Discussion on a coupled fixed point theorem for single-valued operators in b-metric spaces

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Abstract

In this note, an existence and uniqueness theorem of fixed points for single-valued mappings in partially ordered b-metric spaces is established. As a corollary, the contraction constant for a coupled fixed point theorem obtained in a recent paper is relaxed from $[0, \frac{1}{s})$ to $[0, 1)$. Furthermore, a system of integral equation is also discussed. ©2017 All rights reserved.

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1. Introduction

In recent years, there was much attention focused on contractive operators in b-metric spaces. Czerwik [4, 5] introduced the concept of a b-metric space.

Definition 1.1 ([4, 5]). Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \rightarrow \mathbb{R}_+$ is said to be a b-metric if the following conditions are satisfied:

- (i) if $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;
- (iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

A pair (X, d) is called a b-metric space.

Since then, several authors have established the existence and uniqueness theorems of fixed points for single-valued operators in b-metric spaces [2, 3, 6, 11, 13, 15].

On the other hand, there has been growing interest in coupled fixed points for mixed monotone operators. It is well-known that the notion of a coupled fixed point was introduced and studied by

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Opoicev [14] and then by Guo and Lakshmikantham [9]. Their study has not only directly unified the results of increase and decrease operators, but also wide applications in engineering, nuclear physics, biological chemistry technology, etc. [1, 7, 8, 10]. In partially ordered spaces, we usually suppose the concave-convex type conditions or contraction conditions to study the existence and uniqueness of fixed points of mixed monotone operators without compact condition [10, 12, 16–18]. Very recently, Bota et al. [2] proved some coupled fixed point theorems for mixed monotone operators in complete b-metric spaces. The tool was based on the iterative construction of a Cauchy successive approximations sequence. Their main result is the following.

Theorem 1.2 ([2]). *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \times X \rightarrow X$ a continuous mapping with the mixed monotone property on $X \times X$. Assume that the following conditions are satisfied:*

(i) *there exists $k \in [0, \frac{1}{s})$ such that*

$$d(T(x, y), T(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall x \geq u, y \leq v;$$

(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$.*

Then there exist $x, y \in X$ such that $x = T(x, y)$ and $y = T(y, x)$.

Question 1.3. Does the conclusion of Theorem 1.2 remain true for every $k \in [\frac{1}{s}, 1)$?

The purpose of this paper is to present some fixed point theorems for single-valued mappings in partially ordered b-metric spaces, and gives a positive answer to Question 1.3. In essence, our results do not depend on the triangle inequality. Our ideas come from [4, 6, 11]. Furthermore, a system of integral equation which is the same as Theorem 4.1 in [2] is discussed. Finally, we point out that Samet et al. [18] and Roldan et al. [17] have showed that coupled and multi-dimensional fixed point theorems can be obtained as easy consequences of fixed point results in dimension one in the setup of partially metric spaces. Therefore, our results can be applied directly to the coupled fixed points of mixed monotone operators and multi-dimensional fixed points theorems [16].

2. Preliminaries

Before presenting our results, we collect relevant definitions and results which will be needed in the proof of our main results. Denote with $\mathbb{N}$ the set of positive integers.

Let $(X, \leq)$ be a partially ordered set equipped with a metric d and $F : X \times X \rightarrow X$ be a given mapping. We endow the product set $X \times X$ with the following partial order:

$$(x_1, x_2), (y_1, y_2) \in X \times X, \quad (x_1, x_2) \leq_2 (y_1, y_2) \iff x_1 \leq y_1, \quad x_2 \geq y_2.$$

Definition 2.1. F is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and is monotone non-increasing in y , that is, for every $x, y \in X$,

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \implies F(x_1, y) \leq F(x_2, y);$$

$$y_1, y_2 \in X, \quad y_1 \geq y_2 \implies F(x, y_1) \leq F(x, y_2).$$

It is easy to show that the mappings $\eta, \delta : X^2 \times X^2 \rightarrow [0, +\infty)$ defined by

$$\eta((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2);$$

$$\delta((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\},$$

for all $(x_1, x_2), (y_1, y_2) \in X \times X$, are metrics on $X \times X$.

Now, define the mapping $T : X^2 \times X^2$ by

$$T(x, y) = (F(x, y), F(y, x)), \quad \forall (x, y) \in X \times X.$$

Lemma 2.2 ([18]). *The following properties hold:*

- (a) (X, d) is complete if and only if (X^2, η) and (X^2, δ) are complete;
- (b) F has the mixed monotone property if and only if T is monotone non-decreasing with respect to $\leq_2$;
- (c) $(x, y) \in X \times X$ is a coupled fixed point of F if and only if (x, y) is a fixed point of T .

3. Main results

Theorem 3.1. *Let X be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete b -metric space with constant $s \geq 1$. Let $T : X \rightarrow X$ be a nondecreasing mapping such that*

$$d(Tx, Ty) \leq \phi(d(x, y)), \quad \forall x \geq y, \quad (3.1)$$

for each $x, y \in X$, where $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is increasing and satisfies

$$\lim_{n \rightarrow \infty} \phi^n(t) = 0, \quad \forall t > 0.$$

Furthermore, the following conditions are satisfied:

- (H₁) If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x_n \leq x$ for all $n \in \mathbb{N}$;
- (H₂) for $x, y \in X$ there exists $z \in X$ which is comparable to x and y .

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$. Then T has a unique fixed point $x^* \in X$, and $\lim_{n \rightarrow \infty} T^n(x) = x^*$ for each $x \in X$.

Proof. Since T is a non-decreasing function, we obtain by induction that

$$x_0 \leq Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \dots \leq T^nx_0 \leq \dots$$

Now, let $x \in X$ and let $\epsilon > 0$ be arbitrary. Choose $n_0 \in \mathbb{N}$ so that $\phi^{n_0}(\epsilon) < \frac{\epsilon}{2s}$. Put $x_{m+1} = (T^{n_0})x_m$, $m = 0, 1, 2, \dots$. Then, from (3.1) and, as the $T^{n_0m+n_0-i}x_0$ and $T^{n_0m-i}x_0$ ($i = 1, 2, \dots, n_0m$) are comparable, we obtain

$$\begin{aligned} d(x_{m+1}, x_m) &= d((T^{n_0})^{m+1}x_0, (T^{n_0})^mx_0) \\ &\leq \phi(d(T^{n_0m+n_0-1}x_0, T^{n_0m-1}x_0)) \\ &\leq \phi(\phi(d(T^{n_0m+n_0-2}x_0, T^{n_0m-2}x_0))) \\ &= \phi^2(d(T^{n_0m+n_0-2}x_0, T^{n_0m-2}x_0)) \\ &\leq \dots \\ &\leq \phi^{n_0}(d(T^{n_0m}x_0, T^{n_0m-n_0}x_0)) \\ &\leq \phi^{n_0}(\phi(d(T^{n_0m-1}x_0, T^{n_0m-n_0-1}x_0))) \\ &\leq \phi^{n_0}(\phi^2(d(T^{n_0m-2}x_0, T^{n_0m-n_0-2}x_0))) \\ &\leq \dots \\ &\leq \phi^{n_0}(\phi^{n_0m-n_0}(d(T^{n_0}x_0, T^0x_0))) \\ &= \phi^{n_0m}(d(T^{n_0}x_0, x_0)). \end{aligned}$$

So $\lim_{m \rightarrow \infty} d(x_{m+1}, x_m) = 0$.

Next, choose $M \in \mathbb{N}$ so that $d(x_{m+1}, x_m) < \frac{\epsilon}{2s}$ for $m \geq M$. In the following, we claim that if

$d(u, x_M) \leq \epsilon$, then $d(T^{n_0}u, x_M) \leq \epsilon$ with $u \geq x_M$. It follows from (3.1) that

$$d(T^{n_0}u, T^{n_0}x_M) \leq \phi^{n_0}(d(u, x_M)) \leq \phi^{n_0}(\epsilon) < \frac{\epsilon}{2s},$$

and

$$d(T^{n_0}x_M, x_M) < \frac{\epsilon}{2s}.$$

By the related triangle inequality

$$d(T^{n_0}u, x_M) \leq s[d(T^{n_0}u, T^{n_0}x_M) + d(T^{n_0}x_M, x_M)] < s\left(\frac{\epsilon}{2s} + \frac{\epsilon}{2s}\right) = \epsilon.$$

Note that $d(x_{M+1}, x_M) \leq \epsilon$. Our claim implies that $d(T^{n_0}x_{M+1}, x_M) \leq \epsilon$, i.e., $d(x_{M+2}, x_M) \leq \epsilon$, and $d(T^{n_0}x_{M+2}, x_M) \leq \epsilon$, i.e., $d(x_{M+3}, x_M) \leq \epsilon$. Continuing this process, we can deduce that $d(x_{M+i}, x_M) \leq \epsilon$ ($i = 1, 2, \dots$). Therefore, for $j, t \geq m$,

$$d(x_t, x_j) \leq s[d(x_t, x_M) + d(x_M, x_j)] \leq 2s\epsilon.$$

This proves that $\{x_n\}$ is a Cauchy sequence. So there exists $x^* \in X$ such that $\lim_{m \rightarrow \infty} x_m = x^*$. From condition (H_1) , we have $x_m \leq x^*$. Again combining (3.1), we have

$$d(T^{n_0}(x^*), T^{n_0}(x_m)) \leq \phi^{n_0}(d(x^*, x_m)) \rightarrow 0 (m \rightarrow \infty).$$

Thus $d(T^{n_0}(x^*), x^*) = 0$, i.e., $T^{n_0}(x^*) = x^*$. The increasing property of T tells us that $T(x^*) = x^*$. This completes the proof of Theorem 3.1. $\square$

In the same arguments as the proof of Theorem 2.1 and 2.4 in [18], and combining Lemma 2.2, we can obtain the following corollary.

Corollary 3.2. *Let (X, d) be a complete b-metric space with $s \geq 1$ and $T : X \times X \rightarrow X$ a mixed monotone operator on $X \times X$. Assume that the following conditions are satisfied:*

(i) *there exists $k \in [0, 1)$ such that*

$$d(T(x, y), T(u, v)) \leq \frac{k}{2}[d(x, u) + d(y, v)], \quad \forall x \geq u, y \leq v,$$

or

$$d(T(x, y), T(u, v)) \leq k \max\{d(x, u), d(y, v)\}, \quad \forall x \geq u, y \leq v;$$

(ii) *X has the following properties:*

(X₁) *if a non-decreasing sequence $\{x_n\}$ in X converges to some point $x \in X$, then $x_n \leq x$ for all $n \in \mathbb{N}$,*

(X₂) *if a non-increasing sequence $\{y_n\}$ in X converges to some point $y \in X$, then $y_n \geq y$ for all $n \in \mathbb{N}$;*

(iii) *there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$.*

Then T has a coupled fixed point $(x^*, y^*) \in X \times X$. Moreover, if for all $(x, y), (u, v) \in X \times X$, there exists $(z_1, z_2) \in X \times X$ such that $(x, y) \leq_2 (z_1, z_2)$ and $(u, v) \leq_2 (z_1, z_2)$, we have the uniqueness of the coupled fixed point and $x^* = y^*$.

4. An application

In order to compare our results to the ones in [2], we shall consider the same integral equation, that is,

$$\begin{cases} u(t) = h(t) + \int_0^T k(s, t)f(s, u(s), v(s))ds, \\ v(t) = h(t) + \int_0^T k(s, t)f(s, v(s), u(s))ds, \end{cases} \quad (4.1)$$

where $t \in [0, T]$.

Theorem 4.1. Consider the problem (4.1) with $h : [0, T] \rightarrow \mathbb{R}$ continuous and assume that the following conditions are satisfied

- (i) $f : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous and $k : [0, T] \times [0, T] \rightarrow \mathbb{R}_+$ is integrable with respect to the first variable;
- (ii) $f(s, \cdot, \cdot)$ has the mixed monotone property with respect to the last two variable for all $s \in [0, T]$;
- (iii) there exist $\varphi, \psi : [0, T] \rightarrow \mathbb{R}_+$ in $L^1[0, T]$ such that for every $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq y_1$ and $x_2 \geq y_2$ (or reversely), we have

$$|f(s, x_1, x_2) - f(s, y_1, y_2)| \leq \varphi(s)|x_1 - y_1| + \psi(s)|x_2 - y_2| \quad \forall s \in [0, T];$$

$$(iv) \max_{t \in [0, T]} \left(\int_0^T k(s, t) \varphi(s) ds \right)^2 + \max_{t \in [0, T]} \left(\int_0^T k(s, t) \psi(s) ds \right)^2 < \frac{1}{2};$$

- (v) there exist $u_0, v_0 \in C[0, T]$ such that

$$\begin{cases} u_0(t) \leq h(t) + \int_0^T k(s, t) f(s, u_0(s), v_0(s)) ds, \\ v_0(t) \geq h(t) + \int_0^T k(s, t) f(s, v_0(s), u_0(s)) ds, \end{cases}$$

for all $t \in [0, T]$.

Then there exists a unique solution (x^*, y^*) of the system (4.1).

Proof. We consider the following b-metric on X :

$$d(x, y) := \max_{t \in [0, T]} (x(t) - y(t))^2.$$

Notice that d is a b-metric with constant $s = 2$.

Define the operator $F : X \times X \rightarrow X$ by

$$F(u, v)(t) = h(t) + \int_0^T k(s, t) f(s, u(s), v(s)) ds, \quad \forall t \in [0, T].$$

Then problem (4.1) is equivalent to find (x^*, y^*) that is a coupled fixed point of F .

For all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$, we have

$$\begin{aligned} |F(x, y)(t) - F(u, v)(t)|^2 &\leq 2 \max_{t \in [0, T]} \left(\int_0^T k(s, t) \varphi(s) ds \right)^2 \max_{t \in [0, T]} (x(t) - u(t))^2 \\ &\quad + 2 \max_{t \in [0, T]} \left(\int_0^T k(s, t) \psi(s) ds \right)^2 \max_{t \in [0, T]} (y(t) - v(t))^2. \end{aligned}$$

Thus, taking the maximum over $t \in [0, T]$, we obtain

$$d(F(x, y), F(u, v)) \leq c_1 d(x, u) + c_2 d(y, v) \leq (c_1 + c_2) \max\{d(x, u), d(y, v)\},$$

where $c_1 := 2 \max_{t \in [0, T]} \left(\int_0^T k(s, t) \varphi(s) ds \right)^2$ and $c_2 := 2 \max_{t \in [0, T]} \left(\int_0^T k(s, t) \psi(s) ds \right)^2$. Since $c_1 + c_2 < 1$, we see that all the assumptions of Corollary 3.2 are satisfied and the conclusion follows. $\square$

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