Common fixed point theorems for Ćirić type mappings in b-metric spaces without any completeness assumption

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Abstract

In this paper, we establish some common fixed point theorems for four mappings satisfying Ćirić type contractive condition in b-metric spaces without any completeness assumption. Our results improve and generalize the results in the very recent papers ([Z.-Z. Yang, H. Sadati, S. Sedghi, N. Shobe, J. Nonlinear Sci. Appl., 8 (2015), 1022–1031], [V. Ozturk, S. Radenović, SpringerPlus, 5 (2016), 10 pages]). Particularly, the contractive constant $\frac{k}{b^2}$ in the result of Yang et al. is enlarged to $\frac{k}{b}$. Some examples are provided to support our results. ©2017 All rights reserved.

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1. Introduction and preliminaries

The Banach contraction principle is a basic and remarkable result in fixed point theory. Over the years, it has been extended in many different directions and spaces, see [4, 5, 8, 9, 11–14, 17, 21, 23, 24] and the references therein. In 1974, Ćirić [9] introduced the concept of a quasi-contractive mapping, i.e., a self-mapping $T : X \rightarrow X$ satisfies

$$d(Tx, Ty) \leq k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ and for a constant $k \in [0, 1)$, and proved a quasi-contractive mapping must have a unique fixed point in a complete metric space. Recently, Yang [23] obtained a common fixed point theorems for non-compatible self-maps in G-metric spaces. Roldán-López-de-Hierro et al. [21] noticed that the (CLRS)-property is weaker than the non-compatibleness, and improved the results of Yang [23]. On the other hand, Aamri and Moutawakil [1] introduced the (E.A)-property which is also weaker than the non-compatibleness. Using this concept, many authors established some new fixed point results, see, for example, [2, 6, 18–20]. In particular, Ozturk and Radenović [19] established some common fixed point results for four self-mappings in b-metric spaces which improved and generalized the results of

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Ozturk and Turkoglu [20]. Very recently, Yang et al. [24] proved a new common fixed point theorem for non-compatible self-mappings in b -metric space, where the space is not necessarily complete.

Inspired by the ideas of Yang et al. [24], Roldán-López-de-Hierro et al. [21], and Ozturk et al. [19], by using the notion of (CLRS)-property, we establish some common fixed point theorems for quasi-contractive mappings of Ćirić type in b -metric spaces. Our results improve the results of Yang et al. [24] and Ozturk et al. [19]. Particularly, the contractive constant $\frac{k}{b^2}$ in the result of Yang et al. [24] is enlarged to $\frac{k}{b}$. We notice that a gap arises in the proof of the main result of Yang et al. [24]; also refer to [21, Remark 11]. In this work, we also fill this gap; see Remark 2.5.

Let us recall some basic notions and results about b -metric spaces; see, for instance, [7, 10, 24].

Definition 1.1. Let X be a nonempty set and $b \geq 1$ be a given real number. A function $d : X \times X \rightarrow [0, \infty)$ is called a b -metric on X if the following conditions hold:

- (b1) $d(x, y) = 0$ if and only if $x = y$;
- (b2) $d(x, y) = d(y, x)$ for any $x, y \in X$;
- (b3) $d(x, y) \leq b(d(x, z) + d(z, y))$ for any $x, y, z \in X$.

Then the pair (X, d) is called a b -metric space (or metric type space).

Definition 1.2. Let (X, d) be a b -metric space.

- (1) The sequence $\{x_n\}$ converges to $x \in X$ if and only if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$;
- (2) the sequence $\{x_n\}$ is Cauchy if and only if $\lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0$;
- (3) the space (X, d) is complete if and only if every Cauchy sequence in X is convergent.

Remark 1.3. In a b -metric space (X, d) the following assertions hold:

- (i) a convergent sequence has a unique limit;
- (ii) each convergent sequence is Cauchy;
- (iii) in general, a b -metric is not continuous.

In the sequel, let (X, d) be a b -metric space and let $f, S : X \rightarrow X$ be two self-mappings. The following notions and basic results can be found, for instance, in [15, 21, 24].

Definition 1.4. The pair (f, S) is said to be compatible if

$$\lim_{n \rightarrow \infty} d(fSx_n, Sfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = z$ for some $z \in X$.

Definition 1.5. The self-mappings f and S of X are said to be

- (a) R -weakly commuting mappings of type (A_f) if there exists a positive real number R such that $d(fSx, SSx) \leq Rd(fx, Sx)$ for all $x \in X$.
- (b) R -weakly commuting mappings of type (A_S) if there exists a positive real number R such that $d(Sfx, ffx) \leq Rd(Sx, fx)$ for all $x \in X$.

Definition 1.6. The pair (f, S) is said to be weakly compatible if f and S commute at their coincidence points, that is $fSx = Sfx$ whenever $fx = Sx$.

Remark 1.7. It is clear to see that if f and S are R -weakly commuting mappings of type (A_f) or R -weakly commuting mappings of type (A_S) then the pair (f, S) is weakly compatible.

Definition 1.8. The self-mapping f of a b -metric space (X, d) is said to be b -continuous at $x \in X$ if and only if it is b -sequentially continuous at x , that is, $\{f(x_n)\}$ is convergent to $f(x)$ whenever $\{x_n\}$ is convergent to x .

Yang et al. [24] gave the following result.

Theorem 1.9 ([24]). Let (X, d) be a b -metric space and (f, S) be a pair of non-compatible self-mappings with $\overline{fX} \subseteq SX$ (here $\overline{fX}$ denotes the closure of fX). Assume the following conditions are satisfied

$$d(fx, fy) \leq \frac{k}{b^2} \max\{d(Sx, Sy), d(fx, Sx), d(fy, Sy)\}$$

for all $x, y \in X$ and $0 < k < 1$. If (f, S) is a pair of R -weakly commuting mappings of type (A_S) , then f and S have a unique common fixed point (say z) and both f and S are not b -continuous at z .

Definition 1.10 ([19, 20]). The self-mappings f and S on X are said to satisfy the b -(E.A) property if there exists a sequence $\{x_n\}$ such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = t$ for some $t \in X$.

We carry the notion of the (CLRS)-property to b -metric spaces; refer to [3, 21, 22].

Definition 1.11. Let $f, S, T : X \rightarrow X$ be three self-mappings of a b -metric space (X, d) . We say that the pair (f, S) satisfies the common limit in the range of T property (briefly, (CLRT)-property) if there exists a sequence $\{x_n\} \subseteq X$ and a point $u \in X$ such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = Tu \in TX.$$

Particularly, if $T = S$, we say that the pair (f, S) satisfies the (CLRS)-property if there exists a sequence $\{x_n\} \subseteq X$ and a point $u \in X$ such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = Su \in SX. \quad (1.1)$$

Lemma 1.12 ([19]). If (f, S) is not compatible, then (f, S) satisfies the b -(E.A) property.

Lemma 1.13. If (f, S) satisfies the b -(E.A) property and $\overline{fX} \subseteq TX$, then (f, S) satisfies the (CLRT)-property.

2. Main results

The following are main results of this paper.

Theorem 2.1. Let (X, d) be a b -metric space with $b \geq 1$ and $f, g, S, T : X \rightarrow X$ four self-mappings. Suppose that there exists $k \in [0, 1)$ such that

$$d(fx, gy) \leq \frac{k}{b} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\} \quad (2.1)$$

for all $x, y \in X$. If (f, S) satisfies the (CLRT)-property and (g, T) satisfies the (CLRS)-property, then (f, S) and (g, T) have a point of coincidence in X . Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S , and T have a unique common fixed point.

Proof. As (f, S) satisfies the (CLRT)-property and (g, T) satisfies the (CLRS)-property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and $u, v \in X$ such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = Tu \in TX, \quad (2.2)$$

$$\lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} Ty_n = Sv \in SX. \quad (2.3)$$

The proof is broken into the following three steps.

Step 1. We prove that $gu = Tu$, that is (g, T) has a point of coincidence in X . From (2.1), we see that

$$d(fx_n, gu) \leq \frac{k}{b} \max\{d(Sx_n, Tu), d(fx_n, Sx_n), d(gu, Tu), d(fx_n, Tu), d(Sx_n, gu)\}. \quad (2.4)$$

Notice the following two facts.

- (i) $d(fx_n, gu) \leq \frac{k}{b} d(gu, Tu) \leq kd(gu, fx_n) + kd(fx_n, Tu)$ implies $d(fx_n, gu) \leq \frac{k}{1-k} d(fx_n, Tu)$.
(ii) $d(fx_n, gu) \leq \frac{k}{b} d(Sx_n, gu) \leq kd(Sx_n, fx_n) + kd(fx_n, gu)$ implies $d(fx_n, gu) \leq \frac{k}{1-k} d(Sx_n, fx_n)$.

Thus, using (2.4) and $\frac{k}{b} < \frac{k}{1-k}$, we deduce that

$$\begin{aligned} d(fx_n, gu) &\leq \max\left\{\frac{k}{b} d(Sx_n, Tu), \frac{k}{b} d(fx_n, Sx_n), \frac{k}{1-k} d(fx_n, Tu), \frac{k}{b} d(fx_n, Tu), \frac{k}{1-k} d(Sx_n, fx_n)\right\} \\ &\leq \frac{k}{1-k} \max\{d(Sx_n, Tu), d(fx_n, Sx_n), d(fx_n, Tu)\} \\ &\leq \frac{bk}{1-k} [d(Sx_n, Tu) + d(fx_n, Tu)]. \end{aligned}$$

Using (2.2), we have $d(fx_n, gu) \rightarrow 0$ as $n \rightarrow \infty$. By Remark 1.3, we get $gu = Tu$.

Step 2. We prove that $fv = Sv$, that is (f, S) has a point of coincidence in X . From (2.1), we see that

$$d(fv, gy_n) \leq \frac{k}{b} \max\{d(Sv, Ty_n), d(fv, Sv), d(gy_n, Ty_n), d(fv, Ty_n), d(Sv, gy_n)\}. \quad (2.5)$$

Notice the following two facts.

- (i) $d(fv, gy_n) \leq \frac{k}{b} d(fv, Sv) \leq kd(fv, gy_n) + kd(gy_n, Sv)$ implies $d(fv, gy_n) \leq \frac{k}{1-k} d(gy_n, Sv)$.
(ii) $d(fv, gy_n) \leq \frac{k}{b} d(fv, Ty_n) \leq kd(fv, gy_n) + kd(gy_n, Ty_n)$ implies $d(fv, gy_n) \leq \frac{k}{1-k} d(gy_n, Ty_n)$.

Thus, using (2.5) and $\frac{k}{b} < \frac{k}{1-k}$, we deduce that

$$\begin{aligned} d(fv, gy_n) &\leq \max\left\{\frac{k}{b} d(Sv, Ty_n), \frac{k}{1-k} d(gy_n, Sv), \frac{k}{b} d(gy_n, Ty_n), \frac{k}{1-k} d(gy_n, Ty_n), \frac{k}{b} d(Sv, gy_n)\right\} \\ &\leq \frac{k}{1-k} \max\{d(Sv, Ty_n), d(gy_n, Sv), d(gy_n, Ty_n)\} \\ &\leq \frac{bk}{1-k} [d(gy_n, Sv) + d(Sv, Ty_n)]. \end{aligned}$$

Using (2.3), we have $d(fv, gy_n) \rightarrow 0$ as $n \rightarrow \infty$. By Remark 1.3, we get $fv = Sv$.

Step 3. Assume that the pairs (f, S) and (g, T) are weakly compatible. Then we prove f, g, S , and T have a unique common fixed point. Let $q = Tu = gu$ and $p = Sv = fv$. Then we claim that $p = q$. In fact, using (2.1), we have

$$\begin{aligned} d(p, q) &= d(fv, gu) \leq \frac{k}{b} \max\{d(Sv, Tu), d(fv, Sv), d(gu, Tu), d(fv, Tu), d(Sv, gu)\} \\ &= \frac{k}{b} \max\{d(p, q), d(p, p), d(q, q), d(p, q), d(p, q)\} \\ &= \frac{k}{b} d(p, q). \end{aligned}$$

By $\frac{k}{b} < 1$, we get $p = q$. Since the pairs (f, S) and (g, T) are weakly compatible, we have that

$$fp = fSv = Sfv = Sp \text{ and } gq = gTu = Tgu = Tq.$$

Now, we prove $fq = q$. Using (2.1), we have

$$\begin{aligned} d(fp, q) &= d(fp, gu) \leq \frac{k}{b} \max\{d(Sp, Tu), d(fp, Sp), d(gu, Tu), d(fp, Tu), d(Sp, gu)\} \\ &= \frac{k}{b} \max\{d(fp, q), 0, 0, d(fp, q), d(fp, q)\} = \frac{k}{b} d(fp, q). \end{aligned}$$

By $\frac{k}{b} < 1$, we get $fp = q$. Similarly, we have $p = gq$. Using $p = q$, $fp = Sp$, and $gq = Tq$, it follows that $Sp = fp = Tp = gp = p$, that is, p is a common fixed point of f, g, S , and T .

Finally, we show that the common fixed point of f, g, S , and T is unique. Actually, suppose that r is also a common fixed point of f, g, S , and T . Then, by (2.1), we derive that

$$d(p, r) = d(fp, gr) \leq \frac{k}{b} \max\{d(Sp, Tr), d(fp, Sp), d(gr, Tr), d(fp, Tr), d(Sp, gr)\} = \frac{k}{b} d(p, r).$$

From $\frac{k}{b} < 1$, it follows that $p = r$ and the uniqueness is proved. $\square$

If the b -metric d is continuous, then the contractive constant $\frac{k}{b}$ in Theorem 2.1 can be enlarged to k .

Theorem 2.2. *Let (X, d) be a b -metric space with $s \geq 1$, where d is continuous, and $f, g, S, T : X \rightarrow X$ four self-mappings. Suppose that there exists $k \in [0, 1)$ such that*

$$d(fx, gy) \leq k \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\} \quad (2.6)$$

for all $x, y \in X$. If (f, S) satisfies the (CLRT)-property and (g, T) satisfies the (CLRS)-property, then (f, S) and (g, T) have a point of coincidence in X . Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S , and T have a unique common fixed point.

Proof. As (f, S) satisfies the (CLRT)-property and (g, T) satisfies the (CLRS)-property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X and $u, v \in X$ such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = Tu \in TX, \quad \lim_{n \rightarrow \infty} gy_n = \lim_{n \rightarrow \infty} Ty_n = Sv \in SX.$$

We prove that $Tu = gu$ and $Sv = fv$. Using (2.6), we see that

$$d(fx_n, gu) \leq k \max\{d(Sx_n, Tu), d(fx_n, Sx_n), d(gu, Tu), d(fx_n, Tu), d(Sx_n, gu)\}.$$

Letting $n \rightarrow \infty$ in both sides of the above inequality, using the continuity of d , we have that

$$d(Tu, gu) \leq k \max\{d(Tu, Tu), d(Tu, Tu), d(gu, Tu), d(Tu, Tu), d(Tu, gu)\} = kd(Tu, gu).$$

From $k < 1$, we see that $d(Tu, gu) = 0$ and $gu = Tu$. Similarly, we can show that $fv = Sv$.

The remainder of the proof is similar to Step 3 of the proof in Theorem 2.1. $\square$

Theorem 2.3. *Let (X, d) be a b -metric space with $s \geq 1$ and $f, g, S, T : X \rightarrow X$ four self-mappings. Suppose that all hypotheses of Theorem 2.1 hold. Let $p \in X$ be the common fixed point of f, g, S , and T . Then the following statements hold.*

(i) *If (f, S) is a pair of R -weakly commuting mappings of type (A_S) and f or S is b -continuous at p , then*

$$\lim_{n \rightarrow \infty} d(fSx_n, Sfx_n) = 0,$$

where the sequence $\{x_n\}$ is given in (2.2).

(ii) *If (g, T) is a pair of R -weakly commuting mappings of type (A_T) and g or T is b -continuous at p , then*

$$\lim_{n \rightarrow \infty} d(gTy_n, Tgy_n) = 0,$$

where the sequence $\{y_n\}$ is given in (2.3).

Proof. We only prove (i). For this, we consider the following two cases.

(1) If f is b -continuous at p , from $fx_n \rightarrow Tu = p$ and $Sx_n \rightarrow Tu = p$, it follows that

$$ffx_n \rightarrow fp = p \text{ and } fSx_n \rightarrow fp = p.$$

Since f and S are R -weakly commuting mappings of type (A_S) , we have that

$$\begin{aligned} d(Sfx_n, fSx_n) &\leq bd(Sfx_n, ffx_n) + b^2d(ffx_n, p) + b^2d(p, fSx_n) \\ &\leq bRd(Sx_n, fx_n) + b^2d(ffx_n, z) + b^2d(z, fgx_n), \end{aligned}$$

which implies that $d(Sfx_n, fSx_n) \rightarrow 0$ as $n \rightarrow \infty$.

(2) If S is b -continuous at p , from $fx_n \rightarrow Tu = p$ and $Sx_n \rightarrow Tu = p$, it follows that

$$\lim_{n \rightarrow \infty} Sfx_n = Sp = p = gu \text{ and } \lim_{n \rightarrow \infty} SSx_n = Sp = p = gu.$$

We claim that $d(fSx_n, p) \rightarrow 0$. Using this claim, $d(fSx_n, Sfx_n) \leq bd(fSx_n, p) + bd(p, Sfx_n)$ and $d(p, Sfx_n) \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} d(fSx_n, Sfx_n) = 0.$$

Now, we prove our claim. Using (2.1), we have that

$$\begin{aligned} d(fSx_n, p) &= d(fSx_n, gu) \leq \frac{k}{b} \max\{d(SSx_n, Tu), d(fSx_n, SSx_n), d(gu, Tu), d(fSx_n, Tu), d(SSx_n, gu)\} \\ &= \frac{k}{b} \max\{d(SSx_n, p), d(fSx_n, SSx_n), d(fSx_n, p)\}. \end{aligned}$$

From this, we get the following cases.

Case i. If $\max\{d(SSx_n, p), d(fSx_n, SSx_n), d(fSx_n, p)\} = d(SSx_n, p)$, then $d(fgx_n, p) \leq \frac{k}{b}d(SSx_n, p)$.

Case ii. If $\max\{d(SSx_n, p), d(fSx_n, SSx_n), d(fSx_n, p)\} = d(fSx_n, SSx_n)$, then

$$d(fSx_n, p) \leq \frac{k}{b}d(fSx_n, SSx_n) \leq kd(fSx_n, p) + kd(p, SSx_n).$$

This implies that

$$d(fSx_n, p) \leq \frac{k}{1-k}d(p, SSx_n).$$

Case iii. If $\max\{d(SSx_n, p), d(fSx_n, SSx_n), d(fSx_n, p)\} = d(fSx_n, p)$, then $d(fSx_n, p) \leq \frac{k}{b}d(fSx_n, p)$. By $\frac{k}{b} < 1$, we have $d(fSx_n, p) = 0$.

In conclusion,

$$d(fSx_n, p) \leq \max\{\frac{k}{b}d(SSx_n, p), \frac{k}{1-k}d(p, SSx_n)\} = \frac{k}{1-k}d(p, SSx_n) \rightarrow 0.$$

This implies that $d(fSx_n, p) \rightarrow 0$. The proof has been completed. $\square$

If we take $f = g$ and $S = T$ in Theorem 2.1 and Theorem 2.3, by Remark 1.7, we get the following.

Corollary 2.4. Let (X, d) be a b -metric space and let $f, S : X \rightarrow X$ be two self-mappings satisfying the (CLRS)-property. Suppose that there exists $k \in [0, 1)$ such that

$$d(fx, fy) \leq \frac{k}{b} \max\{d(Sx, Sy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(Sx, fy)\}$$

for all $x, y \in X$. If (f, S) is a pair of R -weakly commuting mappings of type (A_S) , then f and S have a unique common fixed point (say z). And if we additionally assume that f or S is b -continuous at z , then

$$\lim_{n \rightarrow \infty} d(Sfx_n, fSx_n) = 0,$$

where the sequence $\{x_n\}$ is given in (1.1).

Remark 2.5.

- (1) From Lemmas 1.12 and 1.13, we see that Theorem 1.9 (that is, [24, Theorem 2.1]) is the special case of Corollary 2.4.
- (2) In [24, Theorem 2.1], the authors assumed that f and g are not compatible, and it is announced that f and S are not continuous at z , where z is the common fixed point of f and S . However, the argument given by the authors to prove that S is not b -continuous at z is not correct. assuming that S is continuous at z , it is proved that $\{Sfx_n\}$ converges to fz , where $fx_n \rightarrow z$, but this does not contradict with S being not b -continuous at z ; also see [21, Remark 11]. By Corollary 2.4, if we assume that f and S are not compatible and $\overline{fX} \subseteq SX$, then f and S cannot be b -continuous at z . This means we fill this gap in [24].

Now, by using Theorem 2.1, we deduce the main results in [19]. To the end, we need the following lemmas.

Lemma 2.6. Let (X, d) be a b -metric space with $s \geq 1$ and $f, g, S, T : X \rightarrow X$ four self-mappings. Then the following statements hold.

- (i) If $fX \subseteq TX$ and fX or TX is closed subset of X , then $\overline{fX} \subseteq TX$.
- (ii) If $gX \subseteq SX$ and gX or SX is closed subset of X , then $\overline{gX} \subseteq SX$.

Lemma 2.7. Let (X, d) be a b -metric space with $s \geq 1$ and let $f, g, S, T : X \rightarrow X$ be four self-mappings with $fX \subseteq TX$ and $gX \subseteq SX$ satisfying the condition (2.1). If $\overline{fX} \subseteq TX$ or $\overline{gX} \subseteq SX$, then the following statements are equivalent.

- (i) (f, S) satisfies the (CLRT) property.
- (ii) (g, T) satisfies the (CLRS) property.

Proof. We prove that (i) implies (ii). To this end, let (f, S) satisfy the (CLRT) property. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = Tu \in TX$$

for some $u \in X$. As $fX \subseteq TX$, there exists a sequence $\{y_n\}$ in X such that $fx_n = Ty_n$. Let us prove $\lim_{n \rightarrow \infty} gy_n = Tu$. Using (2.1), we have

$$\begin{aligned} d(fx_n, gy_n) &\leq \frac{k}{b} \max\{d(Sx_n, Ty_n), d(fx_n, Sx_n), d(gy_n, Ty_n), d(fx_n, Ty_n), d(Sx_n, gy_n)\} \\ &= \frac{k}{b} \max\{d(Sx_n, fx_n), d(gy_n, fx_n), d(Sx_n, gy_n)\}. \end{aligned}$$

From this, we get the following cases.

Case i. If $\max\{d(Sx_n, fx_n), d(gy_n, fx_n), d(Sx_n, gy_n)\} = d(Sx_n, fx_n)$, then $d(fx_n, gy_n) \leq \frac{k}{b} d(Sx_n, fx_n)$.

Case ii. If $\max\{d(Sx_n, fx_n), d(gy_n, fx_n), d(Sx_n, gy_n)\} = d(gy_n, fx_n)$, then $d(fx_n, gy_n) \leq \frac{k}{b} d(gy_n, fx_n)$. By $\frac{k}{b} < 1$, we get $d(fx_n, gy_n) = 0$.

Case iii. If $\max\{d(Sx_n, fx_n), d(gy_n, fx_n), d(Sx_n, gy_n)\} = d(Sx_n, gy_n)$, then $d(fx_n, gy_n) \leq \frac{k}{b} d(Sx_n, gy_n) \leq kd(Sx_n, fx_n) + kd(fx_n, gy_n)$, which implies $d(fx_n, gy_n) \leq \frac{k}{1-k} d(Sx_n, fx_n)$. Thus we obtain $d(fx_n, gy_n) \leq \max\{\frac{k}{b} d(Sx_n, fx_n), \frac{k}{1-k} d(Sx_n, fx_n)\} = \frac{k}{1-k} d(Sx_n, fx_n)$. From $d(Sx_n, fx_n) \leq bd(Sx_n, Tu) + bd(Tu, fx_n)$, it follows that $d(Sx_n, fx_n) \rightarrow 0$ and $d(fx_n, gy_n) \rightarrow 0$. Using $d(gy_n, Tu) \leq bd(gy_n, fx_n) + bd(fx_n, Tu)$, we see that $\lim_{n \rightarrow \infty} gy_n = Tu$.

If $\overline{gX} \subseteq SX$, there exists $v \in X$ such that $Tu = Sv$. Therefore we get

$$\lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} gy_n = Tu = Sv,$$

that is (g, T) satisfies the (CLRS) property.

If $\overline{fX} \subseteq TX$, we prove that (g, T) satisfies the (CLRS) property. We claim that $gu = Tu$. By $gX \subseteq SX$, there exists $v \in X$ such that $gu = Sv$. Therefore we get

$$\lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} gy_n = Tu = gu = Sv,$$

that is (g, T) satisfies the (CLRS) property.

Now we prove our claim. From (2.1), we see that

$$d(fx_n, gu) \leq \frac{k}{b} \max\{d(Sx_n, Tu), d(fx_n, Sx_n), d(Tu, gu), d(fx_n, Tu), d(Sx_n, gu)\}.$$

Notice the following two facts.

- (i) $d(fx_n, gu) \leq \frac{k}{b} d(Tu, gu) \leq kd(Tu, fx_n) + kd(fx_n, gu)$ implies $d(fx_n, gu) \leq \frac{k}{1-k} d(Tu, fx_n)$.
- (ii) $d(fx_n, gu) \leq \frac{k}{b} d(Sx_n, gu) \leq kd(Sx_n, fx_n) + kd(fx_n, gu)$ implies $d(fx_n, gu) \leq \frac{k}{1-k} d(Sx_n, fx_n)$.

Thus, by $\frac{k}{b} < \frac{k}{1-k}$, we deduce that

$$\begin{aligned} d(fx_n, gu) &\leq \max\left\{\frac{k}{b} d(Sx_n, Tu), \frac{k}{b} d(fx_n, Sx_n), \frac{k}{1-k} d(Tu, fx_n), \frac{k}{b} d(fx_n, Tu), \frac{k}{1-k} d(Sx_n, fx_n)\right\} \\ &\leq \frac{k}{1-k} \max\{d(Sx_n, Tu), d(fx_n, Sx_n), d(Tu, fx_n)\} \\ &\leq \frac{bk}{1-k} [d(fx_n, Tu) + d(Tu, Sx_n)]. \end{aligned}$$

Since $d(fx_n, Tu) \rightarrow 0$ and $d(Tu, Sx_n) \rightarrow 0$, we have that $d(fx_n, gu) \rightarrow 0$ as $n \rightarrow \infty$. By Remark 1.3, we get $gu = Tu$.

The proof that (ii) implies (i), follows by a similar argument as one of (i) implies (ii). $\square$

Let $f, g, S, T : X \rightarrow X$ be four self-mappings with $fX \subseteq TX$ and $gX \subseteq SX$ satisfying the condition (2.1). By Lemmas 1.13, 2.6, and 2.7, we see that if one of the subsets fX, gX, SX , and TX is closed in X and one of the pairs (f, S) and (g, T) satisfies the b -(E.A)-property, then (f, S) satisfies the (CLRT)-property and (g, T) satisfies the (CLRS)-property. Thus, by Theorem 2.1, we have the following results.

Corollary 2.8 ([19, Theorem 16]). *Let (X, d) be a b -metric space with $s \geq 1$ and let $f, g, S, T : X \rightarrow X$ be four self-mappings with $fX \subseteq TX$ and $gX \subseteq SX$ satisfying the condition (2.1). Suppose that one of the pairs (f, S) and (g, T) satisfies the b -(E.A)-property and that one of the subsets fX, gX, SX , and TX is closed in X . Then (f, S) and (g, T) have a point of coincidence in X . Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S , and T have a unique common fixed point.*

Corollary 2.9. *Let (X, d) be a b -metric space with $s \geq 1$ and let $f, g, S, T : X \rightarrow X$ be four self-mappings with $\overline{fX} \subseteq TX$ and $\overline{gX} \subseteq SX$ satisfying the condition (2.1). Suppose that one of the conditions (f, S) satisfying the (CLRT)-property and (g, T) satisfying the (CLRS)-property holds. Then (f, S) and (g, T) have a point of coincidence in X . Moreover, if the pairs (f, S) and (g, T) are weakly compatible, then f, g, S , and T have a unique common fixed point.*

Remark 2.10. We see that [19, Theorem 10] and [19, Theorem 15] are the special cases of Corollary 2.8.

The next is an example where we can apply Theorems 2.1 and 2.2 but not Theorem 1.9 or Corollary 2.8.

Example 2.11. Let $X = [1, 10]$ and let $d : X \times X \rightarrow [0, \infty)$ be defined by $d(x, y) = (x - y)^2$. We define mappings f, g, S , and T on X by

$$fx = gx = \begin{cases} 1, & x = 1 \text{ or } x \in (5, 10), \\ 6, & x \in [1, 5], \end{cases}$$

and

$$Sx = Tx = \begin{cases} 1, & x = 1, \\ 9, & x \in (1, 5], \\ \frac{x+1}{6}, & x \in (5, 10). \end{cases}$$

Obviously, d is a b -metric with $b = 2$ and $\overline{fX} = fX = \overline{gX} = gX \not\subseteq SX = TX$. To prove that (f, S) satisfies the (CLRT)-property and (g, T) satisfies the (CLRS)-property, consider a sequence $\{x_n\}$ defined by $x_n = 5 + \frac{1}{n}$. We have

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = 1 = T1 \in TX \text{ and } \lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} Tx_n = 1 = S1 \in SX.$$

On the other hand,

$$d(gfx, ffx) = \begin{cases} (1-1)^2 = 0, & x = 1, \\ (\frac{7}{6}-1)^2 = \frac{1}{36}, & x \in (1, 5], \\ (1-1)^2 = 0, & x \in (5, 10), \end{cases}$$

and

$$d(fx, gx) = \begin{cases} (1-1)^2 = 0, & x = 1, \\ (6-9)^2 = 9, & x \in (1, 5], \\ (\frac{x+1}{6}-1)^2, & x \in (5, 10) \end{cases}$$

for all $x \in X$, and hence there exists $R = 1$ such that

$$d(gfx, ffx) \leq d(fx, gx).$$

That is, (f, S) is R -weakly commuting mappings of type (A_S) and (g, T) is R -weakly commuting mapping of type (A_T) . By Remark 1.7, we get that (f, S) and (g, T) are weakly compatible. Now we prove that the mappings f, g, S , and T satisfy the condition (2.1) of Theorem 2.1 with $k = \frac{13}{16}$. For this, we consider the following cases:

(1) If $x, y \in \{1\} \cup (5, 10)$, then we have

$$d(fx, gy) = d(1, 1) = 0 \leq \frac{k}{b} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\}.$$

(2) If $x, y \in (1, 5]$, then we have

$$d(fx, gy) = d(6, 6) = 0 \leq \frac{k}{b} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\}.$$

(3) If $x \in \{1\} \cup (5, 10)$ and $y \in (1, 5]$, then we have

$$\begin{aligned} d(fx, gy) &= d(1, 6) = 25, \\ d(Sx, Ty) &= \begin{cases} (1-9)^2 = 64, & x = 1, \\ (\frac{x+1}{6}-9)^2, & x \in (5, 10), \end{cases} \\ d(fx, Sx) &= \begin{cases} (1-1)^2 = 0, & x = 1, \\ (\frac{x+1}{6}-1)^2, & x \in (5, 10), \end{cases} \\ d(gy, Ty) &= d(6, 9) = (6-9)^2 = 9, \\ d(fx, Ty) &= d(1, 9) = (9-1)^2 = 64, \end{aligned}$$

and

$$d(Sx, gy) = \begin{cases} (6-1)^2 = 16, & x = 1, \\ (6-\frac{x+1}{6})^2, & x \in (5, 10). \end{cases}$$

Therefore we obtain

$$d(fx, gy) \leq \frac{1}{2} \cdot \frac{13}{16} \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\}$$

for all $x, y \in X$. Thus all conditions of Theorem 2.1 (of course Theorem 2.2) are satisfied. In fact, 1 is a unique point in X such that $f1 = g1 = 1$. However, we can not apply Theorem 1.9 (i.e., [24, Theorem 2.1]) for f and S because $fX \not\subseteq SX$ and

$$d(f1, f5) = 25 > \frac{1}{4} \cdot 64 = \frac{1}{b^2} \max\{d(S1, S5), d(f1, S1), d(S5, f5)\}.$$

Also, we can not apply Corollary 2.8 because $fX \not\subseteq TX$ and $gX \not\subseteq SX$.

The following two examples show that Theorems 2.1 and 2.2 are independent of each other.

Example 2.12. Let X and d be defined as in Example 2.11. Then d is continuous. Define mappings f, g, S , and T on X by

$$fx = gx = \begin{cases} 1, & x = 1 \text{ or } x \in (5, 10), \\ 6, & x \in (1, 5] \end{cases}$$

and

$$Sx = Tx = \begin{cases} 1, & x = 1, \\ 7, & x \in (1, 5], \\ \frac{x+1}{6}, & x \in (5, 10). \end{cases}$$

By the similar arguments as in Example 2.11, it can be checked that all conditions of Theorem 2.2 are satisfied. However, we can not apply Theorem 2.1 because

$$d(f1, g5) = 25 > \frac{1}{2} \cdot 36 = \frac{1}{b} \max\{d(S1, T5), d(f1, S1), d(g5, T5), d(f1, T5), d(S1, g5)\}.$$

Example 2.13. Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \rightarrow [0, \infty)$ be defined by

$$d(m, n) = \begin{cases} 0, & \text{if } m = n, \\ \frac{1}{m}, & \text{if } m \text{ is even and } n = \infty, \\ \frac{1}{n}, & \text{if } n \text{ is even and } m = \infty, \\ |\frac{1}{m} - \frac{1}{n}|, & \text{if } m \text{ and } n \text{ are both even,} \\ 5, & \text{if one of } m, n \text{ is odd and the other is odd (and } m \neq n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

By [16, Theorem 1.5], we know that (X, d) is a b -metric space with $b = \frac{5}{2}$ and d is not continuous. Define mappings f, g, S , and T on X by

$$fn = gn = \begin{cases} 4n, & \text{if } n \in \mathbb{N}, \\ \infty, & \text{if } n = \infty, \end{cases}$$

and

$$Sn = Tn = n \text{ for all } n \in X.$$

Obviously, we have that $\overline{fX} = fX = \overline{gX} = gX \not\subseteq SX = TX = X$ and (f, S) and (g, T) are weakly compatible. Now we prove that (f, S) satisfies the (CLRT)-property or (g, T) satisfies the (CLRS)-property. To this end, consider a sequence $\{x_n\}$ defined by $x_n = n$. We have

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} Sx_n = \infty = T\infty \in TX \text{ and } \lim_{n \rightarrow \infty} gx_n = \lim_{n \rightarrow \infty} Tx_n = \infty = S\infty \in SX.$$

It is easy to check that mappings f, g, S , and T satisfy the condition (2.1) of Theorem 2.1 with $k = \frac{5}{6}$. Thus all conditions of Theorem 2.1 are satisfied. In fact, ∞ is a unique point in X such that $f\infty = g\infty = S\infty = T\infty = \infty$. However, we can not apply Theorem 2.2 because d is not continuous.

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