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Strong and weak convergence theorems for split equilibrium problems and fixed point problems in Banach spaces

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Abstract

In this paper, we give some strong and weak convergence algorithms to find a common element of the solution set of a split equilibrium problem and the fixed point set of a relatively nonexpansive mapping in Banach spaces. Our algorithms only involve the operator A itself and do not need any conditions of the adjoint operator A^* of A and the norm $\|A\|$ of A which are different from the other results in the literature. By applying our main results, we show the existence of a solution of a split feasibility problem in Banach spaces. Finally, we give an example to illustrate the main results of this paper.

Keywords: Split equilibrium problem, equilibrium problem, fixed point, Banach space.
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1. Introduction and preliminaries

Throughout this paper, let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{N}$ denote the set of all positive integers. Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction.

The equilibrium problem for F is to find $z \in C$ such that

$$F(z, y) \geqslant 0, \tag{1.1}$$

for all $y \in C$. The set of all solutions of the problem (1.1) is denoted by $EP(F)$, i.e.,

$$EP(F) = \{z \in C : F(z, y) \geqslant 0, \forall y \in C\}.$$

Many problems in physics, optimization, economics and others can be reduced to find a solution of the problem (1.1) and so the equilibrium problems have been investigated by many authors (see [5, 7, 8, 11–14, 16, 18, 20, 22–27, 30, 32] and the reference therein).

Recently, Kazmi and Rizvi [17] considered a problem, which is called a split equilibrium problem. Let H_1, H_2be two real Hilbert spaces and C, Q be nonempty closed and convex subsets of H_1 and H_2,

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respectively. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_2$ be a bounded linear operator. The split equilibrium problem is to find $x^* \in C$ such that

$$F_1(x^*, x) \geqslant 0, \quad \forall\, x \in C \quad \text{and} \quad y = Ax^* \in Q \quad \text{such that} \quad F_2(y^*, y) \geqslant 0, \quad \forall\, y \in Q. \tag{1.2}$$

Also, they introduced the following iterative algorithm to find a solution of the split equilibrium problem (1.2):

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = P_C(u_n - \lambda_n D u_n), \\ x_{n+1} = \alpha_n + \beta_n + \gamma_n S y_n, \end{cases} \tag{1.3}$$

for each $n \geqslant 1$, where $S : C \to C$ is a nonexpansive mapping, $D : C \to H_1$ is a τ-inverse strongly monotone mapping, A^* is the adjoint of A, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, 2\tau)$, $\{r_n\} \subset (0, \infty)$, $\gamma \in (0, \frac{1}{L})$, where L is the spectral radius of the operator A^*A. Under some suitable conditions on the control sequences, they proved some strong convergence theorems of the algorithm (1.3).

In 2014, Bnouhachem [2] introduced the following iterative method to solve the split equilibrium problem and hierarchical fixed point problem:

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n), \\ y_n = \beta_n S x_n + (1 - \beta_n) u_n, \\ x_{n+1} = P_C[\alpha_n \rho U x_n + (I - \alpha_n \mu F) T y_n], \end{cases} \tag{1.4}$$

for each $n \geqslant 1$, where S, T are two nonexpansive mappings and U is a Lipschitzian mapping and F is a Lipschitz and strongly monotone mapping and A is a bounded linear operator and A^* is the adjoint mapping of A, and proved some strong convergence theorems of the algorithm (1.4) under some certain conditions on the parameters.

In the algorithms (1.3) and (1.4), the bifunction F_2 is required to be upper semi-continuous in the first argument besides satisfying the conditions (A1)-(A4). In order to relax the restriction, Wang et al. [31] introduced a new iterative algorithm to solve the split equilibrium problem as follows:

$$\begin{cases} u_{i,n} = T_{r_n}^{F}(I - \gamma A_i^*(I - T_{r_n}^{F_i})A_i)x_n, \ i = 1, \cdots, N_1, \\ y_n = P_C\big(I - \lambda_n\big(\sum_{i=1}^{N_2} \gamma_i B_i\big)\big)\big(\frac{1}{N_1}\sum_{i=1}^{N_1} u_{i,n}\big), \\ x_{n+1} = \alpha_n v + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i) S_i y_n, \end{cases} \tag{1.5}$$

for each $n \geqslant 1$, where $F : C \times C \to \mathbb{R}$, $F_1, \cdots, F_{N_1} : Q \times Q \to \mathbb{R}$ are bifunctions, $A_1, \cdots, A_{N_1} : H_1 \to H_2$ are linear bounded operators, $B_1, \cdots, B_{N_2} : C \to H_1$ are inverse strongly monotone mappings, for each $i \geqslant 1$, $S_i : C \to C$ is nonexpansive mapping. Under some suitable conditions on the control sequences $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_n\}$, they proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to an element $z = P_\Theta v$, where $\Theta = \cap_{i=1}^{\infty} Fix(S_i) \cap \Gamma \cap \Omega$, where

$$\Gamma = \cap_{i=1}^{N_2} VI(C, B_i), \quad \Omega = \{z \in C : z \in EP(F), A_i z \in EP(F_i), i = 1, \cdots, N_1\}.$$

In fact, in the algorithm (1.5), the bifunctions $F_1, \cdots, F_{N_1}$ are not required to be upper semi-continuous in the first argument.

Recently, split feasibility problems [3, 4, 6, 9, 29, 34, 35], split variational inequality problems [10, 21] and split equilibrium problems [2, 17, 31] have been investigated by many authors. However, most of the results on these kinds of these problems are investigated only in Hilbert spaces, only a few works are considered in Banach spaces. So, in this paper, we consider some results on convergence analysis to solutions of these kinds of problems in Banach spaces.

Let E_1, E_2 be two Banach spaces and C, Q be nonempty closed convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a nonlinear operator. Let $F : C \times C \to \mathbb{R}$ and $H : Q \times Q \to \mathbb{R}$ be two bifunctions. Let Ω denote the set of solutions of the split equilibrium problem on F and H, that is,

$$\Omega = \{z \in C : z \in EP(F), \ Az \in EP(H)\}.$$

In fact, it is difficult to compute the adjoint A^* and the norm $\|A\|$ of $\|A\|$ if the operator A is complex, which is a common topic to solve.

In this paper, we introduce some new strong and weak convergence algorithms to find an element in $\Omega \cap F(S)$, where $F(S)$ is the set of fixed points of a relatively nonexpansive mapping in Banach space E. Our algorithms involve only the operator A, but do not use the adjoint A^* of the operator A and the norm $\|A\|$ of A and so our algorithms can be more convenient and effective to prove our main results. As applications of our main results, we can solve some split feasibility problems in Banach spaces. Finally, we give an example to illustrate the main results in this paper. Our results extend and improve the corresponding results of others in the literature.

2. Preliminaries

Let H be a Hilbert space and C be a nonempty closed subset of H. For any $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leqslant \|x - y\|,$$

for all $y \in C$. Such a mapping P_C is called the metric projection from H onto C. It is well-known that P_C is a firmly nonexpansive mapping from H onto C, i.e.,

$$\|P_C x - P_C y\|^2 \leqslant \langle P_C x - P_C y, x - y\rangle,$$

for all $x, y \in H$. Further, for any $x \in H$ and $z \in C$,

$$z = P_C x \iff \langle x - z, z - y\rangle \geqslant 0,$$

for all $y \in C$.

Let E be a Banach space and E^* be the topological dual space of E. For all $x \in E$ and $x^* \in E^*$, we denote the value of x^* at x by $\langle x, x^*\rangle$. It is known that the normalized duality mapping J on E is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\},$$

for each $x \in E$. Then $J(x)$ is nonempty.

A Banach space E is said to be strictly convex, if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex, if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leqslant 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geqslant \epsilon$. A Banach space E is said to be smooth, if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t},$$

exists for all $x, y \in S(E)$, where $S(E) = \{z \in E : \|z\| = 1\}$. E is said to be uniformly smooth, if the limit exists uniformly in $x, y \in S(E)$. If E is smooth, strictly convex and reflexive, then the duality mapping J is single-valued, one-to-one and onto.

Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Let ϕ be the function on $E \times E$ defined by

$$\phi(x, y) = \|y\|^2 - 2\langle x, Jy\rangle + \|x\|^2,$$

for all $x, y \in E$. The generalized projection Π_C [1] from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y, x),$$

for all $x \in E$. If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ and Π_C is the metric projection P of H onto C.

Let $S : C \to C$ be a nonlinear mapping. We denote the set of fixed points of S by $F(S)$. A point $p \in C$ is said to be an asymptotic fixed point of S, if there exists $\{x_n\}$ in C which converges weakly to p and $\lim_{n\to\infty} \|x_n - Sx_n\| = 0$. Denote the set of all asymptotic fixed points of S by $\hat{S}(S)$. The mapping S is said to be relatively nonexpansive [19], if the following conditions hold:

(1) $F(S)$ is nonempty;

(2) $\phi(p, Sx) \leqslant \phi(p, x)$ for all $p \in F(S)$ and $x \in C$;

(3) $\hat{F}(S) = F(S)$.

If E is a smooth, strictly convex and reflexive Banach space, then the set $F(S)$ of fixed points of the relatively nonexpansive mapping S is closed and convex [19].

Next, the following lemmas are used in the next section:

Lemma 2.1 ([28]). *Let C be a nonempty closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space E. Suppose that $F : C \times C \to \mathbb{R}$ satisfies the following conditions:*

(A1) $F(x, x) = 0$ *for all* $x \in C$;

(A2) F *is monotone, i.e.,* $F(x, y) + F(y, x) \leqslant 0$ *for all* $x, y \in C$;

(A3) *for each* $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leqslant F(x, y)$;

(A4) *for each* $x \in C$, $y \mapsto F(x, y)$ *is convex and lower semi-continuous.*

For any $x \in E$ and $r > 0$, define a mapping $T_r^F : E \to C$ by

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geqslant 0, \forall y \in C\}.$$

Then T_r^F is well-defined and the followings hold:

(1) T_r^F *is single-valued;*

(2) T_r^F *is firmly nonexpansive, i.e., for any* $x, y \in E$,

$$\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leqslant \langle T_r^F x - T_r^F y, Jx - Jy\rangle;$$

(3) $Fix(T_r^F) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 2.2 ([1, 15]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leqslant \phi(x, y),$$

for all $x \in C$ and $y \in E$.

Lemma 2.3 ([1, 15]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then, for any $x \in E$ and $z \in C$ we have*

$$z = \Pi_C x \Leftrightarrow \langle y - z, Jx - Jz\rangle \leqslant 0,$$

for all $y \in C$.

Lemma 2.4 ([15]). *Let E be a smooth and uniformly convex Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are the sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.5 ([33]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leqslant t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leqslant r\}$.

Lemma 2.6 ([15]). *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leqslant \phi(x, y),$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leqslant r\}$.

Lemma 2.7 ([28]). *Let C be a closed convex subset of a smooth, strictly convex and reflexive Banach space E, F be a bifunction from $C \times C \to \mathbb{R}$ satisfying the conditions (A1)-(A4) and let $r > 0$. Then, for any $x \in E$ and $q \in Fix(T_r^F)$,*

$$\phi(q, T_r^F x) + \phi(T_r^F x, x) \leqslant \phi(q, x).$$

3. Strong convergence theorems

Theorem 3.1. *Let E_1 be a uniformly smooth and uniformly convex Banach space and E_2 be a uniformly smooth, strictly convex and reflexive Banach space. Let $A : E_1 \to E_2$ be a linear and continuous operator. Let C and Q be nonempty closed convex subsets of E_1 and E_2, respectively. Let $S : C \to C$ be a relatively nonexpansive mapping and $F : C \times C \to \mathbb{R}$, $H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions (A1)-(A4) with $\Omega \cap F(S) \neq \emptyset$. Define an iterative scheme $\{x_n\}$ by the following manner:*

$$\begin{cases} \text{take } x_1 = x \in E_1, \text{ find } v \in E_1 \text{ such that } Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leqslant n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geqslant 0, \ \forall y \in C, \\ H(Az_n, Ay) + \dfrac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0, \ \forall y \in U_n, \\ y_n = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n), \\ C_n = \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ D_n = \cap_{i=1}^n C_i, \\ x_{n+1} = \Pi_{D_n} x, \end{cases} \tag{3.1}$$

for each $n \geqslant 1$, where $\{r_n\} \subset [r, \infty)$ with $r > 0$, $\{s_n\} \subset [s, \infty)$ with $s > 0$. Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to a point $\Pi_{\Omega\cap Fix(S)}x$, where $\Pi_{\Omega\cap F(S)}$ is the generalized projection of E_1 onto $\Omega \cap Fix(S)$.

Proof. First, we see that, for each $n \geqslant 1$, the sets V_n and U_n are nonempty closed and convex. Now, we show that, for each $n \geqslant 1$, D_n is closed and convex. Since

$$\phi(z, y_n) \leqslant \phi(z, x_n) \iff \|y_n\|^2 - \|x_n\|^2 - 2\langle z, Jy_n - Jx_n\rangle \leqslant 0,$$

each C_n is closed and convex and so each D_n is also closed and convex.

Let $G(x,y) = H(Ax, Ay)$ for all $x, y \in U_n$. Then each G is a bifunction from $U_n \times U_n$ into $\mathbb{R}$ satisfying (A1)-(A4), since A is linear and continuous. We rewrite

$$H(Az_n, Ay) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0,$$

as

$$G(z_n, y) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0,$$

for all $y \in U_n$. Let $p \in \Omega \cap \mathrm{Fix}(S)$. It follows that $p = T^F_{r_n} p$ and $H(Ap, z) \geqslant 0$ for all $z \in Q$. Since $U_n \subset Q$ and $Az \in Q$ for all $z \in U_n$, one has $H(Ap, Az) \geqslant 0$ for all $z \in U_n$. It follows that $G(p, z) \geqslant 0$ for all $z \in U_n$ and so $p = T^G_{s_n} p$. Note that $p \in C$, $u_n = T^F_{r_n} x_n$ and $z_n = T^G_{s_n} u_n$. By Lemma 2.2 and Lemma 2.7, we have $\phi(p, u_n) \leqslant \phi(p, x_n)$ and

$$\phi(p, S\Pi_C z_n) \leqslant \phi(p, \Pi_C z_n) \leqslant \phi(p, z_n) = \phi(p, T^G_{s_n} u_n) \leqslant \phi(p, u_n) \leqslant \phi(p, x_n). \tag{3.2}$$

Thus, by (3.1) and (3.2), we have

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\rangle + \|\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\|^2 \\
&\leqslant \|p\|^2 - 2\alpha_n\langle p, Ju_n\rangle - 2(1-\alpha_n)\langle p, JS\Pi_C z_n\rangle + \alpha_n\|u_n\|^2 + (1-\alpha_n)\|S\Pi_C z_n\|^2 \\
&= \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, S\Pi_C z_n) \\
&\leqslant \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) \\
&= \phi(p, x_n).
\end{aligned}$$

Thus $p \in C_n$ and further $p \in D_n$, for each $n \geqslant 1$. It follows that $\Omega \cap F(S) \subset D_n$ for each $n \geqslant 1$. Hence $\{x_n\}$ is well-defined. By the definitions of x_{n+1} and Π_{D_n}, we have

$$\phi(x_{n+1}, x) \leqslant \phi(z, x),$$

for all $z \in D_n$. Since $x^* = \Pi_{\Omega\cap F(s)} x \in \Omega \cap F(S) \subset D_n$, one has

$$\phi(x_{n+1}, x) \leqslant \phi(x^*, x),$$

and so $\{\phi(x_n, x)\}$ is bounded. Thus $\{x_n\}$ is bounded and so are $\{u_n\}$ and $\{z_n\}$. Since $x_{n+2} = \Pi_{D_{n+1}} x \in D_{n+1} \subset D_n$, we have

$$\phi(x_{n+1}, x) \leqslant \phi(x_{n+2}, x).$$

Thus the limit of $\{\phi(x_n, x)\}$ exists, since $\{\phi(x_n, x)\}$ is bounded. For each $m \geqslant 1$, since $x_{n+m} \in D_{n+m-1} \subset D_{n-1}$, by Lemma 2.2, we have

$$\begin{aligned}
\phi(x_{n+m}, x_n) &= \phi(x_{n+m}, \Pi_{D_{n-1}} x) \\
&\leqslant \phi(x_{n+m}, x) - \phi(x_n, x).
\end{aligned}$$

Since the limit of $\{\phi(x_n, x)\}$ exists, it follows that

$$\lim_{n\to\infty} \phi(x_{n+m}, x_n) = 0, \tag{3.3}$$

for each $m \geqslant 1$. From Lemma 2.4, it follows that

$$\lim_{n\to\infty} \|x_n - x_{n+m}\| = 0, \tag{3.4}$$

for each $m \geqslant 1$. Thus the sequence $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $q \in C$ such that $x_n \to q$ as $n \to \infty$. From (3.3), we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

On the other hand, from (3.1), it follows that

$$\phi(x_{n+1}, z_n) + \phi(x_{n+1}, u_n) \leqslant 2\phi(x_{n+1}, x_n),$$

which, with (3.5), implies that

$$\lim_{n\to\infty} \phi(z_n, x_{n+1}) = \lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

By Lemma 2.4, we have

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = \lim_{n\to\infty} \|u_n - x_{n+1}\| = 0. \tag{3.6}$$

Combining (3.4) with $x_m = x_{n+1}$ and (3.6), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \|u_n - x_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jz_n - Jx_n\| = \lim_{n\to\infty} \|Ju_n - Jx_n\| = 0.$$

Therefore, we have

$$u_n \to q, \quad z_n \to q, \quad \text{as } n \to \infty.$$

Now, we prove that $q = \Pi_{\Omega\cap F(S)}x$. Since $x_{n+1} = \Pi_{D_n}x$ and $\Omega \cap F(S) \subset D_n$, by Lemma 2.3, we have

$$\langle y - x_{n+1}, Jx - Jx_{n+1}\rangle \leqslant 0, \tag{3.7}$$

for all $y \in \Omega \cap F(S)$. Letting $n \to \infty$ in (3.7) and noting that $x_n \to q$, we have

$$\langle y - q, Jx - Jq\rangle \leqslant 0,$$

for all $y \in \Omega \cap F(S)$, which from Lemma 2.3, implies that

$$q = \Pi_{\Omega\cap F(S)}x.$$

This completes the proof. □

If $E_1 = E_2$, $C = Q$ and $A = I$ (the identity mapping) in Theorem 3.1, by the similar proof, we have the following:

Corollary 3.2. *Let* E *be a uniformly smooth and uniformly convex Banach spaces and* C *be a nonempty closed convex subset of* E. *Let* $S : C \to C$ *be a relatively nonexpansive mapping and* $F, H : C \times C \to \mathbb{R}$ *be two bifunctions satisfying the conditions* (A1)-(A4) *with* $EP(F) \cap EP(H) \cap F(S) \neq \emptyset$. *Define an iterative scheme* $\{x_n\}$ *by the following manner:*

$$\begin{cases} x_1 = x \in E_1, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geqslant 0, \ \forall y \in C, \\ H(z_n, y) + \dfrac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0, \ \forall y \in C, \\ y_n = J^{-1}(\alpha_n u_n + (1-\alpha_n)JSz_n), \\ C_n = \{v \in C : \phi(v, y_n) \leqslant \phi(v, x_n)\}, \\ D_n = \cap_{i=1}^n C_i, \\ x_{n+1} = P_{D_n}x, \end{cases} \tag{3.8}$$

for each $n \geqslant 1$, *where* $\{r_n\} \subset [r, \infty)$ *with* $r > 0$ *and* $\{s_n\} \subset [s, \infty)$ *with* $s > 0$. *Then the sequence* $\{x_n\}$ *defined by*

(3.8) *converges strongly to the point* $\Pi_{EP(F)\cap EP(H)\cap F(S)}x$.

Corollary 3.3. *Let* E *be a uniformly smooth and uniformly convex Banach space and* C *be a nonempty closed convex subset of* E. *Let* $S : C \to C$ *be a relatively nonexpansive mapping and* $F : C \times C \to \mathbb{R}$ *be a bifunction satisfying the conditions* (A1)-(A4) *with* $EP(F) \cap F(S) \neq \emptyset$. *Define an iterative scheme* $\{x_n\}$ *by the following manner:*

$$\begin{cases} x_1 = x \in E_1, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geqslant 0, \ \forall y \in C, \\ y_n = J^{-1}(\alpha_n u_n + (1-\alpha_n)JSu_n), \\ C_n = \{v \in C : \phi(v, y_n) \leqslant \phi(v, x_n)\}, \\ D_n = \cap_{i=1}^n C_i, \\ x_{n+1} = P_{D_n}x, \end{cases} \tag{3.9}$$

for each $n \geqslant 1$, *where* $\{r_n\} \subset [r, \infty)$ *with* $r > 0$ *and* $\{s_n\} \subset [s, \infty)$ *with* $s > 0$. *Then the sequence* $\{x_n\}$ *defined by* (3.9) *converges strongly to the point* $\Pi_{EP(F)\cap F(S)}x$.

Remark 3.4. In [28, Theorem 3.1], the sequence $\{\alpha_n\}$ is required to satisfy the condition $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. In Corollary 3.3, there is no any restrictions on $\{\alpha_n\}$ and so Corollary 3.3 improves [28, Theorem 3.1]. The proof of Theorem 3.1 is also simpler than the one of [28, Theorem 3.1].

4. Weak convergence theorems

Lemma 4.1. *Let* E_1 *be a uniformly smooth and uniformly convex Banach space and* E_2 *be a uniformly smooth, strictly convex and reflexive Banach space. Let* $A : E_1 \to E_2$ *be a linear and continuous operator and* C, Q *be nonempty closed convex subsets of* E_1 *and* E_2, *respectively. Let* $S : C \to C$ *be a relatively nonexpansive mapping and* $F : C \times C \to \mathbb{R}$, $H : Q \times Q \to \mathbb{R}$ *be two bifunctions satisfying the conditions* (A1)-(A4) *with* $\Omega \cap F(S) \neq \emptyset$. *Define an iterative scheme* $\{x_n\}$ *by the following manner:*

$$\begin{cases} \text{take } x_1 = x \in E_1, \text{ find } v \in E_1 \text{ such that } Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leqslant n\}, \ U_n = \{x \in V_n : Ax \in Q\}, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geqslant 0, \ \forall y \in C, \\ H(Az_n, Ay) + \dfrac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0, \ \forall y \in U_n, \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n), \end{cases} \tag{4.1}$$

for each $n \geqslant 1$, *where* $\{r_n\} \subset [r, \infty)$ *with* $r > 0$, $\{s_n\} \subset [s, \infty)$ *with* $s > 0$ *and* $\{\alpha_n\} \subset (0, 1)$. *Then the sequence* $\{\Pi_{\Omega\cap F(S)}x_n\}$ *converges strongly to a point* $x^* \in \Omega \cap F(S)$, *where* $\Pi_{\Omega\cap F(S)}$ *is the generalized projection of* E_1 *onto* $\Omega \cap F(S)$.

Proof. For each $p \in \Omega \cap F(S)$, we have

$$\begin{aligned} \phi(p, x_{n+1}) &= \phi(p, J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n)) \\ &= \|p\|^2 - 2\langle p, \alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\rangle + \|\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\|^2 \\ &\leqslant \|p\|^2 - 2\langle p, \alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\rangle + \alpha_n\|u_n\|^2 + (1-\alpha_n)\|S\Pi_C z_n\|^2 \\ &= \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, S\Pi_C z_n) \\ &\leqslant \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, \Pi_C z_n) \\ &\leqslant \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, z_n) \\ &\leqslant \phi(p, u_n) \leqslant \phi(p, x_n). \end{aligned} \tag{4.2}$$

It follows that $\{\phi(p, x_n)\}$ is convergent and so it is bounded, which implies that $\{x_n\}$ is bounded. Further, $\{u_n\}$, $\{z_n\}$ and $\{S\Pi_C z_n\}$ are bounded. Let $y_n = \Pi_{\Omega\cap F(S)} x_n$. Then $\{y_n\}$ is bounded. Since $y_n \in \Omega \cap F(S)$, by (4.2), we have

$$\phi(y_n, x_{n+1}) \leqslant \phi(y_n, x_n). \tag{4.3}$$

By Lemma 2.2, we have

$$\begin{aligned}\phi(y_{n+1}, x_{n+1}) &= \phi(\Pi_{\Omega\cap \mathrm{Fix}(S)} x_{n+1}, x_{n+1}) \\ &\leqslant \phi(y_n, x_{n+1}) - \phi(y_n, y_{n+1}) \\ &\leqslant \phi(y_n, x_{n+1}),\end{aligned}$$

which with (4.3), implies that

$$\phi(y_{n+1}, x_{n+1}) \leqslant \phi(y_n, x_n),$$

and so the limit of $\{\phi(y_n, x_n)\}$ exists.

On the other hand, by (4.3), it follows that, for each $m \geqslant 1$,

$$\phi(y_n, x_{n+m}) \leqslant \phi(y_n, x_{n+m-1}) \leqslant \cdots \leqslant \phi(y_n, x_{n+1}) \leqslant \phi(y_n, x_n).$$

By Lemma 2.2, we have

$$\phi(y_n, y_{n+m}) + \phi(y_{n+m}, x_{n+m}) \leqslant \phi(y_n, x_{n+m}) \leqslant \phi(y_n, x_n),$$

and so

$$\phi(y_n, y_{n+m}) \leqslant \phi(y_n, x_n) - \phi(y_{n+m}, x_{n+m}),$$

for each $m \geqslant 1$. Since the limit of $\{\phi(y_n, x_n)\}$ exists, we have

$$\lim_{n\to\infty} \phi(y_n, y_{n+m}) = 0,$$

for each $m \geqslant 1$. From Lemma 2.4, it follows that

$$\lim_{n\to\infty} \|y_n - y_{n+m}\| = 0,$$

for each $m \geqslant 1$. It follows that $\{y_n\}$ is a Cauchy sequence and hence there exists $x^* \in \Omega \cap \mathrm{Fix}(S)$ such that $\{y_n\}$ converges strongly to x^*. This completes the proof. □

Theorem 4.2. *Let E_1 be a uniformly smooth and uniformly convex Banach space and E_2 be a uniformly smooth, strictly convex and reflexive Banach space. Let C, Q be nonempty closed convex subsets of E_1, E_2, respectively. Let $A : E_1 \to E_2$ be a linear and continuous operator with $Q \subset A(E_1)$, $S : C \to C$ be a relatively nonexpansive mapping and $F : C \times C \to \mathbb{R}$, $H : Q \times Q \to \mathbb{R}$ be two bifunctions satisfying the conditions* (A1)-(A4) *with $\Omega \cap F(S) \neq \emptyset$. Define an iterative scheme $\{x_n\}$ by* (4.1)*. If $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and J is weakly sequentially continuous, then the sequence $\{x_n\}$ converges weakly to the point $x^* \in \Omega \cap F(S)$, where $x^* = \lim_{n\to\infty} \Pi_{\Omega\cap F(S)} x_n$.*

Proof. By Lemma 4.1, $\{u_n\}$ and $\{S\Pi_C z_n\}$ are bounded. Set

$$a = \max\{\sup_{n\geqslant 1} \|u_n\|, \sup_{n\geqslant 1} \|S\Pi_C z_n\|\}.$$

For all $x, y \in B_a = \{x \in E_1 : \|x\|^2 \leqslant a\}$, by Lemma 2.5, there exists a continuous, strictly increasing and convex function g with $g(0) = 0$ such that

$$\|tx + (1-t)y\|^2 \leqslant t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in B_a$ and $t \in [0, 1]$.

For any $p \in \Omega \cap F(S)$, from

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n)) \\
&= \|p\|^2 - 2\langle p, \alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\rangle + \|\alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\|^2 \\
&\leqslant \|p\|^2 - 2\langle p, \alpha_n Ju_n + (1-\alpha_n)JS\Pi_C z_n\rangle + \alpha_n\|u_n\|^2 + (1-\alpha_n)\|S\Pi_C z_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g(\|Ju_n - JS\Pi_C z_n\|) \\
&= \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, S\Pi_C z_n) - \alpha_n(1-\alpha_n)g(\|Ju_n - JS\Pi_C z_n\|) \\
&\leqslant \alpha_n\phi(p, u_n) + (1-\alpha_n)\phi(p, z_n) - \alpha_n(1-\alpha_n)g(\|Ju_n - JS\Pi_C z_n\|) \\
&\leqslant \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, x_n) - \alpha_n(1-\alpha_n)g(\|Ju_n - JS\Pi_C z_n\|) \\
&= \phi(p, x_n) - \alpha_n(1-\alpha_n)g(\|Ju_n - JS\Pi_C z_n\|) \\
&\leqslant \phi(p, x_n),
\end{aligned}$$

it follows that

$$\alpha_n(1-\alpha_n)g(\|Ju_n - JS\Pi_C z_n\|) \leqslant \phi(p, x_n) - \phi(p, x_{n+1}) \to 0.$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} g(\|Ju_n - JS\Pi_C z_n\|) = 0.$$

From the property of g, we have

$$\lim_{n\to\infty} \|Ju_n - JS\Pi_C z_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|u_n - S\Pi_C z_n\| = 0. \tag{4.4}$$

From Lemma 2.6, there exists a continuous, strictly increasing and convex function g_1 with $g_1(0) = 0$ such that

$$g_1(\|x - y\|) \leqslant \phi(x, y),$$

for all $x, y \in B_b$, where

$$b = \max\{\sup_{n\geqslant 1} \|x_n\|, \sup_{n\geqslant 1} \|u_n\|, \sup_{n\geqslant 1} \|z_n\|, \sup_{n\geqslant 1} \|\Pi_C z_n\|\}.$$

From (4.2), we have

$$\phi(p, x_{n+1}) \leqslant \phi(p, u_n) \leqslant \phi(p, x_n).$$

Since the limit of $\{\phi(p, x_n)\}$ exists, the limit of $\{\phi(p, u_n)\}$ also exists and

$$\lim_{n\to\infty} \phi(p, u_n) = \lim_{n\to\infty} \phi(p, x_n).$$

From Lemma 2.7, it follows that

$$\begin{aligned}
g_1(\|u_n - x_n\|) &\leqslant \phi(u_n, x_n) \\
&\leqslant \phi(p, x_n) - \phi(p, u_n) \\
&\leqslant \phi(p, u_{n-1}) - \phi(p, u_n) \\
&\to 0, \quad \text{as } n \to \infty,
\end{aligned}$$

and so

$$\lim_{n\to\infty} g_1(\|u_n - x_n\|) = 0.$$

From the property of g_1 we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.5}$$

From (4.2), we see

$$\phi(p, x_{n+1}) - \phi(p, x_n) \leqslant \alpha_n(\phi(p, u_n) - \phi(p, x_n)) + (1-\alpha_n)(\phi(p, z_n) - \phi(p, x_n)) \leqslant 0.$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ implies that $\limsup_{n\to\infty} \alpha_n < 1$, by using

$$\lim_{n\to\infty} \phi(p, u_n) = \lim_{n\to\infty} \phi(p, x_n),$$

we have

$$\lim_{n\to\infty} \phi(p, z_n) = \lim_{n\to\infty} \phi(p, x_n). \tag{4.6}$$

Similarly, we can obtain

$$\lim_{n\to\infty} \phi(p, S\Pi_C z_n) = \lim_{n\to\infty} \phi(p, \Pi_C z_n) = \lim_{n\to\infty} \phi(p, x_n). \tag{4.7}$$

By Lemma 2.2, we have

$$\phi(\Pi_C z_n, z_n) \leqslant \phi(p, z_n) - \phi(p, \Pi_C z_n).$$

By (4.6) and (4.7), we have

$$\lim_{n\to\infty} \phi(\Pi_C z_n, z_n) = 0.$$

From Lemma 2.4, it follows that

$$\lim_{n\to\infty} \|z_n - \Pi_C z_n\| = 0. \tag{4.8}$$

By Lemma 2.7 and (4.6), we have

$$\begin{aligned} g_1(\|z_n - u_n\|) &\leqslant \phi(z_n, u_n) \\ &\leqslant \phi(p, u_n) - \phi(p, z_n) \\ &\leqslant \phi(p, x_n) - \phi(p, z_n) \to 0, \quad \text{as } n \to \infty. \end{aligned}$$

From the property of g_1, we have

$$\lim_{n\to\infty} \|z_n - u_n\| = 0. \tag{4.9}$$

From (4.4), (4.8), (4.9) and

$$\|S\Pi_C z_n - \Pi_C z_n\| \leqslant \|S\Pi_C z_n - u_n\| + \|u_n - z_n\| + \|z_n - \Pi_C z_n\|,$$

it follows that

$$\lim_{n\to\infty} \|S\Pi_C z_n - \Pi_C z_n\| = 0. \tag{4.10}$$

Since J is uniformly norm-to-norm continuous on bounded sets, it follows from (3.6) and (4.9) that

$$\lim_{n\to\infty} \|Ju_n - Jx_n\| = \lim_{n\to\infty} \|Jz_n - Ju_n\| = 0.$$

From $r_n \geqslant r > 0$ and $s_n \geqslant s > 0$, we have

$$\lim_{n\to\infty} \frac{\|Ju_n - Jx_n\|}{r_n} = \lim_{n\to\infty} \frac{\|Jz_n - Ju_n\|}{s_n} = 0. \tag{4.11}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $x' \in C$. From (4.5), it follows that $\{u_{n_k}\}$ converges weakly x'. By putting $u_n = T^F_{r_n} x_n$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geqslant 0,$$

for all $y \in C$. Replacing n with n_k, it follows from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jx_{n_k}\rangle \geqslant -F(u_{n_k}, y) \geqslant F(y, u_{n_k}),$$

for all $y \in C$. Letting $k \to \infty$, it follows from (4.11) and (A4) that

$$F(y, x') \leqslant 0,$$

for all $y \in C$. For any t with $0 < t \leqslant 1$ and $y \in C$, let $y_t = ty + (1-t)x'$. Since $y \in C$ and $x' \in C$, one has $y_t \in C$ and so $F(y_t, x') \leqslant 0$. Then, by (A1) and (A4), we obtain

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\leqslant tF(y_t, y) + (1-t)F(y_t, x') \\ &\leqslant tF(y_t, y). \end{aligned}$$

It follows that

$$F(y_t, y) \geqslant 0,$$

for all $y \in C$. Letting $t \downarrow 0$, from (A3), we have

$$F(x', y) \geqslant 0,$$

for all $y \in C$. Therefore, $x' \in EP(F)$. By (4.1), we have

$$H(Az_n, Ay) + \frac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0, \tag{4.12}$$

for all $y \in U_n$. Since $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$ and $\lim_{n\to\infty} V_n = \cup_{n=1}^{\infty} V_n = E_1$, one has $U_1 \subset U_2 \cdots \subset U_n \subset \cdots$ and $\lim_{n\to\infty} U_n = \cup_{n=1}^{\infty} U_n = \{x \in E_1 : Ax \in Q\}$. Replacing n with n_k in (4.12) , from (A2), we have

$$\frac{1}{s_{n_k}}\langle y - z_{n_k}, Jz_{n_k} - Ju_{n_k}\rangle \geqslant -H(Az_{n_k}, Ay) \geqslant H(Ay, Au_{n_k}), \tag{4.13}$$

for all $y \in U_{n_k}$. Letting $k \to \infty$, by (4.13) and (A4), we obtain

$$0 \geqslant H(Ay, Ax'),$$

for all $y \in \{x \in E_1 : Ax \in Q\}$. Since $Q \subset A(E_1)$, we have

$$0 \geqslant H(y, Ax'),$$

for all $y \in Q$. By the similar process with $x' \in EP(F)$, we can prove that $Ax' \in EP(H)$. Therefore, $x' \in \Omega$.

Now, we show that $x' \in F(S)$. From (4.5), (4.8) and (4.9), we see that $\Pi_C z_n$ weakly converges to x'. From (4.10), it follows that $x' \in \hat{F}(S) = F(S)$. Let $y_n = \Pi_{\Omega\cap F(S)} x_n$. From Lemma 2.3 and $x' \in \Omega \cap F(S)$, we have

$$\langle y_{n_k} - x', Jx_{n_k} - Jy_{n_k}\rangle \geqslant 0. \tag{4.14}$$

By Lemma 4.1, it follows that $\{y_n\}$ converges strongly to $x^* \in \Omega \cap F(S)$. Since J is weakly sequentially continuous, by letting $k \to \infty$ in (4.14), we have

$$\langle x^* - x', Jx' - Jx^*\rangle \geqslant 0.$$

On the other hand, since J is monotone, we have

$$\langle x^* - x', Jx' - Jx^*\rangle \leqslant 0,$$

and so it follows that

$$\langle x^* - x', Jx' - Jx^*\rangle = 0.$$

Since E_1 is strictly convex, one has $x^* = x'$. Therefore, the sequence $\{x_n\}$ converges weakly to $x^* \in \Omega \cap F(S)$, where $x^* = \lim_{n\to\infty} \Pi_{\Omega\cap F(S)} x_n$. This completes the proof. □

Corollary 4.3. *Let* E *be a uniformly smooth and uniformly convex Banach spaces and* C *be a nonempty closed and convex subset of* E. *Let* $F, H : C \times C \to \mathbb{R}$ *be two bifunctions satisfying the conditions* (A1)-(A4) *and* $S : C \to C$ *be a relatively nonexpansive mapping with* $EP(F) \cap EP(H) \cap F(S) \neq \emptyset$. *Define an iterative scheme* $\{x_n\}$ *by the following manner:*

$$\begin{cases} x_1 = x \in C, \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle \geqslant 0, \ \forall y \in C, \\ H(z_n, y) + \dfrac{1}{s_n}\langle y - z_n, Jz_n - Ju_n\rangle \geqslant 0, \ \forall y \in C, \\ x_{n+1} = J^{-1}(\alpha_n Ju_n + (1-\alpha_n)JSz_n), \end{cases} \tag{4.15}$$

for all $n \geqslant 1$, *where* $\{r_n\} \subset [r, \infty)$ *with* $r > 0$, $\{s_n\} \subset [s, \infty)$ *with* $s > 0$ *and* $\{\alpha_n\} \subset (0, 1)$. *If* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ *and* J *is weakly sequentially continuous, then the sequence* $\{x_n\}$ *generated by* (4.15) *converges weakly to a point* $x^* \in EP(F) \cap EP(H) \cap F(S)$, *where* $x^* = \lim_{n\to\infty} \Pi_{EP(F)\cap EP(H)\cap F(S)} x_n$.

Remark 4.4. Theorem 3.1 and Theorem 4.2 extend the results of Takahashi and Zembayashi [28] from equilibrium problems to split equilibrium problems. In Theorem 3.1 of Takahashi and Zembayashi [28], the sequence $\{\alpha_n\}$ is required to satisfy the condition $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. In our Theorem 3.1, there is no any restrictions on the control condition $\{\alpha_n\}$ and so Theorem 3.1 improves the result of Takahashi and Zembayashi [28]. The proof method of our Theorem 3.1 is also simpler than the one of Takahashi and Zembayashi [28].

5. Applications

Let E_1, E_2 be two Banach spaces and C, Q be nonempty closed convex subsets of E_1, E_2, respectively. Let $A : E_1 \to E_2$ be an operator. The split feasibility problem is to find $x^* \in C$ such that

$$Ax^* \in Q. \tag{5.1}$$

For more results on split feasibility problems, the readers refer to [3, 4, 6, 9, 29, 34, 35].

Now, by Theorem 3.1 and Theorem 4.2, we give the following results on split feasibility problems in Banach spaces:

Theorem 5.1. *Let* E_1 *be a smooth and uniformly convex Banach space and* E_2 *be a smooth, strictly convex and reflexive Banach space. Let* C *and* Q *be nonempty closed convex subsets of* E_1 *and* E_2, *respectively. Let* $A : E_1 \to E_2$ *be a linear and continuous operator. Suppose that* $\Omega \neq \emptyset$, *where* Ω *denotes the solution set of the problem* (5.1). *Define an iterative scheme* $\{x_n\}$ *by the following manner:*

$$\begin{cases} \text{take } x_1 = x \in C, \text{ find } v \in E_1 \text{ such that } Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leqslant n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ z_n = \Pi_{U_n} x_n, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)J\Pi_C z_n), \\ C_n = \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ D_n = \cap_{i=1}^n C_i, \\ x_{n+1} = \Pi_{D_n} x, \end{cases} \tag{5.2}$$

for each $n \geqslant 1$. *Then the sequence* $\{x_n\}$ *defined by* (5.2) *converges strongly to a point* $\Pi_\Omega x$, *where* Π_Ω *is the generalized projection of* E_1 *onto* Ω.

Proof. In Theorem 3.1, let $F(x,y) = 0$ for all $x,y \in C$ and $H(x,y) = 0$ for all $x,y \in Q$. By Lemma 2.3, we have $u_n = \Pi_C x_n$ and $z_n = \Pi_{U_n} u_n$. Since $x_n \in C$ for each $n \geqslant 1$, we have $u_n = x_n$ and hence the algorithm (3.1) is deduced to (5.2) by setting $S = I$ in Theorem 3.1. Therefore, by Theorem 3.1, we can obtain the desired result. This completes the proof. □

Theorem 5.2. *Let E_1 be a smooth and uniformly convex Banach space and E_2 be a smooth, strictly convex and reflexive Banach space. Let C and Q be the nonempty closed and convex subsets of E_1 and E_2, respectively. Let $A : E_1 \to E_2$ be a linear and continuous operator with $Q \subset A(E_1)$. Suppose that $\Omega \neq \emptyset$. Define an iterative scheme $\{x_n\}$ by the following manner:*

$$\begin{cases} \text{take } x_1 = x \in C, \text{ find } v \in E_1 \text{ such that } Av \in Q, \\ V_n = \{x \in E_1 : \|x - v\| \leqslant n\}, \\ U_n = \{x \in V_n : Ax \in Q\}, \\ z_n = \Pi_{U_n} x_n, \\ x_{n+1} = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J \Pi_C z_n), \end{cases} \tag{5.3}$$

for each $n \geqslant 1$. If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and J is weakly sequentially continuous, then the sequence $\{x_n\}$ defined by (5.3) converges weakly to a point $x^* \in \Omega$, where $x^* = \lim_{n\to\infty} \Pi_\Omega x_n$.

Finally, for the sake of simplicity, we give an example in finite dimension Euclidean spaces to illustrate Theorem 3.1 as follows:

Example 5.3. Let $E_1 = \mathbb{R}$ and $E_2 = \mathbb{R}^2$. Let $A : E_1 \to E_2$ be a mapping defined by $Ax = (x, x/2)$ for all $x \in E_1$. Let $C = [0, 10]$ and $Q = [10, +\infty) \times [5, +\infty)$. Let $F(x,y) = x - y$ for any $x, y \in C$ and $H(x,y) = y_1 + y_2 - x_1 - x_2$ for any $x = (x_1, x_2), y = (y_1, y_2) \in Q$. It is obvious that F and H satisfy the conditions (A1)-(A4) and $\Omega = \{10\}$. Take $x_1 = x = 2$, $v = 12$ and for simplicity, set $\alpha_n = \frac{1}{2}$ and $r_n = s_n = 1$ for each $n \geqslant 1$.

For any $x_n \in C$, we need to find $u_n \in C$ such that

$$\begin{aligned} F(u_n, y) + \langle y - u_n, u_n - x_n \rangle &= u_n - y + (y - u_n)(u_n - x_n) \\ &= (u_n - x_n - 1)y + u_n(1 - u_n + x_n) \\ &\geqslant 0, \end{aligned}$$

for all $y \in C$. Hence $u_n = 1 + x_n$ if $x_n \leqslant 9$ and $u_n = 10$ if $x_n > 9$.

For each u_n, we need to find $z_n \in U_n$ such that

$$\begin{aligned} H(Az_n, Ay) + \langle y - z_n, z_n - u_n \rangle &= \frac{3y}{2} - \frac{3z_n}{2} + (y - z_n)(z_n - u_n) \\ &= (z_n - u_n + \frac{3}{2})y - z_n(z_n - u_n + \frac{3}{2}) \\ &\geqslant 0, \end{aligned}$$

for all $y \in U_n$.

If $u_n - \frac{3}{2} \leqslant L_{U_n}$ ($z_n - u_n + \frac{3}{2} \geqslant 0$ for all $z_n \in U_n$), where $L_{U_n} = \min_{x \in U_n} x$, then $z_n = L_{U_n}$ since $(z_n - u_n + \frac{3}{2})y - z_n(z_n - u_n + \frac{3}{2}) \geqslant 0$ for all $y \in U_n$ implies that $z_n \leqslant y$ for all $y \in U_n$. Then $y_n = \frac{u_n + z_n}{2}$. By the simple computation, we obtain some results on $V_n, U_n, D_n, y_n, z_n, u_n$ and x_n as follows:

n	V_n	U_n	D_n	y_n	z_n	u_n	x_n
1	[11,13]	[11,13]	[4.5,10]	7	11	3	2
2	[10,14]	[10,14]	[6.125,10]	7.75	10	5.5	4.5
3	[9,15]	[10,15]	[7.34375,10]	8.5625	10	7.125	6.125
4	[8,16]	[10,16]	[8.2579125,10]	9.171875	10	8.34375	7.34375
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Therefore, by Theorem 3.1, the sequence $\{x_n\}$ converges to an element $x^* \in \Omega$, i.e., $x^* = 10$.

6. Conclusions

In this paper, we introduce some new strong and weak convergence algorithms to solve split equilibrium problems and fixed point problems for relatively nonexpansive mappings in Banach spaces. In our algorithms, we first construct two sets V_n, U_n and transform the bifunction H on $Q \times Q$ to the bifunction HA on the set U_n. The algorithms of this paper only involve the operator A itself and do not use any restrictions on the adjoint A^* and the norm $\|A\|$ of A and so our algorithms can be implemented more effectively.

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References

[1] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., Dekker, New York, **178** (1996), 15–50. 2, 2.2, 2.3
[2] A. Bnouhachem, *Strong convergence algorithm for split equilibrium problems and hierarchical fixed point problems*, Scientific World J., **2014** (2014), 12 pages. 1, 1
[3] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems, **18** (2002), 441–453. 1, 5
[4] C. Byrne, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems, **20** (2004), 103–120. 1, 5
[5] L.-C. Ceng, S. Al-Homidan, Q. H. Ansari, J.-C. Yao, *An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings*, J. Comput. Appl. Math., **223** (2009), 967–974. 1
[6] L.-C. Ceng, Q. H. Ansari, J.-C. Yao, *An extragradient method for solving split feasibility and fixed point problems*, Comput. Math. Appl., **64** (2012), 633–642. 1, 5
[7] L.-C. Ceng, A. Petruşel, J.-C. Yao, *Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, J. Optim. Theory Appl., **143** (2009), 37–58. 1
[8] L.-C. Ceng, J.-C. Yao, *Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, Appl. Math. Comput., **198** (2008), 729–741. 1
[9] Y. Censor, T. Elfving, N. Kopf, T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Problems, **21** (2005), 2071–2084. 1, 5
[10] Y. Censor, A. Gibali, S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms, **59** (2012), 301–323. 1
[11] S.-S. Chang, Y. J. Cho, J. K. Kim, *Approximation methods of solutions for equilibrium problem in Hilbert spaces*, Dynam. Systems Appl., **17** (2008), 503–514. 1
[12] S.-S. Chang, H. W. Joseph Lee, C. K. Chan, *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization*, Nonlinear Anal., **70** (2009), 3307–3319.
[13] V. Colao, G. Marino, H.-K. Xu, *An iterative method for finding common solutions of equilibrium and fixed point problems*, J. Math. Anal. Appl., **344** (2008), 340–352.
[14] P. L. Combettes, S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136. 1
[15] S. Kamimura, W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim., **13** (2002), 938–945. 2.2, 2.3, 2.4, 2.6
[16] P. Katchang, P. Kumam, *A new iterative algorithm of solution for equilibrium problems, variational inequalities and fixed point problems in a Hilbert space*, J. Appl. Math. Comput., **32** (2010), 19–38. 1
[17] K. R. Kazmi, S. H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problem, a variational inequality problem and a fixed point problem*, J. Egyptian Math. Soc., **21** (2013), 44–51. 1, 1
[18] P. Kumam, P. Katchang, *A viscosity of extragradient approximation method for finding equilibrium problems, variational inequalities and fixed point problems for nonexpansive mappings*, Nonlinear Anal. Hybrid Syst., **3** (2009), 475–486. 1
[19] S.-Y. Matsushita, W. Takahashi, *Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2004** (2004), 37–47. 2
[20] A. Moudafi, *Weak convergence theorems for nonexpansive mappings and equilibrium problems*, J. Nonlinear Convex Anal., **9** (2008), 37–43. 1
[21] A. Moudafi, *Split monotone variational inclusions*, J. Optim Theory Appl., **150** (2011), 275–283. 1

[22] S. Plubtieng, R. Punpaeng, *A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **336** (2007), 455–469. 1

[23] S. Plubtieng, R. Punpaeng, *A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings*, Appl. Math. Comput., **197** (2008), 548–558.

[24] X.-L. Qin, S. Y. Cho, S. M. Kang, *Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems*, J. Comput. Appl. Math., **234** (2010), 750–760.

[25] X.-L. Qin, Y. J. Cho, S. M. Kang, *Viscosity approximation methods for generalized equilibrium problems and fixed point problems with applications*, Nonlinear Anal., **72** (2010), 99–112.

[26] S. Takahashi, W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331** (2007), 506–515.

[27] S. Takahashi, W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal., **69** (2008), 1025–1033. 1

[28] W. Takahashi, K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Nonlinear Anal., **70** (2009), 45–57. 2.1, 2.7, 3.4, 4.4

[29] Y.-C. Tang, J.-G. Peng, L.-W. Liu, *A cyclic algorithm for the split common fixed point problem of demicontractive mappings in Hilbert spaces*, Math. Model. Anal., **17** (2012), 57–466. 1, 5

[30] S.-H. Wang, Y. J. Cho, X.-L. Qin, *A new iterative method for solving equilibrium problems and fixed point problems for infinite family of nonexpansive mappings*, Fixed Point Theory Appl., **2010** (2010), 18 pages. 1

[31] S.-H. Wang, X.-Y. Gong, A. A. Abdou, Y. J. Cho, *Iterative algorithm for a family of split equilibrium problems and fixed point problems in Hilbert spaces with applications*, Fixed Point Theory Appl., **2016** (2016), 22 pages. 1, 1

[32] S.-H. Wang, G. Marino, F.-H. Wang, *Strong convergence theorems for a generalized equilibrium problem with a relaxed monotone mapping and a countable family of nonexpansive mappings in a Hilbert space*, Fixed Point Theory Appl., **2010** (2010), 22 pages. 1

[33] H.-K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16** (1991), 1127–1138. 2.5

[34] H.-K. Xu, *Iterative methods for the split feasibility problem in infinite-dimensional Hilbert spaces*, Inverse Problems, **26** (2010), 17 pages. 1, 5

[35] Y.-H. Yao, T. H. Kim, S.-H. Chebbi, H.-K. Xu, *A modified extragradient method for the split feasibility and fixed point problems*, J. Nonlinear Convex Anal., **13** (2012), 383–396. 1, 5