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# BKM's criterion for the 3D nematic liquid crystal flows in Besov spaces of negative regular index

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### Abstract

In this paper, we investigate the blow-up criterion of a smooth solution of the nematic liquid crystal flow in three-dimensional space. More precisely, We prove that if $\int_0^T(\|\omega\|^{\frac{2}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}}+\|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})dt<\infty$, $0<\alpha<2$, then the solution $(u,d)$ can be extended smoothly beyond $t=T$. 

Keywords: Nematic liquid crystal flow, blow-up criteria, regularity criteria, Besov space.
*2010 MSC:* 35Q35, 35B65.

## 1. Introduction

In this paper, we are concerned with the following viscous incompressible flow of nematic liquid crystal in $\mathbb{R}^3$ :

$$\begin{cases}\frac{\partial u}{\partial t}+u\cdot\nabla u-\mu\Delta u+\nabla p=-\lambda\nabla\cdot(\nabla d\odot\nabla d),\\ \frac{\partial d}{\partial t}+u\cdot\nabla d=\gamma(\Delta d+|\nabla d|^2 d),\\ \operatorname{div}u=0,\\ u(x,0)=u_0,\ d(x,0)=d_0.\end{cases}\tag{1.1}$$

Here $u=(u_1(x,t),u_2(x,t),u_3(x,t))$ denotes the velocity of the fluid at a point $x\in\mathbb{R}^3$ and time $t\in[0,T)$; and $d=(d_1(x,t),d_2(x,t),d_3(x,t))$ and $p=p(x,t)$ stand for the macroscopic average of the nematic liquid crystal orientation field and the fluid pressure, respectively. The tensorial notation $\nabla d\odot\nabla d$ denotes the $3\times 3$ matrix whose $(i,j)$-th entry is given by $\partial_{x_i}d\cdot\partial_{x_j}d$, and then $(\nabla d\odot\nabla d)_{ij}=\sum_{k=1}^3\partial x_i d_k\partial x_j d_k$ for any $i,j=1,2,3$. Moreover, it is easy to verify that $\nabla\cdot(\nabla d\odot\nabla d)=\frac{1}{2}\nabla(|\nabla d|^2)+\Delta d\cdot\nabla d$, where $\nabla$ denotes the gradient operator $(\partial_{x_1},\partial_{x_2},\partial_{x_3})$. $u_0$ and $d_0$ are the prescribed initial data of $u$ and $d$, and $u_0$ satisfies the incompressible condition $\operatorname{div}u_0=0$. Clearly $\Delta|d|^2=0$ because of $|d|=1$, we thus have $d\cdot\Delta d=-|\nabla d|^2$.

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$\mu$ is the kinematic viscosity, $\lambda$ is the competition between the kinetic and potential energies, and $\gamma$ is the microscopic elastic relation time for the molecular orientation field.

The system (1.1) is a simplified version of the Ericksen-Leslie model (see [5, 16]), which was first introduced by Lin in [18] and studied by Lin and Liu in their important works in [21, 22] and Coutand and Shkoller in [4]. It is a macroscopic continuum description of the time evolution of the rod-like liquid crystal materials under the influence of both the velocity field $u$ and the macroscopic description of the microscopic orientation configurations $d$. Due to the physical importance and real-world applications, there have been numerous attempts to formulate continuum theories describing the behavior of liquid crystal flows, we refer to the seminal papers [5, 16] and references therein. Mathematically, the system (1.1) can be seen as a variant of the Navier-Stokes problem with respect to the unknown velocity and pressure $(u, p)$ coupled with heat flow of a harmonic map. When $d$ is constant, this model becomes the well-known Navier-Stokes equation, we refer to the papers [1, 8, 13, 14], while when $u = 0$, the system is the harmonic heat flow equation on to a sphere [3, 28].

Besides the physical and real-world applications, there have been numerous attempts to study it from mathematical viewpoint. Many results to system (1.1) have been established (see, for example, [7, 9–11, 17, 19, 20, 23, 24, 26, 27, 29]). Lin et al. [20] proved the global existence of Leray-Hopf type weak solutions for system (1.1) on the bounded domain in $\mathbb{R}^2$ under initial and boundary value conditions. When the space dimension $n \geqslant 3$, Li and Wang [17] obtained the existence of a local strong solution for general initial data and the global strong solution for small initial data. It is well-known that the strong solutions of heat flow of harmonic maps must be blowing up at finite time (see [2]), therefore we can not expect that the system (1.1) has a global smooth solution with general initial data. However, Wen and Ding [29] proved that if $u_0(x) \in H^s(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot u = 0$, and $d_0 \in H^{s+1}(\mathbb{R}^3, \mathbb{S}^2)$ for $s \geqslant 3$, then there exists $T = T(\|u_0\|_{H^s}, \|d_0\|_{H^{s+1}})$ such that system (1.1) has a unique local classical solution in the class

$$\begin{cases} u \in C([0,T]; H^s(\mathbb{R}^3, \mathbb{R}^3)) \cap C^1([0,T]; H^{s-2}(\mathbb{R}^3, \mathbb{R}^3)), \\ d \in C([0,T]; H^{s+1}(\mathbb{R}^3, \mathbb{S}^2)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^3, \mathbb{S}^2)). \end{cases} \tag{1.2}$$

The blow-up criteria of smooth solutions to nematic liquid crystal flow are important topic in the research of global well-posedness. Huang and Wang [11] established a BKM type blow-up criterion for the system (1.1). Namely, if $T^*$ is the maximal time, $0 < T^* < +\infty$, then

$$\int_0^{T^*} (\|\omega\|_\infty + \|\nabla d\|_\infty^2) dt = \infty. \tag{1.3}$$

This result is improved by Liu and Zhao [25], who proved that the smooth solution $(u, d)$ of (1.1) blows up at the time $T^*$, if and only if

$$\int_0^{T^*} \frac{\|\omega\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln(e + \|\omega\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}})}} dt = \infty.$$

This result is also improved by Zhao [30] in terms of two velocity components and molecular orientations. More precisely, let $T^*$ be the maximal existence time of the local strong solution $(u, d)$, then $T^* < +\infty$ if and only if

$$\int_0^{T^*} (\|\nabla_h u^h\|^q_{\dot{B}^0_{p,2p/3}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}) dt = \infty, \quad \text{with } \ 3/p + 2/q = 2, \ \ 3/2 < p < \infty.$$

Recently, a new blow-up criterion in terms of velocity gradient in Besov spaces of negative indices is established by Fan and Zhou [6], i.e.,

$$\nabla u, \Delta^2 d \in L^{\frac{2}{2-\alpha}}(0, T; \dot{B}^{-\alpha}_{\infty,\infty}), \quad \text{with} \quad 0 < \alpha < 1.$$

In this paper, we consider the blow-up criterion of system (1.1) in terms of the vorticity in Besov space of negative index and the orientation field in the homogeneous Besov space, which improves the result of [11, 25].

Before stating our main result, we introduce some function spaces and notations. The norm of the Lebesgue space $\|.\|_{L^p}$ is denoted by $\|.\|_p$. To simplify the notations, we shall use the letter C to denote a generic constant which may vary from line to line, and write $\partial_t u = \frac{\partial u}{\partial t}, \partial_i = \frac{\partial}{\partial x_i}, (u \cdot \nabla) = \sum_{i=1}^3 u_i \partial_i$. Since the concrete values of the constants $\mu, \lambda, \gamma$ play no role in our discussion, to simplify the presentation, we shall assume that $\mu = \lambda = \gamma = 1$ in this paper.

To this end, we state our result as follows.

**Theorem 1.1.** *Let $u_0 \in H^s(\mathbb{R}^3, \mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$ and $d_0 \in H^{s+1}(\mathbb{R}^3, \mathbb{S}^2)$ with $s \geqslant 3$. Suppose that the pair (u, p) is a smooth solution to the equations (1.1) satisfying (1.2). If $(u, d)$ satisfies the condition*

$$\int_0^T (\|\omega(t)\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{2-r}} + \|\nabla d(t)\|^2_{\dot{B}^0_{\infty,\infty}}) dt < \infty, \quad 0 < r < 2,$$

*then the solution $(u, d)$ can be extended smoothly beyond $t = T$.*

## 2. Preliminaries

In this preliminary section, we will present some lemmas which will be used in the proof of Theorem 1.1.

**Lemma 2.1** ([1, Theorem 2.42]). *Let $1 \leqslant q < p < \infty$ and $\alpha$ be a positive real number. Then there exists a constant C such that*

$$\|f\|_{L^p} \leqslant C\|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|f\|^{\theta}_{\dot{B}^{\beta}_{q,q}}, \quad \text{with } \beta = \alpha(\frac{p}{q} - 1) \text{ and } \theta = \frac{q}{p}.$$

In particular, for $q = 2$ and $p = 3$, we have

$$\|f\|^3_{L^3} \leqslant C\|f\|_{\dot{B}^{-r}_{\infty,\infty}} \|f\|^2_{\dot{H}^{\frac{r}{2}}}, \quad \text{with } r > 0. \tag{2.1}$$

For $p = 4$, $q = 2$, $\alpha = \beta = 1$, we have

$$\|f\|_4 \leqslant C\|f\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla f\|^{\frac{1}{2}}_2. \tag{2.2}$$

**Lemma 2.2** (Commutator estimate [12]). *Let $1 < p < \infty$, $s > 0$. Assume that $f \in \dot{W}^{1,p_1} \cap \dot{W}^{s,p_3}$ and $g \in L^{p_4} \cap \dot{W}^{s-1,p_2}$, then there exists constant C independent of $f, g$ such that*

$$\| \wedge^s (fg) - f \wedge^s g\|_p \leqslant C(\|\nabla f\|_{p_1} \|g\|_{\dot{W}^{s-1,p_2}} + \|f\|_{\dot{W}^{s,p_3}} \|g\|_{p_4}) \tag{2.3}$$

*with $p_2, p_3 \in (1, +\infty)$ such that*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

*Here $\wedge := (-\Delta)^{\frac{1}{2}}$ is defined through the Fourier transform, namely*

$$\widehat{\wedge f}(\xi) = |\xi| \hat{f}(\xi).$$

## 3. Proof of Theorem 1.1

In this section we devote to prove Theorem 1.1, and the proof will be divided into two steps.

Step I. $\|u\|_{H^1} + \|\nabla d\|_{H^1}$ estimate.

In this step, we will show:

$$\sup_{0<t<T} (\|\nabla u\|^2_2 + \|\Delta d\|^2_2) \leqslant C. \tag{3.1}$$

Firstly, we estimate $L^2$ norms of $u$ and $\nabla d$. Multiplying both sides of the first equation of (1.1) by $u$ and integrating over $\mathbb{R}^3$, and then applying the identity $\nabla\cdot(\nabla d\odot\nabla d)=\frac{1}{2}\nabla(|\nabla d|^2)+\Delta d\cdot\nabla d$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2+\|\nabla u\|_2^2 = -\int_{\mathbb{R}^3}\nabla d\cdot\Delta d\cdot u dx, \tag{3.2}$$

where we have used the fact $\operatorname{div}u=0$. Multiplying both sides of the second equation of (1.1) by $-\Delta d$ and integrating over $\mathbb{R}^3$, one has

$$\frac{1}{2}\frac{d}{dt}\|\nabla d\|_2^2+\|\Delta d\|_2^2 - \int_{\mathbb{R}^3}(u\cdot\nabla d)\cdot\Delta d dx=-\int_{\mathbb{R}^3}|\nabla d|^2 d\cdot\Delta d dx=\int_{\mathbb{R}^3}|d\cdot\Delta d|^2dx\leqslant\int_{\mathbb{R}^3}|\Delta d|^2dx, \tag{3.3}$$

where we have used the facts $|d|=1$ and $|\nabla d|^2=-d\cdot\Delta d$. Combining (3.2) and (3.3), and integrating in time, we get

$$\sup_{0<t<T}(\|u\|_2^2+\|\nabla d\|_2^2)+\int_0^T\|\nabla u\|_2^2dt\leqslant\|u_0\|_2^2+\|\nabla d_0\|_2^2.$$

Secondly, taking the inner product of $-\Delta u$ with the first equation of (1.1), by integrating by parts and using the incompressibility condition, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2+\|\Delta u\|_2^2\leqslant\int_{\mathbb{R}^3}(u\cdot\nabla)u\cdot\Delta u dx+\int_{\mathbb{R}^3}\nabla d\cdot\Delta d\cdot\Delta u dx. \tag{3.4}$$

For the estimate of $\Delta d$, taking $\Delta$ on both sides of the second equation of (1.1), multiplying $\Delta d$ and integrating over $\mathbb{R}^3$, one has

$$\frac{1}{2}\frac{d}{dt}\|\Delta d\|_2^2+\|\nabla\Delta d\|_2^2\leqslant-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta d dx+\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d dx. \tag{3.5}$$

Combining the estimates (3.4) and (3.5), we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_2^2+\|\Delta d\|_2^2)+\|\Delta u\|_2^2+\|\nabla\Delta d\|_2^2&\leqslant\int_{\mathbb{R}^3}u\cdot\nabla u\cdot\Delta u dx+\int_{\mathbb{R}^3}\nabla d\cdot\Delta d\cdot\Delta u dx\\&\quad-\int_{\mathbb{R}^3}\Delta(u\cdot\nabla d)\cdot\Delta d dx+\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta d dx\\&=I_1+I_2+I_3+I_4.\end{aligned} \tag{3.6}$$

In what follows, we will deal with each term on the right-hand side of (3.6) separately. The first term can be bounded by (2.1) as follows.

$$\begin{aligned}I_1=\int_{\mathbb{R}^3}u\cdot\nabla u\cdot\Delta u dx&\leqslant-\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u\cdot\nabla u\cdot\partial_k u dx\\&\leqslant C\|\omega\|_3^3\leqslant C\|\omega\|_{\dot{B}^{-r}_{\infty,\infty}}\|\omega\|^2_{\dot{H}^{\frac{r}{2}}}\\&\leqslant C\|\omega\|_{\dot{B}^{-r}_{\infty,\infty}}\|\nabla u\|_2^{2-r}\|\nabla\omega\|_2^r\\&\leqslant C\|\omega\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}}\|\nabla u\|_2^2+\frac{1}{6}\|\Delta u\|_2^2.\end{aligned} \tag{3.7}$$

To estimate $I_2, I_3, I_4$, we use (2.2) with $f=\Delta d$ and $|d|=1$, Young inequality and Hölder inequality and the following Gagliardo-Nirenberg inequality:

$$\|\nabla d\|_4^2\leqslant\|d\|_\infty\|\Delta d\|_2.$$

$$\begin{aligned} I_2 = \int_{\mathbb{R}^3} \nabla d \cdot \Delta d \cdot \Delta u dx &\leqslant \|\Delta u\|_2 \|\nabla d\|_4 \|\Delta d\|_4 \\ &\leqslant \frac{1}{6}\|\Delta u\|_2^2 + C\|d\|_\infty \|\Delta d\|_2 \|\nabla d\|_{\dot{B}^0_{\infty,\infty}} \|\nabla \Delta d\|_2 \\ &\leqslant \frac{1}{6}\|\Delta u\|_2^2 + \frac{1}{6}\|\nabla \Delta d\|_2^2 + C\|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}} \|\Delta d\|_2^2, \end{aligned} \tag{3.8}$$

$$\begin{aligned} I_3 = -\int_{\mathbb{R}^3} \Delta(u\cdot\nabla d)\Delta d dx &\leqslant \int_{\mathbb{R}^3} |\Delta u||\nabla d||\Delta d| dx + 2\int_{\mathbb{R}^3} |\nabla u||\nabla^2 d||\Delta d| dx \\ &\leqslant C(\|\Delta u\|_2 \|\nabla d\|_4 \|\Delta d\|_4 + \|\nabla u\|_2 \|\nabla^2 d\|_4 \|\Delta d\|_4) \\ &\leqslant C\|\Delta u\|_2 \|d\|_\infty^{\frac{1}{2}} \|\Delta d\|_2^{\frac{1}{2}} \|\nabla d\|^{\frac{1}{2}}_{\dot{B}^0_{\infty,\infty}} \|\nabla\Delta d\|_2^{\frac{1}{2}} + \|\nabla u\|_2 \|\nabla d\|_{\dot{B}^0_{\infty,\infty}} \|\nabla \Delta d\|_2 \\ &\leqslant \frac{1}{6}\|\Delta u\|_2^2 + \frac{1}{6}\|\nabla\Delta d\|_2^2 + C\|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}} (\|\nabla u\|_2^2 + \|\Delta d\|_2^2), \end{aligned} \tag{3.9}$$

where we have used the fact that $\operatorname{div} u = 0$ implies that $\int_{\mathbb{R}^3} u \cdot \nabla\Delta d \cdot \Delta d dx = 0$.

$$\begin{aligned} I_4 = \int_{\mathbb{R}^3} \Delta(|\nabla d|^2 d)\Delta d dx &\leqslant |\int_{\mathbb{R}^3} \nabla d|\nabla d|^2 \cdot \nabla\Delta d dx| + |\int_{\mathbb{R}^3} d\nabla(|\nabla d|^2)\cdot\nabla\Delta d dx| \\ &\leqslant |\int_{\mathbb{R}^3} \nabla(\nabla d|\nabla d|^2)\Delta d dx| + |\int_{\mathbb{R}^3} d\nabla(|\nabla d|^2)\cdot\nabla\Delta d dx| \\ &\leqslant C|\int_{\mathbb{R}^3} |\nabla d|^2|\nabla^2 d|^2 dx| + C|\int_{\mathbb{R}^3} |\nabla d||\nabla^2 d||\nabla\Delta d| dx| \\ &\leqslant C\|\nabla d\|_4^2 \|\nabla^2 d\|_4^2 + C\|\nabla d\|_4 \|\nabla^2 d\|_4 \|\nabla\Delta d\|_2 \\ &\leqslant \frac{1}{12}\|\nabla\Delta d\|_2^2 + C\|\nabla d\|_4^2 \|\nabla^2 d\|_4^2 \\ &\leqslant \frac{1}{6}\|\nabla\Delta d\|_2^2 + C\|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}} \|\Delta d\|_2^2. \end{aligned} \tag{3.10}$$

Plugging the estimates (3.7)-(3.10) back into (3.6) and absorbing the diffusive terms, one thus can deduce

$$\frac{d}{dt}(\|\nabla u\|_2^2 + \|\Delta d\|_2^2) + \frac{1}{2}(\|\Delta u\|_2^2 + \|\nabla\Delta d\|_2^2) \leqslant C(\|\omega\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})(\|\nabla u\|_2^2 + \|\Delta d\|_2^2).$$

By Gronwall's inequality, we have

$$\begin{aligned} &\|\nabla u\|_2^2 + \|\Delta d\|_2^2 + \int_0^T (\|\Delta u\|_2^2 + \|\nabla\Delta d\|_2^2) dt \\ &\leqslant (\|\nabla u_0\|_2^2 + \|\Delta d_0\|_2^2) \exp\left\{\int_0^T (\|\omega\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}) dt\right\} < C. \end{aligned}$$

Step II. $\|u\|_{H^2} + \|\nabla d\|_{H^2}$ estimate.

In the following, we will show how to deduce $\|u\|_{H^2} + \|\nabla d\|_{H^2}$ estimate from $\|u\|_{H^1} + \|\nabla d\|_{H^1}$ estimate obtained in Step I.

Taking the operation $\nabla^2$ on both sides of the first equation of (1.1), $\nabla^3$ on both sides of the second equation of (1.1), multiplying $(\nabla^2 u, \nabla^3 d)$ to the resulting equations, and integrating over $\mathbb{R}^3$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|\nabla^2 u\|_2^2 + \|\nabla^3 d\|_2^2) + \|\nabla^3 u\|_2^2 + \|\nabla^4 d\|_2^2 &\leqslant -\int_{\mathbb{R}^3} \nabla^2(u\cdot\nabla u)\nabla^2 u dx + \int_{\mathbb{R}^3} \nabla^2(\nabla d\cdot\Delta d)\nabla^2 u dx \\ &\quad -\int_{\mathbb{R}^3} \nabla^3(u\cdot\nabla d)\nabla^3 d dx + \int_{\mathbb{R}^3} \nabla^3(|\nabla d|^2 d)\nabla^3 d dx \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.11}$$

We estimate the right-hand side of (3.11) as follows.

For the first term, the commutator estimate (2.3) together with Gagliardo-Nirenberg inequality and Hölder inequality yield that

$$\begin{aligned} J_1 &= -\int_{\mathbb{R}^3} [\nabla^2(u\cdot\nabla u)-(u\cdot\nabla)\nabla^2 u]\nabla^2 u dx \\ &\leqslant C\|\nabla u\|_2\|\nabla^2 u\|_4^2 \\ &\leqslant C\|\nabla u\|_2\|\nabla^2 u\|_2^{\frac12}\|\nabla^3 u\|_2^{\frac32} \\ &\leqslant C\|\nabla u\|_2^4\|\nabla^2 u\|_2^2+\frac16\|\nabla^3 u\|_2^2, \end{aligned} \tag{3.12}$$

where we have applied the inequality $\|\nabla^2 u\|_4 \leqslant \|\nabla^2 u\|_2^{\frac14}\|\nabla^3 u\|_2^{\frac34}$. Applying (3.1), the second term $J_2$ can be estimated as

$$\begin{aligned} J_2 &= -\int_{\mathbb{R}^3} \nabla(\nabla d\cdot\Delta d)\cdot\nabla^3 u dx \\ &\leqslant C\int_{\mathbb{R}^3}|\nabla^2 d|^2|\nabla^3 u|dx+\int_{\mathbb{R}^3}|\nabla d||\nabla\Delta d||\nabla^3 u|dx \\ &\leqslant C\|\nabla^2 d\|_4^2\|\nabla^3 u\|_2+\|\nabla d\|_4\|\nabla\Delta d\|_4\|\nabla^3 u\|_2 \\ &\leqslant C(\|\nabla^2 d\|_4^4+\|\nabla d\|_4^2\|\nabla\Delta d\|_4^2)+\frac16\|\nabla^3 u\|_2^2 \\ &\leqslant C(\|\nabla^2 d\|_2^{\frac52}\|\nabla^4 d\|_2^{\frac32}+\|\nabla d\|_2^{\frac12}\|\nabla^2 d\|_2^{\frac32}\|\nabla^3 d\|_2^{\frac12}\|\nabla^4 d\|_2^{\frac32})+\frac16\|\nabla^3 u\|_2^2 \\ &\leqslant C(\|\nabla^2 d\|_2^{10}+\|\nabla^2 d\|_2^6\|\nabla^3 d\|_2^2)+\frac16\|\nabla^4 d\|_2^2+\frac16\|\nabla^3 u\|_2^2, \end{aligned} \tag{3.13}$$

where we have used the following Sobolev interpolation inequalities:

$$\|\nabla^2 d\|_4^4 \leqslant \|\nabla^2 d\|_2^{\frac52}\|\nabla^4 d\|_2^{\frac32}, \quad \|\nabla^3 d\|_4^2 \leqslant \|\nabla^3 d\|_2^{\frac12}\|\nabla^4 d\|_2^{\frac32}. \tag{3.14}$$

Again, with the aid of the commutator estimate (2.3), Young inequality and Hölder inequality, we may conclude that

$$\begin{aligned} J_3 &= -\int_{\mathbb{R}^3}[\nabla^3(u\cdot\nabla d)-u\cdot\nabla\nabla^3 d]\nabla^3 d dx \\ &\leqslant C\|\nabla u\|_2\|\nabla^3 d\|_4^2+\|\nabla^3 u\|_2\|\nabla d\|_4\|\nabla^3 d\|_4 \\ &\leqslant C\|\nabla u\|_2\|\nabla^3 d\|_2^{\frac12}\|\nabla^4 d\|_2^{\frac32}+\frac16\|\nabla^3 u\|_2^2+\|\nabla d\|_4^2\|\nabla^3 d\|_4^2 \\ &\leqslant C\|\nabla u\|_2^4\|\nabla^3 d\|_2^2+\frac1{12}\|\nabla^4 d\|_2^2+\frac16\|\nabla^3 u\|_2^2+C\|\nabla^2 d\|_2^6\|\nabla^3 d\|_2^2+\frac1{12}\|\nabla^4 d\|_2^2 \\ &\leqslant C(\|\nabla u\|_2^4+\|\nabla^2 d\|_2^6)(\|\nabla^2 u\|_2^2+\|\nabla^3 d\|_2^2)+\frac16\|\nabla^3 u\|_2^2+\frac16\|\nabla^4 d\|_2^2, \end{aligned} \tag{3.15}$$

where we have used (3.14) and the following Sobolev interpolation inequality:

$$\|\nabla d\|_4 \leqslant \|\nabla d\|_2^{\frac14}\|\nabla^2 d\|_2^{\frac34}. \tag{3.16}$$

Finally, we move to estimate the last term $J_4$, by using the facts that $|d|=1$ and $|\nabla d|^2=-d\cdot\Delta d$, we obtain

$$\begin{aligned} J_4 &= -\int_{\mathbb{R}^3}\Delta(|\nabla d|^2 d)\cdot\Delta^2 d dx \\ &= -\int_{\mathbb{R}^3}[\Delta(|\nabla d|^2)d+2\nabla(|\nabla d|^2)\nabla d+|\nabla d|^2\Delta d]\cdot\Delta^2 d dx \\ &\leqslant \{C(\||\nabla^2 d|^2\|_2+\|\nabla d\cdot\nabla^3 d\|_2+\||\nabla d|^2\cdot|\nabla^2 d|\|_2)+\|d|\Delta d|^2\|_2\}\cdot\|\Delta^2 d\|_2 \\ &\leqslant C(\|\nabla d\|_4\|\nabla^3 d\|_4+\|\nabla^2 d\|_4^2)\|\nabla^4 d\|_2 \\ &\leqslant C(\|\nabla d\|_2^{\frac12}\|\nabla^2 d\|_2^{\frac34}\|\nabla^3 d\|_2^{\frac14}\|\nabla^4 d\|_2^{\frac34}+\|\nabla^2 d\|_2^{\frac54}\|\nabla^4 d\|_2^{\frac34})\|\nabla^4 d\|_2 \\ &\leqslant C(\|\nabla^2 d\|_2^6\|\nabla^3 d\|_2^2+\|\nabla^2 d\|_2^{10})+\frac16\|\nabla^4 d\|_2^2. \end{aligned} \tag{3.17}$$

Inserting (3.12)-(3.13) and (3.15)-(3.17) into (3.11), it yields that

$$\frac{1}{2}\frac{d}{dt}(\|\nabla^2 u\|_2^2+\|\nabla^3 d\|_2^2)+\|\nabla^3 u\|_2^2+\|\nabla^4 d\|_2^2 \leqslant C(\|\nabla u\|_2^4+\|\nabla^2 d\|_2^6)(\|\nabla^2 u\|_2^2+\|\nabla^3 d\|_2^2)+C\|\nabla^2 d\|_2^{10} + \frac{1}{2}\|\nabla^3 u\|_2^2+\frac{1}{2}\|\nabla^4 d\|_2^2.$$

Hence, by the Gronwall's inequality, we have

$$\begin{aligned}&(\|\nabla^2 u\|_2^2+\|\nabla^3 d\|_2^2)+\int_0^T(\|\nabla^3 u\|_2^2+\|\nabla^4 d\|_2^2)dt\\ &\leqslant \left\{\|\nabla^2 u_0\|_2^2+\|\nabla^3 d_0\|_2^2+C\int_0^T\|\nabla^2 d\|_2^{10}dt\right\}\exp\left\{T\sup_{0<t<T}\|\nabla u(t)\|_2^4+\|\nabla^2 d(t)\|_2^6\right\}.\end{aligned}$$

By using the Sobolev embedding $H^2 \hookrightarrow L^\infty(\mathbb{R}^3)$, we can obtain the BKM's criterion (1.3) immediately, which completes the proof of Theorem 1.1.

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