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Hyers-Ulam stability of Pielou logistic difference equation

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Abstract

We investigate Hyers-Ulam stability of the first order difference equation $x_{i+1} = \frac{ax_i+b}{cx_i+d}$, where $ad - bc = 1$, $c \neq 0$ and $|a + d| > 2$. It has Hyers-Ulam stability if the initial point x_0 lies in some definite interval of $\mathbb{R}$. The condition $|a + d| > 2$ implies that the above recurrence is a natural generalization of Pielou logistic difference equation.

Keywords: Hyers-Ulam stability, Pielou logistic difference equation, first order difference equation, linear fractional map, Verhulst-Pearl differential equation.
2010 MSC: 39A30, 39B82.

1. Introduction

The difference equation is the recursively defining a sequence, each of which term is defined as a function of the preceding terms. The difference equation often refers to a specific type of recurrence relation. In particular, if the sequence $\{x_i\}_{i\in\mathbb{N}_0}$ is defined as the relation between the general term x_i and only its first predecessor x_{i-1} with the definite initial term x_0 satisfying the equation $x_{i+1} = g(x_i)$, then it is called first order difference equation.

In 1940, Ulam [11] suggested an important problem of the stability of group homomorphisms:
Given a metric group (G, d) and a function $f : G \to G$ which satisfies the inequality $d(f(xy), f(x)f(y)) \leqslant \varepsilon$ for a positive number ε and for all $x, y \in G$, do there exist a homomorphism $a : G \to G$ and a constant $\delta > 0$ depending only on G and ε such that $d(a(x), f(x)) \leqslant \delta$ for all $x \in G$?

The first positive answer to this question was given by Hyers [3] in 1941 for Cauchy additive equation in Banach spaces.

If the answer is affirmative, the functional equation $a(xy) = a(x)a(y)$ is said to be stable in the sense of Hyers and Ulam (or the equation has the Hyers-Ulam stability). We refer the reader to [3, 4, 10, 11] for the exact definition of Hyers-Ulam stability.

For decades, theory of Hyers-Ulam stability of functional equations or linear differential equations was developed. More recently, Hyers-Ulam stability of difference equations has been given attention. For instance, see [2, 5–9]. However, this stability for difference equations is not yet studied far beyond the linear difference equation as far as we know.

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In this paper, we investigate Hyers-Ulam stability of the first order linear fractional difference equation which is motivated from the discretized function as the solution of Verhulst-Pearl differential equation.

We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{R}$, and $\mathbb{C}$ the set of all positive integers, of all nonnegative integers, of all real numbers, and the set of all complex numbers, respectively.

We would show Hyers-Ulam stability of the first order difference equation of the form

$$x_{i+1} = g(x_i), \tag{1.1}$$

for all integers $i \in \mathbb{N}_0$, where g is the linear fractional map as follows

$$g(x) = \frac{ax+b}{cx+d},$$

where a, b, c, d are real numbers with $ad - bc = 1$, $c \neq 0$ and $|a+d| > 2$. More precisely, we would prove that if a real-valued sequence $\{a_i\}_{i\in\mathbb{N}_0}$ satisfies the inequality

$$|a_{i+1} - g(a_i)| \leqslant \varepsilon,$$

for all $i \in \mathbb{N}_0$, then there exists a solution $\{b_i\}_{i\in\mathbb{N}_0}$ to the difference equation (1.1) and a positive $G(\varepsilon)$ depending only on F and ε such that

$$|b_i - a_i| \leqslant G(\varepsilon),$$

for all $i \in \mathbb{N}_0$ and $\varepsilon \to 0$ implies that $G(\varepsilon) \to 0$.

We remark that the difference equation (1.1) is a discrete form of the functional equation $x(\xi(t)) = H(t, x(t))$, whose stability results have been surveyed in [1].

The Verhulst-Pearl equation is a population growth model which is given as

$$y'(t) = y(t)\big(p - qy(t)\big),$$

for some $p, q > 0$, where y is the size of population at the time t and the positive constant p is the growth rate of population. The nonlinear term, $-qy(t)^2$ is the negative effect on the growth due to the environment. The solution is the map as follows

$$y(t) = \frac{p}{q}\frac{1}{1+\frac{1}{rq}e^{-pt}},$$

for some constant r. Thus we obtain that

$$y(t+1) = \frac{e^p y(t)}{1+\frac{q}{p}(e^p-1)y(t)}.$$

Discretizing the above equation we obtain the following recursive relation

$$y(n+1) = \frac{Ay(n)}{1+Cy(n)},$$

where $A = e^p$ and $C = \frac{q}{p}(e^p - 1)$. This equation is called Pielou logistic difference equation.

The behavior of Pielou logistic difference equation is the same as iterative property of some kind of linear fractional maps. Let the map corresponding Pielou logistic equation be as follows

$$F(x) = \frac{Ax}{Cx+1},$$

where $A > 1$ and $C > 0$.

In the sequel, we consider the matrix representation of linear fractional map, which clarifies the qualitative properties using the trace of matrix representation. For instance, the following map

$$x \mapsto \frac{ax+b}{cx+d},$$

for $ad-bc \neq 0$ has the matrix representation

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

However, since the map $x \mapsto \frac{ax+b}{cx+d}$ is the same as $x \mapsto \frac{pax+pb}{pcx+pd}$, the matrix M is a representative of any matrix pM for all real numbers $p \neq 0$. Thus we may assume that $\det M = 1$ only if $ad-bc > 0$. If $ad-bc < 0$, then we assume that $\det M = -1$. In this paper, we always fix the condition $ad-bc > 0$.

The linear fractional map F for Pielou logistic difference equation has the matrix representation as follows

$$\begin{pmatrix} \sqrt{A} & 0 \\ \frac{C}{\sqrt{A}} & \frac{1}{\sqrt{A}} \end{pmatrix}.$$

The matrix representation of F would be also denoted by F unless it makes confusion. Observe the inequality of the trace: $\mathrm{tr}(F) = \sqrt{A} + \frac{1}{\sqrt{A}} > 2$.

In this paper, we investigate Hyers-Ulam stability of linear fractional maps whose trace is greater than two. These maps generate Pielou logistic difference equation by iteration.

2. Preliminaries

Let g be the linear fractional map

$$g(x) = \frac{ax+b}{cx+d}, \tag{2.1}$$

for real numbers a, b, c and d, where $ad-bc = 1$ and $c \neq 0$.

Recall that $g\left(-\frac{d}{c}\right) = \infty$. Since Hyer-Ulam stability at ∞ is not considered in this article, a suitable proper subinterval in $\mathbb{R}$ should be chosen. The set A is called (forward) invariant under g if $g(A) \subset A$ is satisfied.

In this section, we find a subinterval of $\mathbb{R}$ invariant under g defined in (2.1) if the trace of the matrix representation of g is strictly greater than two.

Lemma 2.1. *Let g be the linear fractional map defined in (2.1). The followings are true for $x \in \mathbb{R}$ and $r > 0$.*

(i) *If $x + \frac{d}{c} > \frac{r}{|c|}$, then $-\frac{1}{r|c|} < g(x) - \frac{a}{c} < 0$;*

(ii) *If $x + \frac{d}{c} < -\frac{r}{|c|}$, then $0 < g(x) - \frac{a}{c} < \frac{1}{r|c|}$.*

Proof. Suppose firstly that $x + \frac{d}{c} > \frac{r}{|c|}$. Then we have

$$\begin{aligned} \frac{r}{|c|} < x + \frac{d}{c} < \infty \Rightarrow 0 < \frac{1}{x+\frac{d}{c}} < \frac{|c|}{r} \\ \Rightarrow -\frac{1}{r|c|} < -\frac{1}{c^2}\frac{1}{x+\frac{d}{c}} < 0 \\ \Rightarrow -\frac{1}{r|c|} < \frac{-ad+bc}{c^2x+cd} < 0 \\ \Rightarrow -\frac{1}{r|c|} < \frac{-acx-ad+acx+bc}{c(cx+d)} < 0 \end{aligned}$$

$$\Rightarrow -\frac{1}{r|c|} < -\frac{a}{c} + \frac{ax+b}{cx+d} < 0.$$

Moreover, we obtain that

$$-\infty < x + \frac{d}{c} < -\frac{r}{|c|} \quad \Rightarrow \quad 0 < \frac{ax+b}{cx+d} - \frac{a}{c} < \frac{1}{r|c|},$$

by the similar calculations. □

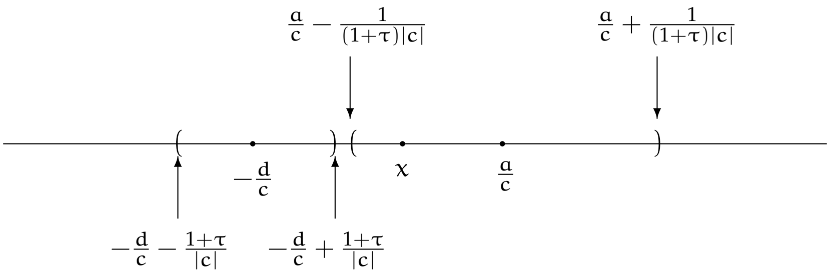
Lemma 2.2. *Let* g *be the linear fractional map defined as* (2.1)*, where* a*,* b*,* c *and* d *are real numbers,* $ad - bc = 1$ *and* $c \neq 0$*. Assume that the matrix representation* g *satisfies* $|\mathrm{tr}(g)| = 2 + \tau$ *for* $\tau > 0$*. If* $\left|x - \frac{a}{c}\right| \leqslant \frac{1}{(1+\tau)|c|}$*, then* $\frac{1+\tau}{|c|} < x + \frac{d}{c}$ *or* $x + \frac{d}{c} < -\frac{1+\tau}{|c|}$*.*

Proof. Since $|\mathrm{tr}(g)| = |a + d| > 2$, we will prove our assertion only for the case of $\frac{a}{c} \neq -\frac{d}{c}$.

Case 1: Assume that $\frac{a}{c} > -\frac{d}{c}$. Since $\frac{a+d}{c} = \frac{a}{c} - \left(-\frac{d}{c}\right) > 0$ and $\mathrm{tr}(g)$ is $a + d$, $\frac{\mathrm{tr}(g)}{c} = \frac{a+d}{c} > 0$. Then we have

$$\frac{a}{c} - \left(-\frac{d}{c}\right) = \frac{a+d}{c} = \frac{\mathrm{tr}(g)}{c} = \left|\frac{\mathrm{tr}(g)}{c}\right| = \frac{2+\tau}{|c|} > \frac{1}{|c|}\left(\frac{1}{1+\tau} + 1 + \tau\right).$$

Using this inequality, we can visualize this case in the following figure.



In view of this figure, we easily see that $x + \frac{d}{c} = x - \left(-\frac{d}{c}\right) > \frac{1+\tau}{|c|}$.

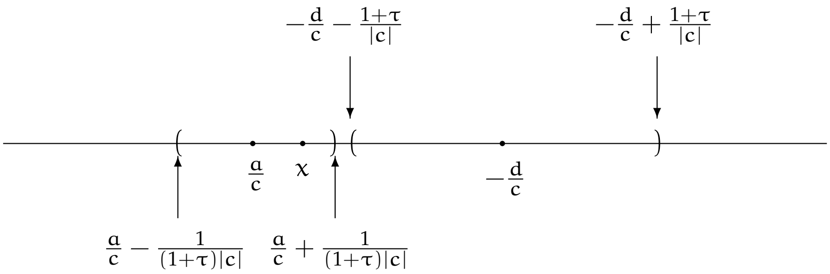
Case 2: Assume that $\frac{a}{c} < -\frac{d}{c}$. By the similar calculations of Case 1, we obtain $\frac{\mathrm{tr}(g)}{c} = \frac{a+d}{c} < 0$. Thus

$$\frac{a}{c} - \left(-\frac{d}{c}\right) = \frac{a+d}{c} = \frac{\mathrm{tr}(g)}{c} = -\left|\frac{\mathrm{tr}(g)}{c}\right| = -\frac{2+\tau}{|c|} < -\frac{1}{|c|}\left(\frac{1}{1+\tau} + 1 + \tau\right),$$

or

$$-\frac{d}{c} - \frac{a}{c} > \frac{1}{|c|}\left(\frac{1}{1+\tau} + 1 + \tau\right).$$

On account of the last inequality, we can visualize this case in the following figure.



By considering this figure, we show that $-\frac{d}{c} - x > \frac{1+\tau}{|c|}$ or $x + \frac{d}{c} < -\frac{1+\tau}{|c|}$. □

Proposition 2.3. *Let* g *be the linear fractional map defined as* (2.1)*, where* a*,* b*,* c *and* d *are real numbers,* $ad - bc = 1$ *and* $c \neq 0$*. Assume that the matrix representation* g *satisfies* $|\mathrm{tr}(g)| = 2 + \tau$*, for* $\tau > 0$*. Then* g *maps the interval* $\left\{x \in \mathbb{R} : \left|x + \frac{d}{c}\right| > \frac{1+\tau}{|c|}\right\}$ *into itself.*

Proof. Denote the following two intervals

$$\left\{x \in \mathbb{R} : \left|x + \frac{d}{c}\right| > \frac{1+\tau}{|c|}\right\}, \quad \left\{x \in \mathbb{R} : \left|x - \frac{a}{c}\right| \leqslant \frac{1}{(1+\tau)|c|}\right\}, \tag{2.2}$$

by S_τ and by T_τ, respectively. Then Lemma 2.2 implies that $T_\tau \subset S_\tau$ and Lemma 2.1 implies that $g(S_\tau) \subset T_\tau$. Hence, the interval S_τ is invariant under g as follows:

$$g(S_\tau) \subset T_\tau \subset S_\tau, \tag{2.3}$$

which completes the proof. □

3. Hyers-Ulam stability

Suppose that a real-valued sequence $\{a_n\}_{n\in\mathbb{N}_0}$ satisfies the inequality

$$|a_{i+1} - F(i, a_i)| \leqslant \varepsilon,$$

for a positive number ε and for all $i \in \mathbb{N}_0$, where $|\cdot|$ is the absolute value of real number. If there exists the sequence $\{b_i\}_{i\in\mathbb{N}_0}$ which satisfies that

$$b_{i+1} = F(i, b_i), \tag{3.1}$$

for each $i \in \mathbb{N}_0$, and $|a_i - b_i| \leqslant G(\varepsilon)$ for all $i \in \mathbb{N}_0$, where the positive number $G(\varepsilon)$ converges to zero as $\varepsilon \to 0$, then we say that the difference equation (3.1) has Hyers-Ulam stability.

Theorem 3.1. *Let* g *be linear fractional map defined as* (2.1) *of which matrix representation satisfies that* $|\mathrm{tr}(g)| = 2 + \tau$ *for* $\tau > 0$*. For any given* $0 < \varepsilon < \frac{\tau}{(1+\tau)|c|}$*, let the real-valued sequence* $\{a_i\}_{i\in\mathbb{N}_0}$ *satisfy the inequality*

$$|a_{i+1} - g(a_i)| \leqslant \varepsilon,$$

for all $i \in \mathbb{N}_0$*. If* a_0 *is in the interval* $S_\tau = \left\{x \in \mathbb{R} : \left|x + \frac{d}{c}\right| > \frac{1+\tau}{|c|}\right\}$*, then there exists a sequence* $\{b_i\}_{i\in\mathbb{N}_0}$ *which satisfies*

$$b_{i+1} = g(b_i),$$

and

$$|b_i - a_i| \leqslant \frac{1}{(1+\tau)^{2i}}|b_0 - a_0| + \sum_{j=0}^{i-1} \frac{\varepsilon}{(1+\tau)^{2j}},$$

for each $i \in \mathbb{N}_0$*.*

Proof. First, we claim that $a_n \in S_\tau$ for all $n \in \mathbb{N}_0$. Recall that $\mathbb{R} \setminus S_\tau$ and T_τ are bounded disjoint intervals because $g(S_\tau) \subset T_\tau \subset S_\tau$ by Lemma 2.2 and Proposition 2.3.

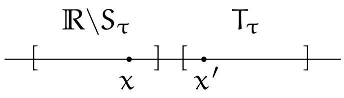
Let $x \in \mathbb{R} \setminus S_\tau$ and $x' \in T_\tau$. From the definitions of S_τ and T_τ, we have

$$-\frac{d}{c} - \frac{1+\tau}{|c|} \leqslant x \leqslant -\frac{d}{c} + \frac{1+\tau}{|c|}, \tag{3.2}$$

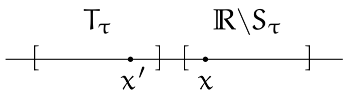
and

$$\frac{a}{c} - \frac{1}{(1+\tau)|c|} \leqslant x' \leqslant \frac{a}{c} + \frac{1}{(1+\tau)|c|}. \tag{3.3}$$

There are only two cases for the location of the bounded disjoint intervals $\mathbb{R} \setminus S_\tau$ and T_τ as we see in the following figure.



or



According to (3.2) and the first figure, we get

$$x \leqslant -\frac{d}{c} + \frac{1+\tau}{|c|} \leqslant \frac{a}{c} - \frac{1}{(1+\tau)|c|} \leqslant x',$$

or by (3.3) and the second figure, we have

$$x' \leqslant \frac{a}{c} + \frac{1}{(1+\tau)|c|} \leqslant -\frac{d}{c} - \frac{1+\tau}{|c|} \leqslant x.$$

Hence, we obtain

$$\begin{aligned} |x - x'| &\geqslant \left|\frac{a+d}{c}\right| - \left(\frac{1+\tau}{|c|} + \frac{1}{(1+\tau)|c|}\right) \\ &= \frac{2+\tau}{|c|} - \frac{1}{|c|}\left(1+\tau+\frac{1}{1+\tau}\right) \\ &= \frac{1}{|c|}\left(1 - \frac{1}{1+\tau}\right) \\ &= \frac{\tau}{(1+\tau)|c|} \\ &> \varepsilon. \end{aligned} \tag{3.4}$$

Since $a_0 \in S_\tau$, $g(a_0) \in T_\tau$ by Lemma 2.1 or (2.3). Moreover, $|g(a_0) - a_1| \leqslant \varepsilon$. Thus, by (3.4), $a_1 \notin \mathbb{R} \setminus S_\tau$, that is, $a_1 \in S_\tau$. Then, by induction, we can show that $a_n \in S_\tau$ for all $n \in \mathbb{N}_0$. Since $g'(x) = \frac{1}{(cx+d)^2}$, $|g'|$ has a uniform upper bound in S_τ as follows

$$|g'(x)| = \frac{1}{|cx+d|^2} = \frac{1}{c^2\left|x + \frac{d}{c}\right|^2} < \frac{1}{(1+\tau)^2} < 1.$$

Thus, g is a Lipschitz map on S_τ with the Lipschitz constant $\frac{1}{(1+\tau)^2}$.

Finally, we can easily apply induction to prove

$$\begin{aligned} |b_i - a_i| &= |g(b_{i-1}) - g(a_{i-1}) + g(a_{i-1}) - a_i| \\ &\leqslant |g(b_{i-1}) - g(a_{i-1})| + |g(a_{i-1}) - a_i| \\ &\leqslant \frac{1}{(1+\tau)^2}|b_{i-1} - a_{i-1}| + \varepsilon \\ &\vdots \\ &\leqslant \frac{1}{(1+\tau)^{2i}}|b_0 - a_0| + \sum_{j=0}^{i-1} \frac{\varepsilon}{(1+\tau)^{2j}}, \end{aligned}$$

for each $i \in \mathbb{N}_0$. □

4. Application

The Pielou logistic difference equation can be treated as the iteration of the linear fractional map

$$F(x) = \frac{Ax}{Cx+1} = \frac{\sqrt{A}x}{\frac{C}{\sqrt{A}}x + \frac{1}{\sqrt{A}}},$$

for $A > 1$ and $C > 0$. (The last expression of F is given in the form of $ad - bc = 1$.) Then $\mathrm{tr}(F) = \sqrt{A} + \frac{1}{\sqrt{A}}$.

The invariant set S_τ under F is as follows:

$$S_\tau = \left\{ x \in \mathbb{R} : \left| x + \frac{1}{C} \right| > \frac{A + 1 - \sqrt{A}}{C} \right\}, \tag{4.1}$$

where $\tau = \sqrt{A} + \frac{1}{\sqrt{A}} - 2 > 0$ by (2.2). Then the Pielou logistic difference equation has Hyers-Ulam stability.

Example 4.1. Let F be the linear fractional map as follows

$$F(x) = \frac{Ax}{Cx + 1},$$

for $A > 1$ and $C > 0$. For every given $0 < \varepsilon < \frac{A\sqrt{A} - 2A + \sqrt{A}}{(A - \sqrt{A} + 1)C}$, let a sequence $\{a_i\}_{i \in \mathbb{N}_0}$ satisfy the inequality

$$|a_{i+1} - F(a_i)| \leqslant \varepsilon,$$

for all $i \in \mathbb{N}_0$. If a_0 is in S_τ defined in (4.1), then there exists a sequence $\{b_i\}_{i \in \mathbb{N}_0}$ which satisfies

$$b_{i+1} = F(b_i),$$

and

$$|a_i - b_i| \leqslant \frac{1}{(1+\tau)^{2i}} |b_0 - a_0| + \sum_{j=0}^{i-1} \frac{\varepsilon}{(1+\tau)^{2j}},$$

for each $i \in \mathbb{N}_0$, where $\tau = \sqrt{A} + \frac{1}{\sqrt{A}} - 2 > 0$.

Proof. By the direct calculation, we obtain

$$\frac{\tau}{(1+\tau)\frac{C}{\sqrt{A}}} = \frac{A\sqrt{A} - 2A + \sqrt{A}}{(A - \sqrt{A} + 1)C}.$$

Then Theorem 3.1 implies that if the inequality $0 < \varepsilon < \frac{A\sqrt{A} - 2A + \sqrt{A}}{(A - \sqrt{A} + 1)C}$ holds and $a_0 \in S_\tau$, then there exists a sequence $\{b_i\}_{i \in \mathbb{N}_0}$ such that

$$b_{i+1} = F(b_i),$$

and

$$|a_i - b_i| \leqslant \frac{1}{(1+\tau)^{2i}} |b_0 - a_0| + \sum_{j=0}^{i-1} \frac{\varepsilon}{(1+\tau)^{2j}},$$

for each $i \in \mathbb{N}_0$, which completes the proof. □

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