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Asymptotically $\mathcal{I}$-Lacunary statistical equivalent of order α for sequences of sets

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Abstract

This paper presents the following definition which is a natural combination of the definition for asymptotically equivalent of order α, where $0 < \alpha \leqslant 1$, $\mathcal{I}$-statistically limit, and $\mathcal{I}$-lacunary statistical convergence for sequences of sets. Let (X, ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α to multiple L, where $0 < \alpha \leqslant 1$, provided that for each $\varepsilon > 0$ and each $x \in X$,

$$\{r \in \mathbb{N} : \frac{1}{h_r^{\alpha}} |\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \geqslant \delta\} \in \mathcal{I},$$

(denoted by $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$) and simply asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α if $L = 1$. In addition, we shall also present some inclusion theorems. The study leaves some interesting open problems.

Keywords: Asymptotical equivalent, sequences of sets, ideal convergence, Wijsman convergence, $\mathcal{I}$-statistical convergence, $\mathcal{I}$-lacunary statistical convergence, statistical convergence of order α.
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1. Introduction

The concept of statistical convergence was introduce by Fast [6] in 1951. A sequence (x_k) of real numbers is said to be statistically convergent to L if for arbitrary $\epsilon > 0$,

$$\frac{1}{n}|\{k < n : |x_k - L| \geqslant \epsilon\}| = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, \cdots, n$ and the vertical bars indicate the number of elements in the enclosed set. In this case we write $st - \lim x = L$ or $x_k \mapsto L(st)$.

By a lacunary $\theta = (k_r)$, $r = 0, 1, 2, \cdots$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by

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$I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. Moreover, the following concept is due to Fridy and Orhan [8].

A sequence (x_k) of real numbers is said to be lacunary statistically convergent to L (or, S_θ-convergent to L), if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geqslant \epsilon\}| = 0,$$

where $|A|$ denotes the cardinality of $A \subset \mathbb{N}$.

Recently in ([5] and [19]), we used ideals to introduce the concepts of $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence which naturally extend the notions of the above mentioned convergence. On the other hand, in [2] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order α, $0 < \alpha \leqslant 1$ was introduced by replacing n by n^α in the denominator in the definition of statistical convergence. One can also see [3, 15] for related works. In 1993 Marouf [12] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Also, in 1997, Li [11] presented and studied asymptotic equivalence of sequences and summability. In 2003, Patterson [14] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for non-negative summability matrices.

The idea of statistical convergence was further extended to $\mathcal{I}$-convergence in [10] using the notion of ideals of $\mathbb{N}$ with many interesting consequences. More investigations in this direction and more applications of ideals can be found in [4, 5, 16–20] where many important references can be found.

In present paper, we use asymptotical equivalent of set sequences to introduce the concept Wijsman asymptotically $\mathcal{I}$-statistical equivalent of order α and Wijsman asymptotically $\mathcal{I}$ -lacunary statistical equivalent of order α for sequences of set. In addition to these definitions, natural inclusion theorems shall also be presented.

2. Definitions and preliminaries

The following definitions and notions will be needed in the sequel.

Definition 2.1 ([12])**.** Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent, if

$$\lim_k \frac{x_k}{y_k} = 1,$$

(denoted by $x \sim y$).

Definition 2.2 ([7])**.** The sequence $x = (x_k)$ has statistic limit L, denoted by $st - \lim x_k = L$, provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n} \{\text{the number of } k \leqslant n : |x_k - L| \geqslant \epsilon\} = 0.$$

The next definition is natural combination of Definitions 2.1 and 2.2.

Definition 2.3 ([14])**.** Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n} \{\text{the number of } k < n : |\frac{x_k}{y_k} - L| \geqslant \epsilon\} = 0,$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent if $L = 1$.

Definition 2.4. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to be an ideal of $\mathbb{N}$, if the following conditions hold:

(a) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(b) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$.

Definition 2.5. A non-empty family $F \subset 2^{\mathbb{N}}$ is said to be a filter of $\mathbb{N}$, if the following conditions hold:

(a) $\phi \notin F$;

(b) $A, B \in F$ implies $A \cap B \in F$;

(c) $A \in F,\ A \subset B$ implies $B \in F$.

If $\mathcal{I}$ is a proper ideal of $\mathbb{N}$ (i.e., $\mathbb{N} \notin \mathcal{I}$), then the family of sets $F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists\, A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$ is a filter of $\mathbb{N}$. It is called the filter associated with the ideal.

Definition 2.6. A proper ideal $\mathcal{I}$ is said to be admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$.

Definition 2.7 ([10])**.** Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a proper admissible ideal in $\mathbb{N}$. Then the sequence (x_k) of elements of $\mathbb{R}$ is said to be $\mathcal{I}$-convergent to $L \in \mathbb{R}$, if for each $\epsilon > 0$ the set $A(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geqslant \epsilon\} \in \mathcal{I}$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Definition 2.8 ([1])**.** Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A, if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A),$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

In [13], statistical convergence of sequences of sets was given by Nuray and Rhoades as follows:

Definition 2.9. Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A, if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \leqslant n : |d(x, A_k) - d(x, A)| \geqslant \varepsilon\}| = 0.$$

In this case we write $st\text{-}\lim_W A_k = A$ or $A_k \to A(WS)$.

We now have

Definition 2.10 ([9, 18])**.** Let (X, ρ) be a metric space and θ be lacunary sequence. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}-$lacunary statistical convergent to A or $S_\theta\,(\mathcal{I}_W)$-convergent to A, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geqslant \varepsilon\}| \geqslant \delta \right\} \in \mathcal{I}.$$

In this case, we write $A_k \to A\,(S_\theta\,(\mathcal{I}_W))$.

3. Main results

In this section we shall give some new definitions and also examine some inclusion relations.

Definition 3.1 ([18])**.** Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-statistical convergent of order α to A or $S(\mathcal{I}_W)$-convergent of order α $(0 < \alpha \leqslant 1)$ to A, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\{n \in \mathbb{N} : \frac{1}{n^\alpha} |\{k \leqslant n : |d(x, A_k) - d(x, A)| \geqslant \varepsilon\}| \geqslant \delta\} \in \mathcal{I}.$$

In this case we write $A_k \to A(S(\mathcal{I}_W)^\alpha)$. The class of all Wijsman $\mathcal{I}$-statistical sequences of order α will be denoted by simply $S(\mathcal{I}_W)^\alpha$.

Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$, we define $d(x;A_k,B_k)$ as follows:

$$d(x;A_k,B_k) = \begin{cases} \dfrac{d(x,A_k)}{d(x,B_k)}, & x \notin A_k \cup B_k, \\ L, & x \in A_k \cup B_k. \end{cases}$$

The next definition is natural combination of Definitions 2.1 and 3.1.

Definition 3.2. Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{I}$-statistical equivalent of order α $(0 < \alpha \leqslant 1)$ to multiple L provided that for each $\epsilon > 0$, $\delta > 0$ and each $x \in X$,

$$\{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leqslant n : |d(x;A_k,B_k) - L| \geqslant \epsilon\}| \geqslant \delta\} \in \mathcal{I},$$

(denoted by $\{A_k\} \overset{S^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$) and simply Wijsman asymptotically $\mathcal{I}$-statistical equivalent of order α if $L = 1$. Furthermore, let $S^L(\mathcal{I}_W)^\alpha$ denote the set of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{S^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

Remark 3.3. If $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A \text{ is a finite subset}\}$, Wijsman asymptotically $\mathcal{I}$-statistical equivalent of order α to multiple L coincides with Wijsman asymptotically statistical equivalent of order α to multiple L. For an arbitrary ideal $\mathcal{I}$ and for $\alpha = 1$ it coincides with Wijsman asymptotically $\mathcal{I}$-statistical equivalent of multiple L. When $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes only Wijsman asymptotically statistical equivalent of multiple L for set sequences, [22, 23].

Definition 3.4 ([18]). Let (X,ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}$-lacunary statistical convergent of order α to A or $S_\theta(\mathcal{I}_W)$-convergent of order α $(0 < \alpha \leqslant 1)$ to A, if for each $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}|\{k \in I_r : |d(x,A_k) - d(x,A)| \geqslant \epsilon\}| \geqslant \delta\} \in \mathcal{I}.$$

In this case we write $A_k \to A(S_\theta(\mathcal{I}_W)^\alpha)$. The class of all $\mathcal{I}$-lacunary statistically convergent sequences of order α will be denoted by $S_\theta(\mathcal{I}_W)^\alpha$.

Remark 3.5. If $\mathcal{I} = \mathcal{I}_{fin} = \{A \subseteq \mathbf{N} : A \text{ is a finite subset}\}$, Wijsman $\mathcal{I}$-lacunary statistical convergent of order α coincides with Wijsman lacunary statistical convergent of order α. For an arbitrary ideal $\mathcal{I}$ and for $\alpha = 1$ it coincides with Wijsman $\mathcal{I}$-lacunary statistical convergent, [18]. When $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ it becomes only Wijsman lacunary statistical convergent for set sequences, [21].

The next definition is natural combination of Definitions 2.1 and 3.4.

Definition 3.6. Let (X,ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α $(0 < \alpha \leqslant 1)$ to multiple L provided that for each $\varepsilon > 0$, $\delta > 0$ and each $x \in X$,

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}|\{k \in I_r : |d(x;A_k,B_k) - L| \geqslant \epsilon\}| \geqslant \delta\} \in \mathcal{I},$$

(denoted by $\{A_k\} \overset{S_\theta^L(\mathcal{I})^\alpha}{\sim} \{B_k\}$) and simply asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α if $L = 1$. Furthermore, let $S_\theta^L(\mathcal{I}_W)^\alpha$ denote the set of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

Remark 3.7. For $\alpha = 1$, the above definition coincides with Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of multiple L, (see, [9, 23]). If we take $\mathcal{I} = \mathcal{I}_{fin}$ and $\alpha = 1$ Wijsman asymptotically lacunary statistical equivalent of multiple L is a special case of Wijsman asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α to multiple L, (see, [22]) .

Theorem 3.8. *Let* $0 < \alpha \leqslant \beta \leqslant 1$. *Then* $S(\mathcal{I}_W)^\alpha \subset S(\mathcal{I}_W)^\beta$.

Proof. Let $0 < \alpha \leqslant \beta \leqslant 1$. Then

$$\frac{|\{k \leqslant n : |d(x; A_k, B_k) - L| \geqslant \epsilon\}|}{n^\beta} \leqslant \frac{|\{k \leqslant n : |d(x; A_k, B_k) - L| \geqslant \epsilon\}|}{n^\alpha},$$

and so for any $\delta > 0$,

$$\{n \in \mathbb{N} : \frac{|\{k \leqslant n : |d(x; A_k, B_k) - L| \geqslant \epsilon\}|}{n^\beta} \geqslant \delta\} \subset \{n \in \mathbb{N} : \frac{|\{k \leqslant n : |d(x; A_k, B_k) - L| \geqslant \epsilon\}|}{n^\alpha} \geqslant \delta\}.$$

Hence if the set on the right hand side belongs to the ideal $\mathcal{I}$ then obviously the set on the left hand side also belongs to $\mathcal{I}$. This shows that $S(\mathcal{I}_W)^\alpha \subset S(\mathcal{I}_W)^\beta$. □

Similarly we can show that

Theorem 3.9. *Let* $0 < \alpha \leqslant \beta \leqslant 1$. *Then*

(i) $S_\theta^L(\mathcal{I}_W)^\alpha \subset S_\theta^L(\mathcal{I}_W)^\beta$.

(ii) *In particular* $S_\theta^L(\mathcal{I}_W)^\alpha \subset S_\theta^L(\mathcal{I}_W)$.

Definition 3.10. Let (X, ρ) be a metric space and θ be lacunary sequence. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly Wijsman asymptotically $\mathcal{I}$-lacunary equivalent of order α ($0 < \alpha \leqslant 1$) to multiple L provided that for each $\varepsilon > 0$ and each $x \in X$,

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \geqslant \epsilon\} \in \mathcal{I},$$

(denoted by $A_k \overset{N_\theta^L(\mathcal{I}_W)^\alpha}{\sim} B_k$) and simply strong asymptotically $\mathcal{I}$-lacunary statistical equivalent of order α if $L = 1$. Further, let $N_\theta^L(\mathcal{I}_W)^\alpha$ denote the set of A_k and B_k such that $\{A_k\} \overset{N_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

We prove the following

Theorem 3.11. *Let* θ *be a lacunary sequence, if* $\{A_k\} \overset{N_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ *then* $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

Proof. If $\epsilon > 0$ and $\{A_k\} \overset{N_\theta^L(\mathcal{I})^\alpha}{\sim} \{B_k\}$, we can write

$$\sum_{k \in I_r} |d(x; A_k, B_k) - L| \geqslant \sum_{k \in I_r, |d(x; A_k, B_k) - L| \geqslant \epsilon} |d(x; A_k, B_k) - L| \geqslant \epsilon |\{k \in I_r : \ |d(x; A_k, B_k) - L| \geqslant \epsilon\}|,$$

and so

$$\frac{1}{\epsilon . h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \geqslant \frac{1}{h_r^\alpha} |\{k \in I_r : \ |d(x; A_k, B_k) - L| \geqslant \epsilon\}|.$$

Then for any $\delta > 0$

$$\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} |\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \geqslant \delta\} \subseteq \{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \sum_{k \in I_r} |d(x; A_k, B_k) - L| \geqslant \epsilon . \delta\} \in \mathcal{I}.$$

This proves the result. □

Remark 3.12. In [23, Theorem 1] it was further proved that

(i) $x \in l_\infty$, the set of the bounded sequences and $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)}{\sim} \{B_k\} \Rightarrow x \overset{N_\theta^L(\mathcal{I}_W)}{\sim} y$;

(ii) $S_\theta^L(\mathcal{I}_W) \cap l_\infty = N_\theta^L(\mathcal{I}_W) \cap l_\infty$.

However whether these results remain true for $0 < \alpha < 1$ is not clear and we leave them as open problems.

We now investigate the relationship between $\{A_k\} \overset{S^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ and $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$.

Theorem 3.13. *Let θ be a lacunary sequence, then*

$$\{A_k\} \overset{S^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\} \textit{ implies } \{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\},$$

if $\liminf_r q_r^\alpha > 1$.

Proof. Suppose first that $\liminf_r q_r^\alpha > 1$. Then there exists $\sigma > 0$ such that $q_r^\alpha \geqslant 1 + \sigma$ for sufficiently large r which implies that

$$\frac{h_r^\alpha}{k_r^\alpha} \geqslant \frac{\sigma}{1+\sigma}.$$

Since $x \overset{S^L(\mathcal{I}_W)^\alpha}{\sim} y$, then for every $\epsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r^\alpha}|\{k \leqslant k_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| &\geqslant \frac{1}{k_r^\alpha}|\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \\ &\geqslant \frac{\sigma}{1+\sigma}.\frac{1}{h_r^\alpha}|\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}|.\end{aligned}$$

Then for any $\delta > 0$, we get

$$\begin{aligned}&\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}|\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \geqslant \delta\} \\ &\subseteq \{r \in \mathbb{N} : \frac{1}{k_r^\alpha}|\{k \leqslant k_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \geqslant \frac{\delta\sigma}{(1+\sigma)}\} \in \mathcal{I}.\end{aligned}$$

This proves the result. □

Remark 3.14. The converse of this result is not clear for $\alpha < 1$ and we leave it as an open problem.

For the next result we assume that the lacunary sequence θ satisfies the condition that for any set $C \in F(\mathcal{I})$, $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(\mathcal{I})$.

Theorem 3.15. *For a lacunary sequence θ satisfying the above condition,*

$$\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\} \quad \textit{implies} \quad \{A_k\} \overset{S^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\},$$

if $B := \sup_r \sum_{i=0}^{r-1} \frac{h_{i+1}^\alpha}{(k_{r-1})^\alpha} < \infty$.

Proof. Suppose that $\{A_k\} \overset{S_\theta^L(\mathcal{I}_W)^\alpha}{\sim} \{B_k\}$ and for $\epsilon, \delta, \delta_1 > 0$ define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{{h_r}^\alpha}|\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| < \delta\},$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n^\alpha}|\{k \leqslant n : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| < \delta_1\}.$$

It is obvious from our assumption that $C \in F(\mathcal{I})$, the filter associated with the ideal $\mathcal{I}$. Further observe that

$$A_j = \frac{1}{{h_j}^{\alpha}} |\{k \in I_j : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| < \delta,$$

for all $j \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now

$$\begin{aligned}
\frac{1}{n^{\alpha}} |\{k \leqslant n : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| &\leqslant \frac{1}{k_{r-1}^{\alpha}} |\{k \leqslant k_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \\
&= \frac{1}{k_{r-1}^{\alpha}} |\{k \in I_1 : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| + \cdots \\
&\quad + \frac{1}{k_{r-1}^{\alpha}} |\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \\
&= \frac{k_1^{\alpha}}{k_{r-1}^{\alpha}} \frac{1}{h_1^{\alpha}} |\{k \in I_1 : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \\
&\quad + \frac{(k_2 - k_1)^{\alpha}}{k_{r-1}^{\alpha}} \frac{1}{h_2^{\alpha}} |\{k \in I_2 : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| + \cdots \\
&\quad + \frac{(k_r - k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}} \frac{1}{h_r^{\alpha}} |\{k \in I_r : |d(x; A_k, B_k) - L| \geqslant \epsilon\}| \\
&= \frac{k_1^{\alpha}}{k_{r-1}^{\alpha}} A_1 + \frac{(k_2 - k_1)^{\alpha}}{k_{r-1}^{\alpha}} A_2 + \cdots + \frac{(k_r - k_{r-1})^{\alpha}}{k_{r-1}^{\alpha}} A_r \\
&\leqslant \sup_{j \in C} A_j . \sup_r \sum_{i=0}^{r-1} \frac{(k_{i+1} - k_i)^{\alpha}}{k_{r-1}^{\alpha}} < B\delta.
\end{aligned}$$

Choosing $\delta_1 = \frac{\delta}{B}$ and in view of the fact that $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$ where $C \in F(\mathcal{I})$ it follows from our assumption on θ that the set T also belongs to $F(\mathcal{I})$ and this completes the proof of the theorem. □

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