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A high-accuracy conservative difference approximation for Rosenau-KdV equation

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Abstract

In this paper, we study the initial-boundary value problem of Rosenau-KdV equation. A conservative two level nonlinear Crank-Nicolson difference scheme, which has the theoretical accuracy $O(\tau^2 + h^4)$, is proposed. The scheme simulates two conservative properties of the initial boundary value problem. Existence, uniqueness, and priori estimates of difference solution are obtained. Furthermore, we analyze the convergence and unconditional stability of the scheme by the energy method. Numerical experiments demonstrate the theoretical results.

Keywords: Rosenau-KdV equation, finite difference scheme, conservative, convergence, stability.
2010 MSC: 65M06, 65N30.

1. Introduction

Consider the following initial-boundary value problem of Rosenau-KdV equation,

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0, \quad x \in (x_L, x_R), \quad t \in (0, T], \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in [x_L, x_R], \tag{1.2}$$

$$u(x_L, t) = u(x_R, t) = 0, \quad u_x(x_L, t) = u_x(x_R, t) = 0, \quad u_{xx}(x_L, t) = u_{xx}(x_R, t) = 0, \quad t \in [0, T], \tag{1.3}$$

where $u_0(x)$ is a known function.

In the study of the dynamics of dense discrete systems, Rosenau [22, 23] proposed the so-called Rosenau equation

$$u_t + u_{xxxxt} + u_x + uu_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0. \tag{1.4}$$

From then on, there are many studies about the existence, the uniqueness and numerical methods for the equation (1.4) (see [5, 6, 7, 14, 16, 18, 21]). As the further consideration of nonlinear wave, Zuo added viscous term to Rosenau equation (1.4) and obtained Rosenau-KdV equation (1.1). Zuo [30] also studied

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the solitary wave solutions and periodic solutions of (1.1). Recently, some researchers [8, 9] discussed the solitary solutions for the generalized Rosenau-KdV equation with usual power law on linearity. In [9], the author also gave the two invariants for the generalized Rosenau-KdV equation. In [29], authors proposed a conservative difference scheme for generalize Rosenau-Kdv equation. Meanwhile, they proved the two conservative laws by discrete energy method and provided numerical experiments.

Since the solitary wave solution for (1.1) is (see [30])

$$u(x,t)=\Big(-\frac{35}{24}+\frac{35}{312}\sqrt{313}\Big)\operatorname{sech}^4\Big[\frac{1}{24}\sqrt{-26+2\sqrt{313}}\Big(x-(\frac{1}{2}+\frac{1}{26}\sqrt{313})t\Big)\Big],$$

the physical boundary condition of Rosenau-KdV equation (1.1) satisfies

$$u(x,t)\to 0,\ \ u_x(x,t)\to 0,\ \ u_{xx}(x,t)\to 0,\ \ (t>0),\ \ |x|\to+\infty. \tag{1.5}$$

Hence, when $-x_L \gg 0$, $x_R \gg 0$, the homogeneous boundary condition (1.3) and the asymptotic condition (1.5) are consistent. The initial boundary value problem (1.1)-(1.3) possesses the following conservative properties (see [9, 12]),

$$Q(t)=\int_{x_L}^{x_R} u(x,t)dx=\int_{x_L}^{x_R} u_0(x)dx=Q(0), \tag{1.6}$$

$$E(t)=\|u\|_{L_2}^2+\|u_{xx}\|_{L_2}^2=E(0), \tag{1.7}$$

where $Q(0)$, $E(0)$ are both constants only depending on initial data.

Many analytic techniques [10, 26] are pretty reasonable methods to understand some nonlinear differential equation. But for models whose exact solutions hardly are found, the numerical method is an alternative choice. Especially, high-accuracy numerical algorithms [2, 3, 4, 12, 25], which maintain the conservative properties of the initial equation, could guarantee the validity of the numerical approximation. Because the high accuracy could guarantee the precision of the approximation, and the conservatives make the approximate solution reflect the physical phenomena better. Many numerical experiments show that conservative difference scheme can simulate the conservative law of initial problem well and it could avoid the nonlinear blow-up (see [13, 20, 19, 24, 27]). Moreover, the conservative scheme is more suitable for long time calculations for large time step. In [15], Li and Vu-Quoc said: "$\cdots$ in some areas, the ability to preserve some invariant properties of the original differential equation is a criterion to judge the success of a numerical simulation". Therefore, constructing a high-order conservative difference scheme is a significant task.

In [12], although a second-order three-level linear conservative difference scheme for problem (1.1)-(1.3) is proposed, it can not compute by itself. By the Richardson extrapolation, a new nonlinear Crank-Nicolson difference scheme, which has the accuracy of $O(\tau^2+h^4)$ without refined mesh is proposed in this paper. Moreover, the scheme can compute by itself and simulate two conservative quantities of the problem well. The existence, the uniqueness and prior estimates of the difference solution are obtained. Convergence and unconditionally stability are proved.

The rest of this paper is organized as follows. In Section 2, we propose a new finite difference scheme for the Rosenau-KdV equation. In Section 3, we prove the existence of the difference solutions by Browder fixed point Theorem and prior estimates are obtained. In Section 4, convergence and unconditionally stability are proved. Finally, some numerical tests are given in Section 5 to verify our theoretical analysis.

2. Finite difference scheme and conservative laws

Let h and τ be the uniform step size in the spatial and temporal direction respectively. Denote $h=\frac{x_R-x_L}{J}$, $x_j=x_L+jh$, $(j=-2,-1,0,1,2,\cdots,J-1,J,J+1,J+2)$, $t_n=n\tau$, $(n=0,1,\cdots,N,\ N=[\frac{T}{\tau}])$. Throughout this paper, we denote C as a general positive constant, which may have different values in different occurrences. Let $u_j^n \equiv u(x_j, t_n)$ be the exact solution of $u(x,t)$ at (x_j, t_n) and $U_j^n \approx u(x_j, t_n)$ be the approximation solution of $u(x,t)$ at (x_j, t_n), respectively. Let $e_j^n = u_j^n - U_j^n$ and define

$$Z_h^0 = \{U = (U_j) \,|\, U_{-2} = U_{-1} = U_0 = U_J = U_{J+1} = U_{J+2} = 0,\ j = -2, -1, 0, 1, 2, \cdots, J, J+1, J+2\}$$

$$(U_j^n)_x = \frac{U_{j+1}^n - U_j^n}{h},\quad (U_j^n)_{\bar{x}} = \frac{U_j^n - U_{j-1}^n}{h},\quad (U_j^n)_{\hat{x}} = \frac{U_{j+1}^n - U_{j-1}^n}{2h},$$

$$(U_j^n)_{\ddot{x}} = \frac{U_{j+2}^n - U_{j-2}^n}{4h},\quad (U_j^n)_t = \frac{U_j^{n+1} - U_j^n}{\tau},\quad U_j^{n+\frac{1}{2}} = \frac{U_j^{n+1} + u_j^n}{2},$$

$$\langle U^n, V^n\rangle = h\sum_{j=1}^{J-1} U_j^n V_j^n,\quad \|U^n\|^2 = \langle U^n, U^n\rangle,\quad \|U^n\|_\infty = \max_{1\leqslant j\leqslant J-1} |U_j^n|.$$

Consider the following finite difference scheme for problem (1.1)-(1.3)

$$\begin{gathered}(U_j^n)_t + \frac{5}{3}(U_j^n)_{xx\bar{x}\bar{x}t} - \frac{2}{3}(U_j^n)_{x\bar{x}\hat{x}\hat{x}t} + \frac{4}{3}(U_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{1}{3}(U_j^{n+\frac{1}{2}})_{\ddot{x}} + \frac{3}{2}(U_j^{n+\frac{1}{2}})_{x\bar{x}\hat{x}} - \frac{1}{2}(U_j^{n+\frac{1}{2}})_{x\bar{x}\ddot{x}} \\ + \frac{4}{9}\Big\{U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\hat{x}} + [(U_j^{n+\frac{1}{2}})^2]_{\hat{x}}\Big\} - \frac{1}{9}\Big\{U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\ddot{x}} + [(U_j^{n+\frac{1}{2}})^2]_{\ddot{x}}\Big\} = 0, \\ j = 1, 2, \cdots, J-1,\quad N = 1, 2, \cdots, N-1,\end{gathered} \tag{2.1}$$

$$U_j^0 = u_0(x_j),\quad j = 0, 1, 2, \cdots, J, \tag{2.2}$$

$$U^n \in Z_h^0,\quad (U_0^n)_{\hat{x}} = (U_J^n)_{\hat{x}},\quad (U_0^n)_{x\bar{x}} = (U_J^n)_{x\bar{x}},\quad n = 0, 1, 2, \cdots, N. \tag{2.3}$$

By the homogeneous boundary condition (1.3) and the asymptotic physical boundary condition (1.5), the discrete boundary condition (2.3) is reasonable. To analyze conveniently, define:

$$\phi(U_j^{n+\frac{1}{2}}) = \frac{4}{9}\Big\{U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\hat{x}} + [(U_j^{n+\frac{1}{2}})^2]_{\hat{x}}\Big\},$$

$$\kappa(U_j^{n+\frac{1}{2}}) = \frac{1}{9}\Big\{U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\ddot{x}} + [(U_j^{n+\frac{1}{2}})^2]_{\ddot{x}}\Big\}.$$

Lemma 2.1. *For any two discrete functions $U, V \in Z_h^0$, we have*

$$\langle U_x, V\rangle = -\langle U, V_{\bar{x}}\rangle,$$

and

$$\langle V, U_{x\bar{x}}\rangle = -\langle U_x, V_x\rangle,$$

from summation by parts (see [28]). Thus,

$$\langle U, U_{\hat{x}}\rangle = 0,\ \ \langle U, U_{\ddot{x}}\rangle = 0,\ \ \|U_x\|^2 = \|U_{\bar{x}}\|^2,\ \ \langle U_{x\bar{x}}, U\rangle = -\langle U_x, U_x\rangle = -\|U_x\|^2. \tag{2.4}$$

And if $(U_0)_{x\bar{x}} = (U_J)_{x\bar{x}} = 0$, then

$$\langle U_{xx\bar{x}\bar{x}}, U\rangle = \|U_{xx}\|^2.$$

Lemma 2.2 (See [17])**.** *For all $U \in Z_h^0$, by Cauchy-Schwarz inequality and summation by parts (see [25]), we have*

$$\|U_{\ddot{x}}\|^2 \leqslant \|U_{\hat{x}}\|^2 \leqslant \|U_x\|^2.$$

The following theorem shows how the difference scheme (2.1)-(2.3) simulates the conservative law (1.6) and (1.7) numerically.

Theorem 2.3. *The difference scheme* (2.1)-(2.3) *is conservative for the following discrete energy,*

$$Q^n = h\sum_{j=1}^{J-1} U_j^n = Q^{n-1} = \cdots = Q^0, \tag{2.5}$$

$$E^n = \|U^n\|^2 + \frac{5}{3}\|U_{xx}^n\|^2 - \frac{2}{3}\|U_{x\hat{x}}^n\|^2 = E^{n-1} = \cdots = E^0, \tag{2.6}$$

for $n = 1, 2, \cdots, N$.

Proof. Multiplying h in the two sides of (2.1) and summing up for j from 1 to $J-1$, from boundary (2.3) and Lemma 2.1, we obtain

$$h\sum_{j=1}^{J-1}(U_j^n)_t = 0. \tag{2.7}$$

From the definition of Q^n, (2.5) is deduced from (2.7).

Taking the inner product of (2.1) with $2U^{n+\frac{1}{2}}$, from boundary (2.3) and Lemma 2.1, we get

$$\begin{aligned}&\|U^n\|_t^2 + \frac{5}{3}\|U_{xx}^n\|_t^2 - \frac{2}{3}\|U_{x\hat{x}}^n\|_t^2 + \frac{8}{3}\langle U_{\hat{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle - \frac{2}{3}\langle U_{\ddot{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle + 3\langle U_{x\bar{x}\hat{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle \\ &-\langle U_{x\bar{x}\ddot{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle + 2\langle \phi(U^{n+\frac{1}{2}}), U^{n+\frac{1}{2}}\rangle - 2\langle \kappa(U^{n+\frac{1}{2}}), U^{n+\frac{1}{2}}\rangle = 0.\end{aligned} \tag{2.8}$$

Since

$$\langle U_{\hat{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle = 0, \ \ \langle U_{\ddot{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle = 0, \ \ \langle U_{x\bar{x}\hat{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle = 0, \ \ \langle U_{x\bar{x}\ddot{x}}^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\rangle = 0, \tag{2.9}$$

$$\begin{aligned}\langle \phi(U^{n+\frac{1}{2}}), U^{n+\frac{1}{2}}\rangle &= \frac{4}{9}h\sum_{j=1}^{J-1}\left\{U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\hat{x}} + [(U_j^{n+\frac{1}{2}})^2]_{\hat{x}}\right\}U_j^{n+\frac{1}{2}} \\ &= \frac{4}{9}h\sum_{j=1}^{J-1}(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\hat{x}} + \frac{4}{9}h\sum_{j=1}^{J-1}[(U_j^{n+\frac{1}{2}})^2]_{\hat{x}}U_j^{n+\frac{1}{2}} \\ &= \frac{4}{9}h\sum_{j=1}^{J-1}(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{4}{9}h\sum_{j=1}^{J-1}(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\hat{x}} \\ &= 0,\end{aligned} \tag{2.10}$$

$$\begin{aligned}\langle \kappa(U^{n+\frac{1}{2}}), U^{n+\frac{1}{2}}\rangle &= \frac{1}{9}\sum_{j=1}^{J-1}\left\{U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\ddot{x}} + [(U_j^{n+\frac{1}{2}})^2]_{\ddot{x}}\right\}U_j^{n+\frac{1}{2}} \\ &= \frac{1}{9}h\sum_{j=1}^{J-1}(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\ddot{x}} + \frac{1}{9}h\sum_{j=1}^{J-1}[(U_j^{n+\frac{1}{2}})^2]_{\ddot{x}}U_j^{n+\frac{1}{2}} \\ &= \frac{1}{9}h\sum_{j=1}^{J-1}(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\ddot{x}} - \frac{1}{9}h\sum_{j=1}^{J-1}(U_j^{n+\frac{1}{2}})^2(U_j^{n+\frac{1}{2}})_{\ddot{x}} \\ &= 0.\end{aligned} \tag{2.11}$$

Substituting (2.9), (2.10), (2.11) into (2.8), we get

$$(\|U^{n+1}\|^2 - \|U^n\|^2) + \frac{5}{3}(\|U_{xx}^{n+1}\|^2 - \|U_{xx}^n\|^2) - \frac{2}{3}(\|U_{x\hat{x}}^{n+1}\|^2 - \|U_{x\hat{x}}^n\|^2) = 0. \tag{2.12}$$

From the definition of E^n, we get (2.6) by deducing (2.12).

□

3. Solvability and prior estimation

In order to prove the solvability of difference scheme, we present the following Browder fixed point theorem [1].

Lemma 3.1 (Browder fixed point theorem)**.** *Let H be a finite dimensional inner product space. Suppose that $g : H \to H$ is a continuous operator and there exists $\alpha > 0$ such that $\langle g(x), x\rangle > 0$ for all $x \in H$ with $\|x\| = \alpha$. Then there exists $x^* \in H$ such that $g(x^*) = 0$ and $\|x^*\| \leqslant \alpha$.*

Theorem 3.2. *There exists $U^n \in Z_h^0 (1 \leqslant n \leqslant N)$ which satisfies difference scheme (2.1)-(2.3).*

Proof. We use the mathematical induction to prove our theorem. From (2.2), the difference solution exists for $n = 0$. Suppose that there exist $U^0, U^1, \cdots, U^n$ satisfy difference scheme (2.1)-(2.3) for $n \leqslant N-1$. Next we prove there exists U^{n+1} satisfying difference scheme (2.1)-(2.3).

Let g be an operator on Z_h^0 defined by

$$\begin{aligned} g(v) = 2v - 2U^n &+ \frac{10}{3} v_{xx\bar{x}\bar{x}} - \frac{10}{3} U^n_{xx\bar{x}\bar{x}} - \frac{4}{3} v_{x\bar{x}\hat{x}\hat{x}} + \frac{4}{3} U^n_{x\bar{x}\hat{x}\hat{x}} + \frac{4}{3}\tau v_{\hat{x}} - \frac{1}{3}\tau v_{\ddot{x}} \\ &+ \frac{3}{2} v_{x\bar{x}\hat{x}} - \frac{1}{2}\tau v_{x\bar{x}\ddot{x}} + \tau\phi(v) - \tau\kappa(v). \end{aligned} \tag{3.1}$$

Taking the inner product of (3.1) with v, similar to (2.9), (2.10), (2.11), we have

$$\langle v_{\hat{x}}, v\rangle = 0, \quad \langle v_{\ddot{x}}, v\rangle = 0, \quad \langle v_{x\bar{x}\hat{x}}, v\rangle = 0, \quad \langle v_{x\bar{x}\ddot{x}}, v\rangle = 0, \quad \langle \phi(v), v\rangle = 0, \quad \langle \kappa(v), v\rangle = 0,$$

and

$$\langle g(v), v\rangle = 2\|v\|^2 - 2\langle U^n, v\rangle + \frac{10}{3}\|v_{xx}\|^2 - \frac{10}{3}\langle U^n_{xx}, v_{xx}\rangle - \frac{4}{3}\|v_{x\hat{x}}\|^2 + \frac{4}{3}\langle U^n_{x\hat{x}}, v_{x\hat{x}}\rangle. \tag{3.2}$$

By Lemma 2.1 and Lemma 2.2, we obtain easily that

$$\langle U^n_{x\hat{x}}, v_{x\hat{x}}\rangle = \frac{1}{4}\langle U^n_{xx}, v_{xx}\rangle + \frac{1}{4}\langle U^n_{xx}, v_{x\bar{x}}\rangle + \frac{1}{4}\langle U^n_{x\bar{x}}, v_{xx}\rangle + \frac{1}{4}\langle U^n_{x\bar{x}}, v_{x\bar{x}}\rangle, \tag{3.3}$$

$$\|U^n_{x\bar{x}}\|^2 = \|U^n_{xx}\|^2, \quad \|v_{x\bar{x}}\|^2 = \|v_{xx}\|^2, \quad \|v^n_{x\hat{x}}\|^2 \leqslant \|v_{xx}\|^2. \tag{3.4}$$

Substituting (3.3) into (3.2), by Cauchy-Schwarz inequality and (3.4), we have

$$\begin{aligned} \langle g(v), v\rangle &\geqslant 2\|v\|^2 - 2\langle U^n, v\rangle + 2\|v_{xx}\|^2 - 3\langle U^n_{xx}, v_{xx}\rangle + \frac{1}{3}\langle U^n_{xx}, v_{x\bar{x}}\rangle + \frac{1}{3}\langle U^n_{x\bar{x}}, v_{xx}\rangle + \frac{1}{3}\langle U^n_{x\bar{x}}, v_{x\bar{x}}\rangle \\ &\geqslant 2\|v\|^2 - (\|U^n\|^2 + \|v\|^2) + 2\|v_{xx}\|^2 - \frac{3}{2}(\|U^n_{xx}\|^2 + \|v_{xx}\|^2) - \frac{1}{2}(\|U^n_{xx}\|^2 + \|v_{xx}\|^2) \\ &\geqslant \|v\|^2 - (\|U^n\| + 2\|U^n_{xx}\|^2). \end{aligned}$$

Therefore, for any $v \in Z_h^0$, if $\|v\|^2 = \|U^n\|^2 + 2\|U^n_{xx}\|^2 + 1$, then $\langle g(v), v\rangle > 0$. From Lemma 3.1, there exists $v^* \in Z_h^0$ such that $g(v^*) = 0$. Let $U^{n+1} = 2v^* - U^n$, and U^{n+1} is the solution of difference scheme (2.1)-(2.3). □

Next, we present some prior estimates for the solutions of difference scheme (2.1)-(2.3).

Theorem 3.3. *Suppose $u_0 \in H_0^2[x_L, x_R]$, then the solutions of difference scheme (2.1)-(2.3) satisfy:*

$$\|U^n\| \leqslant C, \quad \|U^n_x\| \leqslant C, \quad \|U^n_{xx}\| \leqslant C, \quad \|U^n\|_\infty \leqslant C, \quad \|U^n_x\|_\infty \leqslant C, \quad (n = 1, 2, \cdots, N).$$

Proof. It follows from Lemma 2.2 that

$$\|U^n_{x\hat{x}}\|^2 \leqslant \|U^n_{xx}\|^2.$$

By Theorem 2.3, we have

$$\|U^n\|^2 + \|U^n_{xx}\|^2 \leqslant E^n = \|U^n\|^2 + \frac{5}{3}\|U^n_{xx}\|^2 - \frac{2}{3}\|U^n\|^2 = E^0 = C,$$

that is,

$$\|U^n\| \leqslant C, \|U^n_{xx}\| \leqslant C.$$

From (2.4) and Cauchy-Schwarz inequality, we get

$$\|U_x^n\|^2 \leqslant \|U^n\|.\|U_{xx}^n\| \leqslant \frac{1}{2}(\|U^n\|^2 + \|U_{xx}^n\|^2), \tag{3.5}$$

that is,

$$\|U_x^n\| \leqslant C.$$

Finally, by discrete Sobolev inequality [28], we have $\|U^n\|_\infty \leqslant C$, $\|U_x^n\|_\infty \leqslant C$. □

4. Convergence, stability and uniqueness of solution

The truncation error of difference scheme (2.1)-(2.3) is

$$\begin{aligned} r_j^n = (u_j^n)_t + \frac{5}{3}(u_j^n)_{xx\bar{x}\bar{x}t} - \frac{2}{3}(u_j^n)_{x\bar{x}\hat{x}\hat{x}t} + \frac{4}{3}(u_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{1}{3}(u_j^{n+\frac{1}{2}})_{\ddot{x}} + \frac{3}{2}(u_j^{n+\frac{1}{2}})_{x\bar{x}\hat{x}} \\ - \frac{1}{2}(u_j^{n+\frac{1}{2}})_{x\bar{x}\ddot{x}} + \phi(u_j^{n+\frac{1}{2}}) - \kappa(u_j^{n+\frac{1}{2}}), \quad j = 1, 2, \cdots, J-1, \quad n = 1, 2, \cdots, N-1, \end{aligned} \tag{4.1}$$

$$u_j^0 = u_0(x), \quad j = 1, 2, \cdots, J-1, \tag{4.2}$$

$$u_n \in Z_h^0, (u_0^n)_{\hat{x}} = (u_J^n)_{\hat{x}}, \quad (u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}}, \quad n = 0, 1, 2, \cdots, N. \tag{4.3}$$

By Taylor expansion, as $h, \tau \to 0$,

$$|r_j^n| = O(\tau^2 + h^4). \tag{4.4}$$

Lemma 4.1 (See [12]). *Suppose $u_0 \in H_0^2[x_L, x_R]$, then the solution of problem (1.1)-(1.3) satisfies:*

$$\|u\|_{L_2} \leqslant C, \quad \|u_x\|_{L_2} \leqslant C, \quad \|u_{xx}\|_{L_2} \leqslant C, \quad \|u\|_{L_\infty} \leqslant C, \quad \|u_x\|_{L_\infty} \leqslant C.$$

Theorem 4.2. *Suppose $u_0 \in H_0^2[x_L, x_R]$, then the solution U^n of difference scheme (2.1)-(2.3) converges to the solution of problem (1.1)-(1.3) in the sense of norm $\|\cdot\|_\infty$, and the convergent rate is $O(\tau^2 + h^4)$.*

Proof. Subtracting (4.1), (4.2), (4.3) from (2.1)-(2.3), we have

$$\begin{aligned} r_j^n = (e_j^n)_t + \frac{5}{3}(e_j^n)_{xx\bar{x}\bar{x}t} - \frac{2}{3}(e_j^n)_{x\bar{x}\hat{x}\hat{x}t} + \frac{4}{3}(e_j^{n+\frac{1}{2}})_{\hat{x}} - \frac{1}{3}(e_j^{n+\frac{1}{2}})_{\ddot{x}} + \frac{3}{2}(e_j^{n+\frac{1}{2}})_{x\bar{x}\hat{x}} - \frac{1}{2}(e_j^{n+\frac{1}{2}})_{x\bar{x}\ddot{x}} \\ + \phi(u_j^{n+\frac{1}{2}}) - \phi(U_j^{n+\frac{1}{2}}) - \kappa(u_j^{n+\frac{1}{2}}) + \kappa(U_j^{n+\frac{1}{2}}), \quad j = 1, 2, \cdots, J-1, \quad n = 1, 2, \cdots, N-1, \end{aligned} \tag{4.5}$$

$$e_j^0 = 0, \quad j = 1, 2, \cdots, J-1, \tag{4.6}$$

$$e^n \in Z_h^0, \quad (e_0^n)_{\hat{x}} = (e_J^n)_{\hat{x}}, \quad (e_0^n)_{x\bar{x}} = (e_J^n)_{x\bar{x}}, \quad n = 0, 1, 2, \cdots, N. \tag{4.7}$$

Taking the inner product of (4.5) with $2e^{n+\frac{1}{2}}$, from (4.7) and Lemma 2.1, we get

$$\begin{aligned} \langle r^n, 2e^{n+\frac{1}{2}} \rangle = \|e^n\|_t^2 + \frac{5}{3}\|e_{xx}^n\|_t^2 - \frac{2}{3}\|e_{x\hat{x}}^n\|_t^2 + \frac{8}{3}\langle e_{\hat{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle - \frac{2}{3}\langle e_{\ddot{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle \\ + 3\langle e_{x\bar{x}\hat{x}}^{n+\frac{1}{2}}), e^{n+\frac{1}{2}} \rangle - \langle e_{x\bar{x}\ddot{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle + 2\langle \phi(u^{n+\frac{1}{2}}) - \phi(U^{n+\frac{1}{2}}), e^{n+\frac{1}{2}} \rangle \\ - 2\langle \kappa(u^{n+\frac{1}{2}}) - \kappa(U^{n+\frac{1}{2}}), e^{n+\frac{1}{2}} \rangle. \end{aligned} \tag{4.8}$$

Similar to (2.9), we have

$$\langle e_{\hat{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle = 0, \quad \langle e_{\ddot{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle = 0, \quad \langle e_{x\bar{x}\hat{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle = 0, \quad \langle e_{x\bar{x}\ddot{x}}^{n+\frac{1}{2}}, e^{n+\frac{1}{2}} \rangle = 0. \tag{4.9}$$

From Theorem 3.3, Lemma 4.1, Lemma 2.2 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\langle \phi(u^{n+\frac{1}{2}})-\phi(U^{n+\frac{1}{2}}), e^{n+\frac{1}{2}}\rangle \\
&= \frac{4h}{9}\sum_{j=1}^{J-1}[u_j^{n+\frac{1}{2}}(u_j^{n+\frac{1}{2}})_{\hat{x}} - U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\hat{x}}]e_j^{n+\frac{1}{2}} + \frac{4h}{9}\sum_{j=1}^{J-1}[(u_j^{n+\frac{1}{2}})^2-(U_j^{n+\frac{1}{2}})^2]_{\hat{x}}e_j^{n+\frac{1}{2}} \\
&= \frac{4h}{9}\sum_{j=1}^{J-1}[u_j^{n+\frac{1}{2}}(e_j^{n+\frac{1}{2}})_{\hat{x}} + e_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\hat{x}}]e_j^{n+\frac{1}{2}} - \frac{4h}{9}\sum_{j=1}^{J-1}[e_j^{n+\frac{1}{2}}(u_j^{n+\frac{1}{2}}+U_j^{n+\frac{1}{2}})](e_j^{n+\frac{1}{2}})_{\hat{x}} \\
&\leqslant C(\|e^{n+1}\|^2+\|e^n\|^2+\|e_{\hat{x}}^{n+1}\|^2+\|e_{\hat{x}}^n\|^2) \\
&\leqslant C(\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2),
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\langle \kappa(u^{n+\frac{1}{2}})+\kappa(U^{n+\frac{1}{2}}), e^{n+\frac{1}{2}}\rangle \\
&= \frac{h}{9}\sum_{j=1}^{J-1}[u_j^{n+\frac{1}{2}}(u_j^{n+\frac{1}{2}})_{\check{x}} - U_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\check{x}}]e_j^{n+\frac{1}{2}} + \frac{h}{9}\sum_{j=1}^{J-1}[(u_j^{n+\frac{1}{2}})^2-(U_j^{n+\frac{1}{2}})^2]_{\check{x}}e_j^{n+\frac{1}{2}} \\
&= \frac{h}{9}\sum_{j=1}^{J-1}[u_j^{n+\frac{1}{2}}(e_j^{n+\frac{1}{2}})_{\check{x}} + e_j^{n+\frac{1}{2}}(U_j^{n+\frac{1}{2}})_{\check{x}}]e_j^{n+\frac{1}{2}} - \frac{h}{9}\sum_{j=1}^{J-1}[e_j^{n+\frac{1}{2}}(u_j^{n+\frac{1}{2}}+U_j^{n+\frac{1}{2}})](e_j^{n+\frac{1}{2}})_{\check{x}} \\
&\leqslant C(\|e^{n+1}\|^2+\|e^n\|^2+\|e_{\check{x}}^{n+1}\|^2+\|e_{\check{x}}^n\|^2) \\
&\leqslant C(\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2),
\end{aligned} \tag{4.11}$$

$$\langle r^n, 2e^{n+\frac{1}{2}}\rangle = \langle r^n, e^{n+1}+e^n\rangle \leqslant \|r^n\|^2+\|e^{n+1}\|^2+\|e^n\|^2. \tag{4.12}$$

Substituting (4.9), (4.10), (4.11), (4.12) into (4.8), we have

$$\|e^n\|_t^2+\frac{5}{3}\|e_{xx}^n\|_t^2-\frac{2}{3}\|e_{x\hat{x}}^n\|_t^2 \leqslant \|r^n\|^2+C(\|e^{n+1}\|^2+\|e^n\|^2+\|e_x^{n+1}\|^2+\|e_x^n\|^2). \tag{4.13}$$

Similar to (3.5), we have

$$\|e_x^n\|^2 \leqslant \frac{1}{2}(\|e^n\|^2+\|e_{xx}^n\|^2), \quad \|e_x^{n+1}\|^2 \leqslant \frac{1}{2}(\|e^{n+1}\|^2+\|e_{xx}^{n+1}\|^2), \tag{4.14}$$

and (4.13) can be rewritten as

$$\|e^n\|_t^2+\frac{5}{3}\|e_{xx}^n\|_t^2-\frac{2}{3}\|e_{x\hat{x}}^n\|_t^2 \leqslant \|r^n\|^2+C(\|e^{n+1}\|^2+\|e^n\|^2+\|e_{xx}^{n+1}\|^2+\|e_{xx}^n\|^2). \tag{4.15}$$

Letting $B^n = \|e^n\|^2+\frac{5}{3}\|e_{xx}^n\|^2-\frac{2}{3}\|e_{x\hat{x}}^n\|^2$, and summing up (4.15) from 0 to $n-1$:

$$B^n \leqslant B^0 + C\tau\sum_{l=0}^{n-1}\|r^l\|^2 + C\tau\sum_{l=0}^{n}(\|e^l\|^2+\|e_{xx}^l\|^2). \tag{4.16}$$

By (4.4) and (4.6), we get

$$\tau\sum_{l=0}^{n-1}\|r^l\|^2 \leqslant n\tau \max_{0\leqslant l\leqslant n-1}\|r^l\|^2 \leqslant T\cdot O(\tau^2+h^4)^2,$$

$$B^0 = O(\tau^2+h^4)^2.$$

Similar to (3.4), we have

$$\|e^n_{x\hat{x}}\| \leqslant \|e^n_{xx}\|,$$

then it follows from (4.16) that

$$\|e^n\|^2 + \|e^n_{xx}\|^2 \leqslant B^n \leqslant O(\tau^2 + h^4)^2 + C\tau \sum_{l=0}^{n} (\|e^l\|^2 + \|e^l_{xx}\|^2).$$

By discrete Gronwall inequality [28], we obtain

$$\|e^n\| \leqslant O(\tau^2 + h^4), \quad \|e^n_{xx}\| \leqslant O(\tau^2 + h^4).$$

From (4.14), we have

$$\|e^n_x\| \leqslant O(\tau^2 + h^4).$$

Finally, by discrete Sobolev inequality [28], we get:

$$\|e^n\|_\infty \leqslant O(\tau^2 + h^4).$$

□

According to Theorem 4.2, we have the following theorems.

Theorem 4.3. *Under the condition of Theorem 4.2, the solution U^n of difference scheme (2.1)-(2.3) is stable in the sense of norm $\|\cdot\|_\infty$.*

Theorem 4.4. *The solution of difference scheme (2.1)-(2.3) is unique.*

5. Numerical simulations

The scheme (2.1)-(2.3) is a nonlinear system of equations which can be solved with Newton iteration. In our experiments, we take $x_L = -70$, $x_R = 100$, $T = 40$ and

$$u_0(x) = \Big(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\Big)\operatorname{sech}^4\Big(\frac{1}{24}\sqrt{-26 + 2\sqrt{313}}x\Big).$$

For some different values of τ and h, we list errors at several different time in Table 1 and verify the accuracy of the difference scheme in Table 2 by using the method of [11]. The numerical simulation of two conservative quantities (1.6) and (1.7) is listed in Table 3.

Table 1: The errors estimates of numerical solution with various h and τ.

	$\tau = 0.4, h = 0.2$		$\tau = h = 0.1$		$\tau = 0.025, h = 0.05$	
	$\|e^n\|$	$\|e^n\|_\infty$	$\|e^n\|$	$\|e^n\|_\infty$	$\|e^n\|$	$\|e^n\|_\infty$
$t = 10$	5.70350e-3	2.24638e-3	3.58600e-4	1.41368e-4	2.24209e-5	8.83941e-6
$t = 20$	1.04463e-2	3.92536e-3	6.57457e-4	2.47330e-4	4.11090e-5	1.54677e-5
$t = 30$	1.44616e-2	5.27150e-3	9.10990e-4	3.32470e-4	5.69649e-5	2.07909e-5
$t = 40$	1.80102e-2	6.42816e-3	1.13536e-3	4.05860e-4	7.09986e-5	2.53821e-5

Table 2: The verification of the convergence rate $O(\tau^2 + h^4)$.

	$\|e^n(h,\tau)\| \big/ \|e^{4n}(\frac{h}{2},\frac{\tau}{4})\|$			$\|e^n(h,\tau)\|_\infty \big/ \|e^{4n}(\frac{h}{2},\frac{\tau}{4})\|_\infty$		
	$\tau = 0.4$ $h = 0.2$	$\tau = 0.1$ $h = 0.1$	$\tau = 0.025$ $h = 0.05$	$\tau = 0.4$ $h = 0.2$	$\tau = 0.1$ $h = 0.1$	$\tau = 0.025$ $h = 0.05$
$t = 10$	–	15.9049	15.9939	–	15.8902	15.9929
$t = 20$	–	15.8890	15.9930	–	15.8709	15.9900
$t = 30$	–	15.8746	15.9921	–	15.8555	15.9911
$t = 40$	–	15.8629	15.9913	–	15.8383	15.9899

Table 3: Numerical simulations on the conservation invariant Q^n and E^n.

	$\tau = 0.1, h = 0.1$		$\tau = 0.025, h = 0.05$	
	Q^n	E^n	Q^n	E^n
$t = 0$	5.498173680817	1.984390174779	5.498173680817	1.989782937260
$t = 10$	5.498173679973	1.989782938890	5.498173679780	1.989782939868
$t = 20$	5.498173679905	1.989782938887	5.498173679065	1.989782939212
$t = 30$	5.498173660051	1.989782938891	5.498173676733	1.989782938309
$t = 40$	5.498174221000	1.989782938904	5.498173712324	1.989782938511

From these computational results, it shows that our proposed algorithm is efficient and reliable.

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