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Multiple weighted estimates for vector-valued commutators of multilinear square functions

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Abstract

Let T be the multilinear square function with a kernel of Dini's type and T_q be the vector-valued version of T. In this paper, we obtain the weighted strong type and weighted end-point weak type estimates for the commutators of T_q respectively if the kernels satisfies $L \log L^l$-Dini type conditions. ©2017 All rights reserved.

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1. Introduction

Let $K_t(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. For any $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$, and each $f_j \in C_c^\infty(\mathbb{R}^n)$, the multilinear square function is given by

$$T(\vec{f})(x) = \left(\int_0^\infty \Big| \int_{(\mathbb{R}^n)^m} K_t(x, y_1, \ldots, y_m) \prod_{j=1}^m f_j(y_j) dy_1 \ldots dy_m \Big|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

This kind of operator has important applications in PDEs and other fields, we refer to [2, 4–7, 9, 12, 13] and the references therein. For instance, Fabes et al. [5] obtained a collection of multilinear Littlewood-Paley estimates, which they then applied to two problems in partial differential equations. The first problem is the estimation of the square root of an elliptic operator in divergence form, and the second is the estimation of solutions to the Cauchy problem for nondivergence form parabolic equations.

In order to state our main results, let us recall some notations and definitions.

Suppose that $\varpi(t) : [0, \infty) \mapsto [0, \infty)$ is a nondecreasing function with $0 < \varpi(1) < \infty$. For $a > 0$, we say that $\varpi \in \mathrm{Dini}(a)$, if

$$|\varpi|_{\mathrm{Dini}(a)} = \int_0^1 \varpi^a(t) \frac{dt}{t} < \infty.$$

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Suppose that there is a positive constant A such that the kernel K_t satisfies the following conditions.
Size condition:

$$\left(\int_0^\infty |K_t(x, y_1, \cdots, y_m)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \leqslant \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}}.$$

Smoothness condition:

$$\left(\int_0^\infty |K_t(z, y_1, \cdots, y_m) - K_t(x, y_1, \cdots, y_m)|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \leqslant \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}} \varpi\left(\frac{|z - x|}{\sum_{j=1}^m |x - y_j|}\right), \tag{1.1}$$

whenever $|z - x| \leqslant \frac{1}{2} \max_{j=1}^m |x - y_j|$, and

$$\left(\int_0^\infty |K_t(x, y_1, \dots, y_j, \dots, y_m) - K_t(x, y_1, \dots, y_j', \dots, y_m)|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$
$$\leqslant \frac{A}{(\sum_{j=1}^m |x - y_j|)^{mn}} \varpi\left(\frac{|y_j - y_j'|}{\sum_{j=1}^m |x - y_j|}\right),$$

whenever $|y_j - y_j'| \leqslant \frac{1}{2} \max_{j=1}^m |x - y_j|$.

In this paper, we always assume that T can be extended to bounded operators from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q for some $1 < q, q_1, \cdots, q_m < \infty$ with $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$.

Remark 1.1. When $\varpi(x) = x^\gamma$ for some $\gamma > 0$, the boundedness of multilinear square function were studied by Xue et al. [12].

Remark 1.2. Let T be a multilinear square function with a kernel satisfying Dini(1) condition. Si et al. [11] showed that T is bounded from $L^1 \times \cdots \times L^1$ to $L^{\frac{1}{m}, \infty}$. Secondly, they obtained that, if each $p_i > 1$, then T is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^p(\nu_{\vec{\omega}})$ and if there is a $p_i = 1$, then T is bounded from $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ to $L^{p,\infty}(\nu_{\vec{\omega}})$, where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$ (see Section 2 Definition 2.1 for the definition).

The main purpose of this paper is to study the boundedness properties of vector-valued commutator $T_{\Pi\vec{b},q}$ generated by T and BMO functions. Let $\vec{b} = (b_1, \dots, b_l) \in (\mathrm{BMO})^l$. For any given positive integer l with $1 \leqslant l \leqslant m$, the commutators associated with T are defined by

$$T_{\Pi\vec{b}}(\vec{f})(x) = \left(\int_0^\infty \Big|\int_{(\mathbb{R}^n)^m} \prod_{j=1}^l [b_j(x) - b_j(y_j)] K_t(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j) dy_1 \dots dy_m\Big|^2 \frac{dt}{t}\right)^{1/2}.$$

The vector-valued version of the commutator $T_{\Pi\vec{b}}$ can be defined by

$$T_{\Pi\vec{b},q}(\vec{f})(x) = |T_{\Pi\vec{b}}(\vec{f})(x)|_q = \left(\sum_{k=1}^\infty |T_{\Pi\vec{b}}(f_{1k}, \cdots, f_{mk})(x)|^q\right)^{1/q},$$

where $\vec{f} = (f_1, \dots, f_m)$ with $f_i = \{f_{ik}\}_{k=1}^\infty$ for $i = 1, \cdots, m$.

We get the following strong and end-point estimates for $T_{\Pi\vec{b},q}$.

Theorem 1.3. *Let $1/m < p < \infty, \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \cdots, p_m < \infty, 1/m < q < \infty$ and $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ with $1 < q_1, \cdots, q_m < \infty$. If $\vec{\omega} \in A_{\vec{p}}$, $\vec{b} \in (\mathrm{BMO})^l$ and ϖ satisfies*

$$|\varpi|_{L\log L^l-\mathrm{Dini}} = \int_0^1 \varpi(t)(1 + \log^l \frac{1}{t}) \frac{dt}{t} < \infty, \tag{1.2}$$

then

$$\|T_{\Pi\vec{b},q}(\vec{f})\|_{L^p(\nu_{\vec{\omega}})} \leqslant C \prod_{j=1}^l \|b_j\|_{\mathrm{BMO}} \prod_{j=1}^m \||f_j|_{q_j}\|_{L^{p_j}(w_j)}.$$

Theorem 1.4. *Let* $1/m < q < \infty$ *and* $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ *with* $1 < q_1, \cdots, q_m < \infty$. *If* $\vec{\omega} \in A_{(1,\cdots,1)}$, $\vec{b} \in (BMO)^l$ *and* ϖ *satisfies the condition* (1.2), *then*

$$\nu_{\vec{\omega}}\left(\left\{x \in \mathbb{R}^n : T_{\Pi\vec{b},q}(\vec{f})(x) > t^m\right\}\right) \leqslant C\left(\prod_{j=1}^m \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j|_{q_j}(y_j)}{t}\right)\omega_j(y_j)dy_j\right)^{1/m},$$

where $\Phi^{(m)} = \overbrace{\Phi \circ \cdots \circ \Phi}^{m}$ *and* $\Phi(t) = t(1 + \log^+ t)$.

Throughout this paper, C denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line.

2. Proofs of Theorem 1.3 and Theorem 1.4

Let us begin with the definition of multiple-weights introduced by Lerner et al. [8].

Definition 2.1 ([8])**.** Let $1 \leqslant p_1, \cdots, p_m < \infty$, and $1/p = 1/p_1 + \cdots + 1/p_m$. Given $\vec{\omega} = (\omega_1, \cdots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \prod_{i=1}^m \omega_i^{\frac{p}{p_i}}\right)^{\frac{1}{p}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}} < \infty,$$

when $p_i = 1$, $\left(\frac{1}{|Q|}\int_Q \omega_i^{1-p_i'}\right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Definition 2.2. For $1 \leqslant l \leqslant m$, the maximal operators $\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)$ and $\mathcal{M}(|\vec{f}|_q)$ are respectively defined by

$$\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x) = \sup_{Q\ni x} \prod_{j=1}^l \||f_j|_{q_j}\|_{L(\log L),Q} \prod_{j=l+1}^m \frac{1}{|Q|}\int_Q |f_j|_{q_j}$$

and

$$\mathcal{M}(|\vec{f}|_q)(x) = \sup_{Q\ni x} \prod_{j=1}^m \frac{1}{|Q|}\int_Q |f_j|_{q_j},$$

where the supremum is taken over all the cubes containing x.

We can control multilinear square function by using maximal operators in the following way.

Lemma 2.3 ([11])**.** *Let* $\varpi \in \text{Dini}(1)$ *and* $0 < \delta < \frac{1}{m}$. *For any compact supported* f_j, $j = 1, \ldots, m$, *we have*

$$M^\sharp_\delta T(\vec{f})(x) \leqslant C\mathcal{M}(\vec{f})(x).$$

The above lemma can be used to establish the vector-valued inequalities for multilinear square function. The vector-valued multilinear square function T_q associated is defined by

$$T_q(\vec{f})(x) = |T(\vec{f})(x)|_q = \left(\sum_{k=1}^\infty |T(f_{1k}, \cdots, f_{mk})(x)|^q\right)^{1/q},$$

where $\vec{f} = (f_1, \ldots, f_m)$ with $f_i = \{f_{ik}\}_{k=1}^\infty$ for $i = 1, \cdots, m$.

Theorem 2.4. *Let* $1 \leqslant p_1, \cdots, p_m < \infty, 1 < q_1, \cdots, q_m < \infty$ *and* $0 < p, q < \infty$ *such that* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}, \frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$. *Suppose that* $\varpi \in \text{Dini}(1)$, *then we have*

(i) *If* $1 < p_1, ..., p_m < \infty$ *and* $\omega \in A_{p_1} \cap \cdots \cap A_{p_m}$, *then*

$$\|T_q(\vec{f})\|_{L^p(\omega)} \leqslant C \prod_{j=1}^{m} \||f_j|_{q_j}\|_{L^{p_j}(\omega)}.$$

(ii) *If at least one* $p_j = 1$ *and* $\omega \in A_1$, *then*

$$\|T_q(\vec{f})\|_{L^{p,\infty}(\omega)} \leqslant C \prod_{j=1}^{m} \||f_j|_{q_j}\|_{L^{p_j}(\omega)}.$$

Proof. By using Lemma 2.3 and a standard argument, we can deduce that

$$\|T\vec{f}\|_{L^p(\omega)} \leqslant C \| \prod_{j=1}^{m} M(f_j) \|_{L^p(\omega)}.$$

We apply the extrapolation theorems ([3, Theorem 2.1]) to $(T\vec{f}, \prod_{j=1}^{m} Mf_j) \in \mathcal{F}$, then by the vector-valued inequality for the Hardy-Littlewood maximal operator M ([1, Theorem 3.1]) we get the conclusion of Theorem 2.4. □

To prove Theorems 1.3 and 1.4, we need the following preliminary lemmas.

Lemma 2.5. *Let* $0 < \delta < 1/m$, $1/m < q < \infty$ *and* $1/q = 1/q_1 + \cdots + 1/q_m$ *with* $1 < q_1, \cdots, q_m < \infty$. *If* ϖ *satisfies the condition* (1.2), *then*

$$M^{\sharp}_{\delta}(T_q(\vec{f}))(x) \leqslant C\mathcal{M}(|\vec{f}|_q)(x)$$

holds for any smooth vector function $\{\vec{f}_k\}_{k=1}^{\infty}$ *and any* $x \in \mathbb{R}^n$.

Proof. For any fixed $x \in \mathbb{R}^n$, let Q be a cube which contains x with side length $l(Q)$. We decompose $\vec{f} = \vec{f}^0 + \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} \vec{f}^{\vec{\alpha}}$, where $\vec{f}^0 = \vec{f}\chi_{8\sqrt{n}Q} = (f_1\chi_{8\sqrt{n}Q}, \cdots, f_m\chi_{8\sqrt{n}Q})$ and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m), \alpha_i = 0$ or ∞. Set $C = |c|_q = \left(\sum_{k \geqslant 1} |c_k|^q \right)^{1/q}$. Then we have

$$\begin{aligned} \left(\frac{1}{|Q|} \int_Q \left| |T_q(\vec{f})(y)|^{\delta} - |C|^{\delta} \right| dy \right)^{\frac{1}{\delta}} &\leqslant C \left(\frac{1}{|Q|} \int_Q \left| T_q(\vec{f}^0)(y) \right|^{\delta} dy \right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|} \int_Q \Big| \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} T(\vec{f}^{\vec{\alpha}})(y) - c \Big|_q^{\delta} dy \right)^{\frac{1}{\delta}} \\ &= U_1 + U_2. \end{aligned}$$

For U_1, we apply Kolmogorov's inequality and Theorem 2.4 to get

$$\left(\frac{1}{|Q|} \int_Q \left| T_q(\vec{f}^0)(y) \right|^{\delta} dy \right)^{\frac{1}{\delta}} \leqslant C \|T_q(\vec{f}^0)\|_{L^{1/2,\infty}(Q, \frac{dx}{|Q|})} \leqslant C \prod_{j=1}^{2} \frac{1}{|Q|} \int_Q |f_j|_{q_j}(z) dz \leqslant C\mathcal{M}(|\vec{f}|_q)(x).$$

For U_2, we choose $c = \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} c^{\vec{\alpha}}$, where $c^{\vec{\alpha}} = T(f_{1k}^{\alpha_1}, \cdots, f_{mk}^{\alpha_m})(x)$. It is obvious that $U_2 \leqslant \sum_{\vec{\alpha}, \vec{\alpha} \neq \vec{0}} U_{2\vec{\alpha}}$, where

$$U_{2\vec{\alpha}} = \left(\frac{1}{|Q|} \int_Q |T(f_{1k}^{\alpha_1}, \cdots, f_{mk}^{\alpha_m})(y) - T(f_{1k}^{\alpha_1}, \cdots, f_{mk}^{\alpha_m})(x)|_q^{\delta} dy \right)^{\frac{1}{\delta}}.$$

Next, we estimate the typical term $U_{2\vec{\alpha}}$ with $\alpha_1 = \cdots = \alpha_l = \infty$ and $\alpha_{l+1} = \cdots = \alpha_m = 0$. Let $\mathcal{Q}_s = 2^{s+3}\sqrt{n}Q \setminus 2^{s+2}\sqrt{n}Q$ for $s = 1, 2, \cdots$. For $x, y \in Q$ and any $y_j \in \mathcal{Q}_s$ with $j \neq l+1, \cdots, m$, one has $|y - y_j| \geqslant 2^s\sqrt{n}l(Q)$, then $\left(\int_0^{\infty} |K_t(y, \vec{y}) - K_t(x, \vec{y})|^2 \frac{dt}{t} \right)^{1/2} \leqslant \frac{C\varpi(2^{-s})}{|2^s\sqrt{n}Q|^m}$. We have the following estimates

$$|T(f_{1k}^{\infty}, \cdots, f_{lk}^{\infty}, f_{(l+1)k}^{0} \cdots f_{(m)k}^{0})(y) - T(f_{1k}^{\infty}, \cdots, f_{lk}^{\infty}, f_{(l+1)k}^{0} \cdots f_{(m)k}^{0})(x)|_q$$

$$
\begin{aligned}
&\leqslant C\Big|\left(\int_0^\infty \Big|\int_{(\mathbb{R}^n)^m} K_t(y,y_1,\dots,y_m)\prod_{j=1}^{l} f_{jk}^\infty(y_j)\prod_{j=l+1}^{m} f_{jk}^0(y_j)d\vec{y}\Big|^2\frac{dt}{t}\right)^{\frac12}\\
&\quad-\left(\int_0^\infty \Big|\int_{(\mathbb{R}^n)^m} K_t(x,y_1,\dots,y_m)\prod_{j=1}^{l} f_{jk}^\infty(y_j)\prod_{j=l+1}^{m} f_{jk}^0(y_j)d\vec{y}\Big|^2\frac{dt}{t}\right)^{\frac12}\Big|_q\\
&\leqslant C\left(\sum_{k=1}^\infty\Big|\int_{(\mathbb{R}^n\setminus 8\sqrt{n}Q)^m}\left(\int_0^\infty |K_t(y,\vec{y})-K_t(x,\vec{y})|^2\frac{dt}{t}\right)^{1/2}\prod_{j=1}^{l}|f_{jk}^\infty(y_j)|\prod_{j=l+1}^{m}|f_{jk}^0(y_j)|d\vec{y}\Big|^q\right)^{1/q}\\
&\leqslant C\prod_{j=l+1}^{m}\int_{8\sqrt{n}Q}|f_j(y_j)|_{q_j}dy_j\int_{(\mathbb{R}^n\setminus 8\sqrt{n}Q)^l}\frac{\overline{\omega}\left(\frac{|y-x|}{\sum_{j=1}^m|y-y_j|}\right)}{\left(\sum_{j=1}^m|y-y_j|\right)^{mn}}\prod_{j=1}^{l}|f_j(y_j)|_{q_j}dy_1\cdots dy_l\\
&\leqslant C\sum_{s=1}^\infty\overline{\omega}(2^{-s})\prod_{j=l+1}^{m}\int_{2^{s+3}\sqrt{n}Q}|f_j(y_j)|_{q_j}dy_j\frac{1}{|2^{(s+3)}\sqrt{n}Q|^m}\int_{(2^{s+3}\sqrt{n}Q)^l}\prod_{j=1}^{l}|f_j(y_j)|_{q_j}dy_j\\
&\leqslant C|\overline{\omega}|_{L\log L^l-\mathrm{Dini}}\prod_{j=1}^{m}\frac{1}{2^{(s+3)n}|\sqrt{n}Q|}\int_{2^{s+3}\sqrt{n}Q}|f_j(y_j)|_{q_j}dy_j\\
&\leqslant C|\overline{\omega}|_{L\log L^l-\mathrm{Dini}}\mathcal{M}(|\vec{f}|_q)(x).
\end{aligned}
$$

Since $0<\delta<1/2$, by Hölder's inequality, we obtain that

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q |T(f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0\cdots f_{(m)k}^0)(y)-T(f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0\cdots f_{(m)k}^0)(x)|_q^\delta dy\right)^{\frac1\delta}\\
&\quad\leqslant C\frac{1}{|Q|}\int_Q |T(f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0\cdots f_{(m)k}^0)(y)-T(f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0\cdots f_{(m)k}^0)(x)|_q dy\\
&\quad\leqslant C|\overline{\omega}|_{\mathrm{Dini}-L\log L^l}\mathcal{M}(|\vec{f}|_q)(x).
\end{aligned}
$$

Hence the proof of Lemma 2.5 is complete. □

For positive integers m and j with $1\leqslant j\leqslant m$, we denote by $\mathcal{C}_j^m$ the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements.

Lemma 2.6. *Let $0<\delta<\varepsilon<1/m$. If $\overline{\omega}$ satisfies the condition (1.2), then*

$$
\begin{aligned}
M_\delta^\sharp(T_{\Pi\vec{b},q}\vec{f})(x)\leqslant C\prod_{j=1}^{l}\|b_j\|_{BMO}\left(\mathcal{M}_{L(\log L)}^{l}(|\vec{f}|_q)(x)+M_\varepsilon(T_q\vec{f})(x)\right)\\
+C\sum_{j=1}^{l-1}\sum_{\sigma\in\mathcal{C}_j^l}\prod_{i\in\sigma}\|b_i\|_{BMO}M_\varepsilon(T_{\Pi b_{\sigma'},q}\vec{f})(x)
\end{aligned}
$$

holds for any smooth vector function $\{\vec{f}_k\}_{k=1}^\infty$ and for any $x\in\mathbb{R}^n$, where $\sigma'=\{1,\cdots,l\}\setminus\sigma$.

Proof. For any cube Q centered at x, we can obtain that

$$
\begin{aligned}
\left(\frac{1}{|Q|}\int_Q\Big||T_{\Pi\vec{b},q}(\vec{f})(z)|^\delta-|C|^\delta\Big|dz\right)^{1/\delta}&\leqslant C\left(\frac{1}{|Q|}\int_Q\Big|T_{\Pi\vec{b}}(\vec{f})(z)-c\Big|_q^\delta dz\right)^{1/\delta}\\
&\leqslant C\left(\frac{1}{|Q|}\int_Q\Big|(b_1(z)-\lambda_1)\cdots(b_l(z)-\lambda_l)T(\vec{f})(z)\Big|_q^\delta dz\right)^{1/\delta}
\end{aligned}
$$

$$+C\sum_{i=1}^{l-1}\sum_{\sigma\in\mathcal{C}_i^l}\left(\frac{1}{|Q|}\int_Q\left(\prod_{j\in\sigma}|b_j(z)-\lambda_j|T_{\Pi b_{\sigma'},q}\vec{f}(z)\right)^\delta dz\right)^{1/\delta}$$

$$+C\left(\frac{1}{|Q|}\int_Q\left|T((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f})(z)-c\right|_q^\delta dz\right)^{1/\delta}$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III},$$

where $C=\left|c\right|_q$ and $\lambda_j=\frac{1}{|8\sqrt{n}Q|}\int_{8\sqrt{n}Q}b_j(z)dz$ for $j=1,\cdots,l$.

Now we give the estimate for the above terms.

We can choose $1<p_1,\cdots,p_l<\infty$ with $\frac{1}{p_1}+\cdots+\frac{1}{p_l}+\frac{1}{\varepsilon}=\frac{1}{\delta}$. Since $0<\delta<\varepsilon<1/m$, Hölder's inequality gives

$$\mathrm{I}\leqslant C\prod_{j=1}^{l}\|b_j\|_{BMO}M_\varepsilon(T_q\vec{f})(x).$$

Similarly, we have

$$\mathrm{II}\leqslant C\sum_{i=1}^{l-1}\sum_{\sigma\in\mathcal{C}_i^l}\prod_{j\in\sigma}\|b_j\|_{BMO}M_\varepsilon(T_{\Pi b_{\sigma'},q}\vec{f})(x).$$

Next, we estimate III. Let $\vec{f}^{\vec{\alpha}}=f_1^{\alpha_1}\cdots f_m^{\alpha_m}$ and $\vec{f}_j=f_j^0+\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}\vec{f}^{\vec{\alpha}}$, where $\vec{f}_j^0=\vec{f}_j\chi_{8\sqrt{n}Q}$ and $\vec{\alpha}=(\alpha_1,\ldots,\alpha_m)$, $\alpha_i=0$ or ∞. Then, we have

$$\left|T(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)\vec{f})(z)-c\right|_q\leqslant T_q(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)\vec{f}^0)(z)$$
$$+C\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}|(T(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)\vec{f}^{\alpha}))(z)-(T(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)\vec{f}^{\alpha}))(x)|_q,$$

where $c=\sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}}(T(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)f_1^{\alpha_1}\cdots f_m^{\alpha_m}))(x)$.

By using Kolmogorov's inequality and Theorem 2.4, we obtain that

$$\left(\frac{1}{|Q|}\int_Q\left|T_q(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)\vec{f}^0)(z)\right|^\delta dz\right)^{1/\delta}\leqslant C\|T_q(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)\vec{f}^0)\|_{L^{1/m,\infty}(Q,\frac{dz}{|Q|})}$$
$$\leqslant C\prod_{j=1}^{l}\frac{1}{|Q|}\int_Q|b_j(z)-\lambda_j||f_j(z)|_{q_j}dz\prod_{j=l+1}^{m}\frac{1}{|Q|}\int_Q|f_j(z)|_{q_j}dz$$
$$\leqslant C\prod_{j=1}^{l}\|b_j\|_{BMO}\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x).$$

Now, the arguments in Lemma 2.5 can be applied. If $\alpha_1=\cdots=\alpha_l=\infty$ and $\alpha_{l+1}=\cdots=\alpha_m=0$, by the Minkowski inequality and the smoothness condition (1.1), we have

$$\left|T(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0,\cdots,f_{mk}^0)(z)-T(\prod_{i=1}^{l}(b_i(\cdot_i)-\lambda_i)f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0,\cdots,f_{mk}^0)(x)\right|_q$$
$$\leqslant C\left|\left(\int_0^\infty\left|\int_{(\mathbb{R}^n)^m}|K_t(z,\vec{y})-K_t(x,\vec{y})||\prod_{i=1}^{l}(b_i(y_i)-\lambda_i)|\prod_{j=1}^{l}|f_{jk}^\infty(y_j)|\prod_{j=l+1}^{m}|f_{jk}^0(y_j)|d\vec{y}\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}\right|_q$$

$$
\begin{aligned}
&\leqslant C\Big|\int_{(\mathbb{R}^n)^m}\left(\int_0^\infty |K_t(z,\vec{y})-K_t(x,\vec{y})|^2\frac{dt}{t}\right)^{1/2}|\prod_{i=1}^{l}(b_i(y_i)-\lambda_i)|\prod_{j=1}^{l}|f_{jk}^{\infty}(y_j)|\prod_{j=l+1}^{m}|f_{jk}^{0}(y_j)|d\vec{y}|^2\frac{dt}{t}\Big)^{\frac{1}{2}}\Big|_q\\
&\leqslant C\sum_{k=1}^{\infty}\int_{(Q_k)^l}\varpi(2^{-k})\frac{|\prod_{i=1}^{l}(b_i(y_i)-\lambda_i)||f_1(y_1)|_{q_1}\cdots|f_l(y_l)|_{q_l}dy_1\cdots dy_l}{(|z-y_1|+\cdots+|z-y_m|)^{mn}}\prod_{j=l+1}^{m}\int_{8Q}|f_j(y_j)|_{q_j}dy_j\\
&\leqslant C\sum_{k=1}^{\infty}\frac{\varpi(2^{-k})}{(2^k|\sqrt{n}Q|^{\frac{1}{n}})^{mn}}\prod_{j=1}^{l}\int_{2^{k+3}\sqrt{n}Q}|b_j(y_j)-\lambda_j||f_j(y_j)|_{q_j}dy_j\prod_{j=l+1}^{m}\int_{2^{k+3}\sqrt{n}Q}|f_j(y_j)|_{q_j}dy_j\\
&\leqslant C\sum_{k=1}^{\infty}\varpi(2^{-k})k^l\prod_{j=1}^{l}\|b_j\|_{BMO}\||f_j|_{q_j}\|_{L(\log L),2^{k+3}\sqrt{n}Q}\prod_{j=l+1}^{m}\frac{1}{2^{(k+3)n}|\sqrt{n}Q|}\int_{2^{k+3}\sqrt{n}Q}|f_j|_{q_j}dy_j\\
&\leqslant C|\varpi|_{L\log L^l-\mathrm{Dini}}\prod_{j=1}^{l}\|b_j\|_{BMO}\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x),
\end{aligned}
$$

where in the last inequality we used the fact that ϖ satisfies the condition (1.2).

For the other cases, the estimates are similar. Then we proved Lemma 2.6. □

By modifying the proof of Theorem 3.1 in [10], we get the following.

Lemma 2.7. *Let* $0<p<\infty, 1/m<q<\infty$, *and* $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$ *with* $1<q_1,\cdots,q_m<\infty$. *Suppose that* $\vec{b}\in(BMO)^l$, $w\in A_\infty$ *and* ϖ *satisfies the condition* (1.2), *then*

$$\int_{\mathbb{R}^n}|T_{\Pi\vec{b},q}\vec{f}|^p w(x)dx\leqslant C\prod_{j=1}^{l}\|b_j\|^p_{BMO}\int_{\mathbb{R}^n}\left(\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x)\right)^p w(x)dx, \tag{2.1}$$

and

$$
\begin{aligned}
&\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)}\omega\left(\left\{y\in\mathbb{R}^n:|T_{\Pi\vec{b},q}\vec{f}(y)|>t^m\right\}\right)\\
&\qquad\leqslant C\sup_{t>0}\frac{1}{\Phi^{(m)}(1/t)}\omega\left(\left\{y\in\mathbb{R}^n:\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(y)>t^m\right\}\right),
\end{aligned}
$$

hold for any smooth function $\vec{f}$ *with compact support.*

The following multiple weighted estimates for $\mathcal{M}^l_{L(\log L)}$ can be found in [10].

Lemma 2.8 ([10])**.** *Let* $1/m<q<\infty$ *and* $1<q_1,\cdots,q_m<\infty$ *with* $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$.

(i) *Let* $1/m<p<\infty$ *and* $1<p_1,\cdots,p_m<\infty$ *with* $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, *and* $\vec{\omega}$ *satisfies the* $A_{\vec{p}}$ *condition. Then*

$$\left(\int_{\mathbb{R}^n}\left|\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x)\right|^p\nu_{\vec{\omega}}(x)dx\right)^{1/p}\leqslant C\left(\int_{\mathbb{R}^n}||f_j|_{q_j}(x)|^{p_j}\omega_j(x)dx\right)^{1/p_j}.$$

(ii) *Let* $\vec{\omega}\in A_{(1,\cdots,1)}$. *Then*

$$\nu_{\vec{\omega}}\left(\left\{x\in\mathbb{R}^n:\mathcal{M}^l_{L(\log L)}(|\vec{f}|_q)(x)>t^m\right\}\right)\leqslant C\left(\prod_{j=1}^{m}\int_{\mathbb{R}^n}\Phi^{(m)}\left(\frac{|f_j|_{q_j}(y_j)}{t}\right)\omega_j(y_j)dy_j\right)^{1/m},$$

where $\Phi(t)=t(1+\log^+t)$ *and* $\Phi^{(m)}=\overbrace{\Phi\circ\cdots\circ\Phi}^{m}$.

Proof of Theorem 1.3. Combining (2.1) and Lemma 2.8, we get Theorem 1.3. □

Proof of Theorem 1.4. Using the properties of function Φ, Lemma 2.7, and Lemma 2.8, we can obtain Theorem 1.4. Since the arguments are almost the same as the proof of Theorem 3.16 in [8], we omit the details here. □

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