(\mathcal{A}x_n - \mathcal{B}y_n)\|^2} - \epsilon), n \in \Pi,$$

otherwise $\gamma_n = \gamma$ (γ being any nonnegative value), where the index set $\Pi = \{n : \mathcal{A}x_n - \mathcal{B}y_n \neq 0\}$. Let the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\lambda_n\}$, and $\{\eta_n\}$ satisfy the following conditions:

(i) $\lambda_n \subset [a, b] \subset (0, \frac{1}{L})$ *and* $\eta_n \subset [c, d] \subset (0, \frac{1}{K})$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{(x_n, y_n)\}$ converges strongly to $(x^\star, y^\star) \in \Omega$.

We also have the following algorithm for finding a common element of the set of solution of an equilibrium problem, solution of a variational inequality problem, and common fixed point of a nonexpansive semigroup.

Theorem 3.5. *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $\mathcal{T} := \{T(t) : t \geqslant 0\}$ be u.a.r. nonexpansive semigroup on $\mathcal{H}$. Let $F : \mathcal{H} \to \mathcal{H}$ be a monotone and L-Lipschitz continuous operator on C and $\Phi : C \times C \to \mathbb{R}$ be a function satisfying conditions (A1)-(A4). Suppose $\Omega = \mathrm{Fix}(\mathcal{T}) \bigcap VI(C, F) \bigcap EP(\Phi) \neq \emptyset$. Let $\{x_n\}$ be sequence generated by x_0, $\vartheta \in \mathcal{H}$, and by*

$$\begin{cases} z_n = U^{\Phi}_{\kappa_n} x_n, \\ u_n = P_C(z_n - \lambda_n F(z_n)), \\ v_n = P_C(z_n - \lambda_n F(u_n)), \\ x_{n+1} = \alpha_n \vartheta + \beta_n v_n + \delta_n T(r_n) v_n, \quad \forall n \geqslant 0. \end{cases}$$

Let the sequences $\{r_n\}$, $\{\kappa_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, and $\{\lambda_n\}$ satisfy the following conditions:

(i) $\alpha_n + \beta_n + \delta_n = 1$ *and* $\liminf_n \beta_n \delta_n > 0$;
(ii) $\lim_{n\to\infty} r_n = \infty$, $\liminf_n \kappa_n > 0$, *and* $\lambda_n \subset [a, b] \subset (0, \frac{1}{L})$;
(iii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$.

Remark 3.6. In [12], the authors presented some algorithm for solving split variational inequality problem for inverse strongly monotone operators, but in this paper (Theorem 3.4) we consider some algorithm for solving split equality variational inequality problem for monotone and Lipschitz continuous operators which does not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.

Remark 3.7. Theorem 3.1 generalized the main result of Moudafi [34, 35]. Indeed, we present an algorithm for solving split equality problem for nonexpansive semigroups and monotone and Lipschitz continuous operator which does not require any knowledge of the operator norms. We also present a strong convergence theorem which is more desirable than weak convergence.

Remark 3.8. In [42], Zhao presented a convergence theorem for solving split equality fixed-point problem of quasi-nonexpansive mapping. In this paper we extend the result for solving split equality problem with equilibrium problem, variational inequality problem, and fixed point problem of nonexpansive semigroups.

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