Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 3099–3108
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

On a non-autonomous stochastic Lotka-Volterra competitive system

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Communicated by A. Atangana

Abstract

In this paper, we consider a general non-autonomous Lotka-Volterra competitive model with random perturbations. Sufficient conditions for stochastic permanence and extinction are established. Particularly, when these conditions are applied to a stochastic logistic equation, these conditions are sufficient and necessary. Some figures are also worked out to illustrate the main results. Some recent results are extended. Moreover, our results reveal that different types of stochastic noises have different effects on the permanence and extinction of the population.

Keywords: Competitive system, random perturbations, permanence, extinction.
2010 MSC: 92D25, 60H10, 60H30.

1. Introduction

In the natural world, it is a usual phenomena that several species compete for the limited resources. Therefore it is important to study the multi-species competitive models. A famous non-autonomous Lotka-Volterra competitive system can be expressed as follows

$$\frac{dx_i(t)}{dt} = x_i(t)[r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)], \quad i = 1, ..., n, \tag{1.1}$$

where $x_i(t)$ is the size of the ith population at time t, $r_i(t)$ is the growth rate of the ith species at time t, $a_{ii}(t) > 0$ is the introspecific competition rate, and $a_{ij}(t) > 0$ is the interspecific competition rate, $i, j = 1, 2, ..., n,\ i \neq j$. Owing to its theoretical and practical significance, system (1.1) has been extensively investigated and many important properties of the global dynamics of solutions have been obtained. For example, persistence and extinction of (1.1) were considered in [1, 2, 35, 37]. Zhao et al. [38] investigated permanence and global attractivity of model (1.1). Model (1.1) with time delay was analyzed in [13, 17]. [3, 15] studied the effect of impulses on model (1.1). Model (1.1) with stage structure was considered by [24].

On the other hand, in the real world, population systems are inevitably affected by environmental noises. Then it is important to study stochastic population systems to reveal the effect of random noise

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doi:10.22436/jnsa.010.06.24

Received 2017-03-06

on the dynamics of the system. In fact, stochastic population models have been received great attention recently, see e.g. [4, 6–12, 16, 18–23, 25–30, 32, 33, 36, 39, 40]. Particularly, under the assumption that the growth rate $r_i(t)$ is affected by random noise, with

$$r_i(t) \to r_i(t) + \beta_i(t)\dot{W}_1(t),$$

where $\dot{W}_1(t)$ stands for the white noise, Li and Mao [18] proposed and investigated the following stochastic competitive model

$$dx_i(t) = x_i(t)\left(r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right)dt + \beta_i(t)x_i(t)dW_1(t),\ i = 1, ..., n, \tag{1.2}$$

where $W_1(t)$ is a standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The authors [18] investigated stochastic permanence, extinction, and global attractivity of model (1.2). From then on, model (1.2) and its generalizations have been investigated by many authors. Bao et al. [7] investigated model (1.2) with Lévy jumps. Jiang et al. [16] considered persistence and extinction of model (1.2) in autonomous case.

However, model (1.2) is based on the assumption that only the growth rates r_i are affected by the stochastic noise. Then an important and interesting topic arises naturally: what happens if both r_i and a_{ij} are perturbed by the stochastic noise? As a matter of fact, Bandyopadhyay and Chattopadhyay [6] has pointed out that if the parameters in the models are assumed to be deterministic irrespective of environmental fluctuations, there would be some limitations in mathematical modeling of ecological systems, at the same time, it would be difficult to fit data perfectly and to predict the future dynamics of the system accurately. May [34] has claimed that due to environmental noise, the birth rate, competition coefficients, and other parameters in the system should be stochastic. Motivated by these, in this paper, we assume that a_{ij} is also perturbed stochastic noise, with

$$-a_{ij}(t) + \alpha_{ij}(t)\dot{W}_2(t).$$

Then model (1.2) becomes

$$\begin{aligned} dx_i(t) &= x_i(t)\left(r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\right)dt \\ &\quad + \beta_i(t)x_i(t)dW_1(t) + \sum_{j=1}^{n} \alpha_{ij}(t)x_i(t)x_j(t)dW_2(t),\ i = 1, ..., n, \end{aligned} \tag{1.3}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, P)$; $r_i(t)$, $a_{ij}(t)$, $\beta_i(t)$, and $\alpha_{ij}(t)$ are continuous and bounded functions on $[0, +\infty)$ and $a_{ij}(t) \geqslant 0$ for $i, j = 1, ..., n$.

In the investigation of population models, permanence and extinction are two important topics. However, as far as we know, no results related to permanence and extinction of model (1.3) have been reported. The aim of this paper is to study these problems. We will show that when the noise is small, the population system is stochastically permanent. At the same time, we will prove that a sufficiently large noise will force all the populations become extinct. In particular, we shall establish the sufficient and necessary conditions for stochastic permanence and extinction to a general stochastic logistic equation. Some recent results will be generalized. Moreover, we will show that a different type of stochastic noise has a different effect on the permanence and extinction of the population.

2. Main results

For the sake of simplicity, define

$$R_+^n = \{a = (a_1, .., a_n) : a_i > 0,\ 1 \leqslant i \leqslant n\}, \quad f^u = \sup_{t \geqslant 0} f(t), \quad f^l = \inf_{t \geqslant 0} f(t).$$

Lemma 2.1. *If* $\min_{1\leqslant i,j\leqslant n} a_{ij}^l > 0$, *then for any given initial value* $x(0) \in R_+^n$, *there is a unique solution* $x(t)$ *to* (1.3) *on* $t \geqslant 0$ *and the solution will remain in* R_+^n *almost surely (a.s.).*

Proof. The proof is a slight modification of in [18] and therefore is omitted. □

From now on, we always suppose that $\min_{1\leqslant i,j\leqslant n} a_{ij}^l > 0$.

Definition 2.2 ([18]). If for all $\varepsilon \in (0,1)$, there exists a pair of positive constants $\xi_1 = \xi_1(\varepsilon)$ and $\xi_2 = \xi_2(\varepsilon)$ such that for any initial data $x(0) \in R_+^n$, the solution satisfies

$$\liminf_{t\to+\infty} P\Big\{|x(t)| \geqslant \xi_1\Big\} \geqslant 1-\varepsilon, \ \ \liminf_{t\to+\infty} P\Big\{|x(t)| \leqslant \xi_2\Big\} \geqslant 1-\varepsilon,$$

then model (1.3) is said to be stochastically permanent.

Lemma 2.3. *Suppose that* $x(t)$ *is an arbitrary solution of* (1.3), *then for every* $0 < p < 1$, $i = 1, 2\ldots, n$, *there is a constant* $K = K(p) > 0$ *such that*

$$\limsup_{t\to+\infty} E\Big[x_i^p(t)\Big] \leqslant K(p), \ \ t \geqslant 0. \tag{2.1}$$

Proof. The proof is standard but for the completeness of the paper we only give a brief one. Define

$$V(x) = \sum_{i=1}^n e^t x_i^p$$

for $x \in R_+^n$, where $p < 1$. Applying Itô's formula ([31]) results in

$$\begin{aligned}
dV(x) &= e^t \sum_{i=1}^n x_i^p dt + e^t \sum_{i=1}^n \Big\{p x_i^{p-1} dx_i + 0.5p(p-1)x_i^{p-2}(dx_i)^2\Big\} \\
&= e^t p \sum_{i=1}^n x_i^p \Big[1/p + r_i(t) - \sum_{j=1}^n a_{ij}(t)x_j + 0.5(p-1)\beta_i^2(t) \\
&\quad + 0.5(p-1)\Big(\sum_{j=1}^n \alpha_{ij}(t)x_j\Big)^2\Big] dt + e^t p \sum_{i=1}^n \beta_i(t)x_i^p dW_1(t) + e^t p \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}(t)x_i^p x_j dW_2(t). \\
&\leqslant e^t p \sum_{i=1}^n x_i^p \Big[1/p + r_i(t) - a_{ii}(t)x_i\Big] dt + e^t p \sum_{i=1}^n \beta_i(t)x_i^p dW_1(t) + e^t p \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}(t)x_i^p x_j dW_2(t). \\
&\leqslant e^t \sum_{i=1}^n K_i(p) dt + e^t p \sum_{i=1}^n \beta_i(t)x_i^p dW_1(t) + e^t p \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij}(t)x_i^p x_j dW_2(t),
\end{aligned}$$

where

$$K_i(p) = \Big(\frac{p}{p+1}\Big)^{p+1} \Big(\frac{1/p + |r_i|^u}{a_{ii}^l}\Big)^p.$$

In other words, we have shown that

$$e^t E\Big[\sum_{i=1}^n x_i^p(t)\Big] \leqslant \sum_{i=1}^n x_i^p(0) + E\int_0^t e^s \sum_{i=1}^n K_i(p) ds = \sum_{i=1}^n x_i^p(0) + \sum_{i=1}^n K_i(p)(e^t - 1).$$

Consequently

$$\limsup_{t\to+\infty} E\Big[\sum_{i=1}^n x_i^p(t)\Big] \leqslant \sum_{i=1}^n K_i(p) =: K(p).$$

This completes the proof. □

Now, let us impose an assumption.

(H1): $\min_{1\leqslant i\leqslant n} b_i^l > 0$, where $b_i(t) = r_i(t) - 0.5\beta_i^2(t)$, $1 \leqslant i \leqslant n$. That is to say, the intensities of the stochastic noises in the system are not too large.

Theorem 2.4. *If* **(H1)** *is satisfied, then* (1.3) *is stochastically permanent.*

Proof. Let

$$U(x) = \sum_{i=1}^{n} x_i, \quad V_1(x) = 1/U^2(x)$$

for $x \in R_+^n$. It then follows from Itô's formula that

$$dV_1(x) = -\frac{2}{U^3(x)}\sum_{i=1}^{n} x_i\Big(r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j\Big)dt + \frac{3}{U^4(x)}\Big[\Big(\sum_{i=1}^{n}\beta_i(t)x_i\Big)^2 + \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2(t)x_ix_j\Big)^2\Big]dt$$
$$-\frac{2}{U^3(x)}\Big[\sum_{i=1}^{n}\beta_i(t)x_i dW_1(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)x_ix_j dW_2(t)\Big].$$

By **(H1)**, there is a positive constant λ satisfying

$$\min_{1\leqslant i\leqslant n} b_i^l > \lambda \max_{1\leqslant i\leqslant n}(\beta_i^2)^u.$$

Let

$$V_2(x) = (1 + V_1(x))^\lambda, \quad x \in R_+^n.$$

Then

$$dV_2(x) = LV_2(x)dt - \lambda(1+V_1(x))^{\lambda-1}\frac{2}{U^3(x)} \times \Big[\sum_{i=1}^{n}\beta_i(t)x_i dW_1(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)x_ix_j dW_2(t)\Big],$$

where

$$\begin{aligned}
LV_2(x) = {} & \lambda(1+V_1(x))^{\lambda-2}\Big\{-(1+V_1(x))\frac{2}{U^3(x)}\sum_{i=1}^{n}x_i\Big(r_i(t) - \sum_{j=1}^{n}a_{ij}(t)x_j\Big)\\
& + (1+V_1(x))\frac{3}{U^4(x)}\Big[\sum_{i=1}^{n}\beta_i^2(t)x_i^2 + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}^2(t)x_i^2x_j^2\Big]\\
& + \frac{\lambda-1}{2}\frac{4}{U^6(x)}\Big[\Big(\sum_{i=1}^{n}\beta_i(t)x_i\Big)^2 + \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)x_ix_j\Big)^2\Big]\Big\}\\
= {} & \lambda(1+V_1(x))^{\lambda-2}\Big\{-\frac{2}{U^3(x)}\sum_{i=1}^{n}x_ir_i(t) - \frac{2}{U^5(x)}\sum_{i=1}^{n}x_ir_i(t)\\
& + \frac{2}{U^3(x)}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(t)x_ix_j + \frac{2}{U^5(x)}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}(t)x_ix_j\\
& + \Big[\frac{3}{U^4(x)} + \frac{3}{U^6(x)} + \frac{2(\lambda-1)}{U^6(x)}\Big]\Big[\Big(\sum_{i=1}^{n}\beta_i(t)x_i\Big)^2 + \Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(t)x_ix_j\Big)^2\Big]\Big\}\\
\leqslant {} & \lambda(1+V_1(x))^{\lambda-2}\Big\{-\frac{2}{U^6(x)}\sum_{i=1}^{n}x_i^2r_i(t) + \frac{\max_{1\leqslant i,j\leqslant n} a_{ij}^u}{U(x)} + \frac{\max_{1\leqslant i,j\leqslant n} a_{ij}^u}{U^3(x)}
\end{aligned}$$

$$+\frac{3\max\limits_{1\leqslant i\leqslant n}(\beta_i^2)^u}{U^2(x)}+3\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u+\frac{2\lambda+1}{U^6(x)}\sum_{i=1}^n\beta_i^2(t)x_i^2+\frac{(2\lambda+1)\max\limits_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u}{U^2(x)}\Bigg\}$$

$$\begin{aligned}&=\lambda(1+V_1(x))^{\lambda-2}\Bigg\{-\frac{2}{U^6(x)}\sum_{i=1}^n x_i^2\Big[b_i(t)-\lambda\beta_i^2(t)\Big]+V_1^{1.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u\\&\quad+V_1(x)\Big[3\max_{1\leqslant i\leqslant n}(\beta_i^2)^u+(2\lambda+1)\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u\Big]+V_1^{0.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+3\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u\Bigg\}\\&\leqslant\lambda(1+V_1(x))^{\lambda-2}\Bigg\{-2V_1^2(x)[\min_{1\leqslant i\leqslant n}b_i^l-\lambda\max_{1\leqslant i\leqslant n}(\beta_i^2)^u]+V_1^{1.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u\\&\quad+V_1(x)\Big[3\max_{1\leqslant i\leqslant n}(\beta_i^2)^u+(2\lambda+1)\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u\Big]+V_1^{0.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+3\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u\Bigg\}.\end{aligned}$$

Let ν be sufficiently small such that

$$0<0.5\nu/\lambda<\min_{1\leqslant i\leqslant n}b_i^l-\lambda\max_{1\leqslant i\leqslant n}(\alpha_i^2)^u.$$

Let

$$V_3(x(t))=e^{\nu t}V_2(x(t)).$$

An application of Itô's formula, gives

$$\begin{aligned}dV_3(x)&=\nu e^{\nu t}V_2(x)dt+e^{\nu t}dV_2(x)\\&=LV_3(x)dt-\lambda e^{\nu t}(1+V_1(x))^{\lambda-1}\frac{2}{U^3(x)}\times\Bigg[\sum_{i=1}^n\beta_i(t)x_i dW_1(t)+\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}(t)x_ix_j dW_2(t)\Bigg],\end{aligned}$$

where

$$\begin{aligned}LV_3(x)&\leqslant\lambda e^{\nu t}(1+V_1(x))^{\lambda-2}\Bigg\{\nu(1+V_1(x))^2/\lambda-2V_1^2(x)[\min_{1\leqslant i\leqslant n}b_i^l-\lambda\max_{1\leqslant i\leqslant n}(\beta_i^2)^u]\\&\quad+V_1^{1.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+V_1(x)\Big[3\max_{1\leqslant i\leqslant n}(\beta_i^2)^u+(2\lambda+1)\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u\Big]\\&\quad+V_1^{0.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+3\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u\Bigg\}\\&=\lambda e^{\nu t}(1+V_1(x))^{\lambda-2}\Bigg\{-2V_1^2(x)[\min_{1\leqslant i\leqslant n}b_i^l-\lambda\max_{1\leqslant i\leqslant n}(\beta_i^2)^u-0.5\nu/\lambda]\\&\quad+V_1^{1.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+V_1(x)\Big[3\max_{1\leqslant i\leqslant n}(\beta_i^2)^u+(2\lambda+1)\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u+2\nu/\lambda\Big]\\&\quad+V_1^{0.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+3\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u+\nu/\lambda\Bigg\}=e^{\nu t}J(x),\end{aligned}$$

where

$$\begin{aligned}J(x)&=\lambda(1+V_1(x))^{\lambda-2}\Bigg\{-2V_1^2(x)[\min_{1\leqslant i\leqslant n}b_i^l-\lambda\max_{1\leqslant i\leqslant n}(\beta_i^2)^u-0.5\nu/\lambda]\\&\quad+V_1^{1.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+V_1(x)\Big[3\max_{1\leqslant i\leqslant n}(\beta_i^2)^u+(2\lambda+1)\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u+2\nu/\lambda\Big]\\&\quad+V_1^{0.5}(x)\max_{1\leqslant i,j\leqslant n}a_{ij}^u+3\max_{1\leqslant i,j\leqslant n}(\alpha_{ij}^2)^u+\nu/\lambda\Bigg\}.\end{aligned}$$

Clearly, $J(x)$ has an upper bound in R_+^n, so we define $C_1:=\sup\limits_{x\in R_+^n}J(x)<+\infty$. Consequently,

$$dV_3(x(t))\leqslant C_1e^{\nu t}dt-\lambda e^{\nu t}(1+V_1(x))^{\lambda-1}\frac{2}{U^3(x)}\Bigg[\sum_{i=1}^n\beta_i(t)x_i dW_i(t)+\sum_{i=1}^n\sum_{j=1}^n\alpha_{ij}(t)x_ix_j dW_2(t)\Bigg].$$

Integrating both sides of the above inequality and then taking expectations, we have

$$E\Big[V_3(x(t))\Big] = E\Big[e^{\nu t}(1+V_1(x(t)))^{\lambda}\Big] \leqslant \Big(1+V_1(x(0))\Big)^{\lambda} + C_1 e^{\nu t}/\nu.$$

That is to say,

$$\limsup_{t\to+\infty} E\Big[V_1^{\lambda}(x(t))\Big] \leqslant C_1/\nu.$$

Thanks to

$$\Big(\sum_{i=1}^{n} x_i(t)\Big)^{\lambda} \leqslant \Big(n \max_{1\leqslant i\leqslant n} x_i(t)\Big)^{\lambda} = n^{\lambda}\Big(\max_{1\leqslant i\leqslant n} x_i^2(t)\Big)^{0.5\lambda} \leqslant n^{\lambda}|x(t)|^{\lambda}.$$

Then we have

$$\limsup_{t\to+\infty} E\Big[|x(t)|^{-2\lambda}\Big] \leqslant n^{2\lambda}C_1/\nu =: C.$$

For arbitrary fixed $\varepsilon > 0$, denote $\xi_1 = \varepsilon^{0.5/\lambda}/C^{0.5/\lambda}$, according to Chebyshev's inequality ([31]), one can see that

$$P\Big\{|x(t)| < \xi_1\Big\} \leqslant \xi_1^{2\lambda} E\Big[1/|x(t)|^{2\lambda}\Big].$$

Consequently, $\liminf_{t\to+\infty} P\Big\{|x(t)| \geqslant \xi_1\Big\} \geqslant 1-\varepsilon$.

To complete the proof, it suffices to prove that for arbitrary fixed $\varepsilon > 0$, we can find a constant $\xi_2 > 0$ such that $\liminf_{t\to+\infty} P\Big\{|x(t)| \leqslant \xi_2\Big\} \geqslant 1-\varepsilon$. In fact, by (2.1) and Chebyshev's inequality, one can prove this assertion easily. □

We are in the position to study the extinction of (1.3).

Theorem 2.5. *If* $\limsup_{t\to+\infty} t^{-1}\int_0^t b_i(s)ds < 0$, *then the population* $x_i(t)$, *represented by model* (1.3) *goes to extinction a.s., i.e.,* $\lim_{t\to+\infty} x_i(t) = 0$ *a.s.,* $1 \leqslant i \leqslant n$.

Proof. By virtue of Itô's formula,

$$d\ln x_i = \Big[b_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_i - 0.5\Big(\sum_{j=1}^{n}\alpha_{ij}(t)x_j\Big)^2\Big]dt + \beta_i(t)dW_1(t) + \sum_{j=1}^{n}\alpha_{ij}(t)x_j dW_2(t), \quad 1 \leqslant i \leqslant n.$$

Integrating both sides from 0 to t,

$$\begin{aligned}\ln x_i(t) - \ln x_i(0) = \int_0^t \Big[b_i(s) - \sum_{j=1}^{n} a_{ij}(s)x_i(s) - 0.5\Big(\sum_{j=1}^{n}\alpha_{ij}(t)x_j(s)\Big)^2\Big]ds \\ + M_i(t) + N_i(t), \quad 1 \leqslant i \leqslant n.\end{aligned} \tag{2.2}$$

Here

$$M_i(t) = \int_0^t \beta_i(s)dW_1(s),\ N_i(t) = \int_0^t \sum_{j=1}^{n}\alpha_{ij}(s)x_j(s)dW_2(s), \quad 1 \leqslant i \leqslant n.$$

Clearly, $M_i(t)$ is a local martingale with quadratic variation

$$\langle M_i(t), M_i(t)\rangle = \int_0^t \beta_i^2(s)ds \leqslant (\beta_i^2)^u t.$$

It then follows from the strong law of large numbers for local martingales (see, e.g., [31]) that

$$\lim_{t\to+\infty} M_i(t)/t = 0 \quad \text{a.s.}, 1 \leqslant i \leqslant n. \tag{2.3}$$

At the same time,

$$\langle N_i(t), N_i(t)\rangle = \int_0^t \Big(\sum_{j=1}^n \alpha_{ij}(s)x_j(s)\Big)^2 ds.$$

According to the exponential martingale inequality (see, e.g., [31]), one can observe that

$$P\Big\{\sup_{0\leqslant t\leqslant k}\Big[N_i(t) - \frac{1}{2}\langle N_i(t), N_i(t)\rangle\Big] > 2\ln k\Big\} \leqslant 1/k^2, \quad 1 \leqslant i \leqslant n.$$

Thanks to the Borel-Cantalli lemma (see, e.g., [31]), for almost all $\omega \in \Omega$, there exists a random integer $k_0 = k_0(\omega)$ such that for $k \geqslant k_0$,

$$\sup_{0\leqslant t\leqslant k}\Big[N_i(t) - \frac{1}{2}\langle N_i(t), N_i(t)\rangle\Big] \leqslant 2\ln k,\ 1 \leqslant i \leqslant n.$$

Therefore,

$$N_i(t) \leqslant 2\ln k + 0.5\int_0^t \Big(\sum_{j=1}^n \alpha_{ij}(s)x_j(s)\Big)^2 ds$$

for all $0 \leqslant t \leqslant k,\ k \geqslant k_0$ almost surely. When these inequalities are used in (2.2), we have

$$\begin{aligned}
\ln x_i(t) - \ln x_i(0) &= \int_0^t \Big[b_i(s) - \sum_{j=1}^n a_{ij}(s)x_i(s) - 0.5\Big(\sum_{j=1}^n \alpha_{ij}(s)x_j(s)\Big)^2\Big] ds \\
&\quad + M_i(t) + 2\ln k + 0.5\int_0^t \Big(\sum_{j=1}^n \alpha_{ij}(s)x_j(s)\Big)^2 ds \\
&\leqslant \int_0^t b_i(s)ds + M_i(t) + 2\ln k,\ \ 1 \leqslant i \leqslant n
\end{aligned}$$

for all $0 \leqslant t \leqslant k,\ k \geqslant k_0$ almost surely. That is to say, for $0 < k-1 \leqslant t \leqslant k$,

$$t^{-1}[\ln x_i(t) - \ln x_i(0)] \leqslant t^{-1}\int_0^t b_i(s)ds + \frac{2\ln k}{t} + M_i(t)/t \leqslant t^{-1}\int_0^t b_i(s)ds + \frac{2n\ln k}{k-1} + M_i(t)/t.$$

According to (2.3),

$$\limsup_{t\to+\infty} t^{-1}\ln x_i(t) \leqslant \limsup_{t\to+\infty} t^{-1}\int_0^t b_i(s)ds.$$

That is to say, if $\limsup_{t\to+\infty} t^{-1}\int_0^t b_i(s)ds < 0$, then $\lim_{t\to+\infty} x_i(t) = 0$. □

Remark 2.6. Biologically, Theorem 2.5 means that sufficiently large stochastic random noises can force the species become extinct.

Remark 2.7. Li and Mao [18] have studied model (1.2) and have shown that

(A) if **(H1)** is satisfied, then (1.2) is stochastically permanent;

(B) if $\limsup_{t\to+\infty} t^{-1}\int_0^t b_i(s)ds < 0$, then $x_i(t)$, represented by model (1.2), goes to extinction a.s., i.e., $\lim_{t\to+\infty} x_i(t) = 0$ a.s., $1 \leqslant i \leqslant n$.

Clearly, model (1.2) is a special case of model (1.3) (i.e., $\alpha_{ij}(t) \equiv 0$ for all $1 \leqslant i, j \leqslant n$). Therefore our Theorems 2.4 and 2.5 extend the results of [18].

To finish this section, we consider the following stochastic logistic equation:

$$dx(t) = x(t)\Big(r - a x(t)\Big)dt + \beta x(t) dW_1(t) + \alpha x^2 dW_2(t), \quad x(0) = x_0 > 0, \tag{2.4}$$

where $a > 0$. By Theorems 2.4 and 2.5, we obtain the following sufficient and necessary conditions for stochastic permanence and extinction.

Corollary 2.8. *For Eq.* (2.4)*:*

(I) *if* $r - 0.5\beta^2 < 0$, *then the population* $x(t)$ *is extinctive a.s.;*
(II) *if* $r - 0.5\beta^2 > 0$, *then the population* $x(t)$ *is stochastically permanent.*

Remark 2.9. Corollary 2.8 reveals an interesting and important result: different types of stochastic noises have different effects on the permanence and extinction of the population. Note that the permanence and extinction of species depend only on the value of $r - 0.5\beta^2$. Therefore, the stochastic noise on r is unfavorable for the permanence of the population while the stochastic noises on a has no impact on the permanence and extinction of the population.

3. Example and numerical simulations

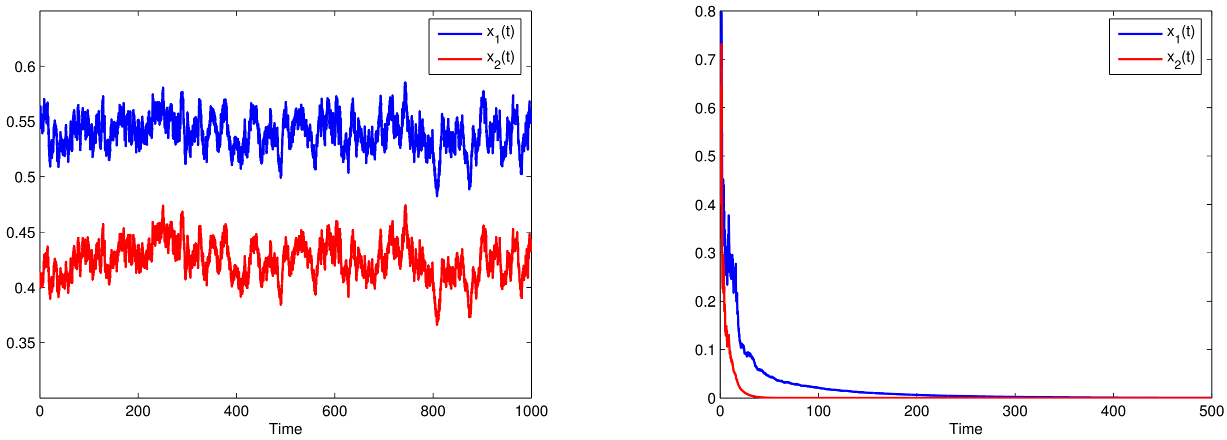
In this section we use the Milstein method (see, e.g., Higham [14]) to substantiate our main results. For simplicity, we choose $n = 2$.

Consider the discretization equations:

$$\begin{aligned}
x_1^{(k+1)} &= x_1^{(k)} + x_1^{(k)}\Big[r_1(k\Delta t) - a_{11}(k\Delta t)x_1^{(k)} - a_{12}(k\Delta t)x_2^{(k)}\Big]\Delta t \\
&\quad + \beta_1(k\Delta t)x_1^{(k)}\sqrt{\Delta t}\gamma^{(k)} + \frac{\beta_1^2(k\Delta t)}{2}x_1^{(k)}((\gamma^{(k)})^2 - 1)\Delta t \\
&\quad + \alpha_{11}(k\Delta t)(x_1^{(k)})^2\sqrt{\Delta t}\eta^{(k)} + \frac{\alpha_{11}^2(k\Delta t)}{2}(x_1^{(k)})^2((\eta^{(k)})^2 - 1)\Delta t \\
&\quad + \alpha_{12}(k\Delta t)x_1^{(k)}x_2^{(k)}\sqrt{\Delta t}\eta^{(k)} + \frac{\alpha_{12}^2(k\Delta t)}{2}x_1^{(k)}x_2^{(k)}((\eta^{(k)})^2 - 1)\Delta t, \\
x_2^{(k+1)} &= x_2^{(k)} + x_2^{(k)}\Big[r_2(k\Delta t) - a_{21}(k\Delta t)x^{(k)} - a_{22}(k\Delta t)x_2^{(k)}\Big]\Delta t \\
&\quad + \beta_2(k\Delta t)x_2^{(k)}\sqrt{\Delta t}\gamma^{(k)} + \frac{\beta_2^2(k\Delta t)}{2}x_2^{(k)}((\gamma^{(k)})^2 - 1)\Delta t \\
&\quad + \alpha_{21}(k\Delta t)x_1^{(k)}x_2^{(k)}\sqrt{\Delta t}\eta^{(k)} + \frac{\alpha_{21}^2(k\Delta t)}{2}x_1^{(k)}x_2^{(k)}((\eta^{(k)})^2 - 1)\Delta t \\
&\quad + \alpha_{22}(k\Delta t)(x_2^{(k)})^2\sqrt{\Delta t}\eta^{(k)} + \frac{\alpha_{22}^2(k\Delta t)}{2}(x_2^{(k)})^2((\eta^{(k)})^2 - 1)\Delta t,
\end{aligned}$$

where γ_k and η_k, $k = 1, 2, ..., n$, are the Gaussian random variables $N(0, 1)$.

In Fig. 1, we choose $r_1(t) = 0.36 + 0.01\sin t$, $r_2(t) = 0.33 + 0.05\sin t$, $a_{11}(t) = 0.2 + 0.01\sin t$, $a_{22}(t) = 0.22 + 0.01\sin t$, $a_{12}(t) = 0.1 + 0.02\sin t$, $a_{21}(t) = 0.05 + 0.02\sin t$, $\alpha_{11}(t) = 0.5 + 0.2\sin t$, $\alpha_{12}(t) = 0.3 + 0.1\sin t$, $\alpha_{21}(t) = 0.7 + 0.12\sin t$, and $\alpha_{22}(t) = 0.46 + 0.3\sin t$. The only difference between conditions of Fig. 1 (a) and Fig. 1 (b) is that the values of β_1^2 and β_2^2 are different. In Fig. 1 (a), we choose $\beta_1^2(t)/2 = \beta_2^2(t)/2 = 0.21 + 0.02\sin t$, then **(H1)** holds. In view of Theorem 2.4, system (1.3) is stochastically permanent. Fig. 1 (a) confirms this. In Fig. 1 (b), we choose $\beta_1^2(t)/2 = 0.38 + 0.02\sin t$, $\beta_2^2(t)/2 = 0.4 + 0.02\sin t$, then $\limsup_{t\to+\infty} t^{-1}\int_0^t b_1(s)ds = -0.02 < 0$ and $\limsup_{t\to+\infty} t^{-1}\int_0^t b_2(s)ds = -0.07 < 0$. By virtue of Theorem 2.5, both x_1 and x_2 go to extinction. See Fig. 1 (b).



(a) (b)

Figure 1: Solutions of system (1.3) for $n = 2$, $r_1(t) = 0.36 + 0.01 \sin t$, $r_2(t) = 0.33 + 0.05 \sin t$, $a_{11}(t) = 0.2 + 0.01 \sin t$, $a_{22}(t) = 0.22 + 0.01 \sin t$, $a_{12}(t) = 0.1 + 0.02 \sin t$, $a_{21}(t) = 0.05 + 0.02 \sin t$, $\alpha_{11}(t) = 0.5 + 0.2 \sin t$, $\alpha_{12}(t) = 0.3 + 0.1 \sin t$, $\alpha_{21}(t) = 0.7 + 0.12 \sin t$ and $\alpha_{22}(t) = 0.46 + 0.3 \sin t$, $x_1(0) = 0.55$, $x_2(0) = 0.4$, and step size $\Delta t = 0.001$. The horizontal axis represents the time t. (a) is with $\beta_1^2(t)/2 = \beta_2^2(t)/2 = 0.21 + 0.02 \sin t$; (b) is with $\beta_1^2(t)/2 = 0.38 + 0.02 \sin t$, $\beta_2^2(t)/2 = 0.4 + 0.02 \sin t$.

4. Conclusions and further research

This paper is devoted to the permanence and extinction of a general Lotka-Volterra competitive model with random perturbations. We show that when the noise is sufficiently small the population system is stochastically permanent; at the same time, we prove that a sufficiently large noise will force all the populations become extinct. Particularly, for a stochastic logistic equation, sufficient and necessary conditions for stochastic permanence and extinction are established. Some recent results are generalized.

Some interesting topics deserve further investigation. One could study more realistic but more complex models, for example, stochastic systems under regime switching (see e.g., [30, 40]), or with Lévy jumps (see e.g., [29]), or with reaction-diffusion ([5]). Also it is interesting to study n-dimensional stochastic food chain model or cooperative system, and we leave these for future work.

Acknowledgment

The author thanks the editor and the anonymous referees for their careful reading and valuable comments, which greatly improved the presentation of the paper.

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