Available online at www.isr-publications.com/jnsa
J. Nonlinear Sci. Appl., 10 (2017), 2798–2815
Research Article

ISSN: 2008-1898

Journal of Nonlinear Sciences and Applications

Journal Homepage: www.tjnsa.com - www.isr-publications.com/jnsa

Fourier series of sums of products of Bernoulli functions and their applications

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Communicated by Y. J. Cho

Abstract

We consider three types of sums of products of Bernoulli functions and derive their Fourier series expansions. In addition, we express each of them in terms of Bernoulli functions.

Keywords: Fourier series, Bernoulli polynomials, Bernoulli functions.
2010 MSC: 11B68, 11S80, 05A19, 05A30.

1. Introduction

Let $B_m(x)$ be the Bernoulli polynomials given by the generating function

$$\frac{t}{e^t-1}e^{xt} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!}, \quad (\text{see } [1, 2, 7, 9, 15, 17]).$$

For any real number x, we let

$$<x>= x-[x] \in [0,1),$$

denote the fractional part of x.

Here we will consider the following three types of sums of products of Bernoulli functions and derive their Fourier series expansions. Further, we will express each of them in terms of Bernoulli functions $B_m(<x>)$.

(1) $\alpha_m(<x>) = \sum_{k=0}^{m} B_k(<x>)B_{m-k}(<x>)$, $(m \geqslant 1)$;

(2) $\beta_m(<x>) = \sum_{k=0}^{m} \frac{1}{k!(m-k)!}B_k(<x>)B_{m-k}(<x>)$, $(m \geqslant 1)$;

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doi:10.22436/jnsa.010.05.46

Received 2016-12-19

(3) $\gamma_m(<x>) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(<x>)B_{m-k}(<x>)$, $(m \geqslant 2)$.

For elementary facts about Fourier analysis, the reader may refer to any book (for example, see [6, 8, 18]).

As to $\gamma_m(<x>)$, we note that the following polynomial identity follows immediately from (4.3) and (4.4).

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x) B_{m-k}(x)\\&= \frac{2}{m^2}\left(B_m + \frac{1}{2}\right) + \frac{2}{m}\sum_{k=1}^{m-2} \frac{1}{m-k}\binom{m}{k} B_{m-k}B_k(x) + \frac{2}{m}H_{m-1}B_m(x), \quad (m \geqslant 2),\end{aligned} \tag{1.1}$$

where $H_m = \sum_{j=1}^{m} \frac{1}{j}$ are the harmonic numbers.

Simple modification of (1.1) yields

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k}(x) B_{2m-2k}(x) + \frac{2}{2m-1} B_1(x) B_{2m-1}(x)\\&= \frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k}B_{2m-2k}(x) + \frac{1}{m}H_{2m-1}B_{2m}(x)\\&+ \frac{2}{2m-1}B_1(x)B_{2m-1}, \quad (m \geqslant 2).\end{aligned} \tag{1.2}$$

Letting $x = 0$ in (1.2) gives a slightly different version of the well-known Miki's identity (see [14]):

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} B_{2k}B_{2m-2k}\\&= \frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k}B_{2m-2k} + \frac{1}{m}H_{2m-1}B_{2m}, \quad (m \geqslant 2).\end{aligned} \tag{1.3}$$

Setting $x = \frac{1}{2}$ in (1.3) with $\overline{B}_m = \left(\frac{1-2^{m-1}}{2^{m-1}}\right) B_m = \left(2^{1-m} - 1\right) B_m = B_m\left(\frac{1}{2}\right)$, we have

$$\begin{aligned}&\sum_{k=1}^{m-1} \frac{1}{2k(2m-2k)} \overline{B}_{2k}\overline{B}_{2m-2k}\\&= \frac{1}{m}\sum_{k=1}^{m} \frac{1}{2k}\binom{2m}{2k} B_{2k}\overline{B}_{2m-2k} + \frac{1}{m}H_{2m-1}\overline{B}_{2m}, \quad (m \geqslant 2),\end{aligned} \tag{1.4}$$

which is the Faber-Pandharipande-Zagier identity (see [4]). Some of the different proofs of Miki's identity can be found in [3, 5, 14, 16]. Dunne-Schubert in [3] uses the asymptotic expansion of some special polynomials coming from the quantum field theory computations, Gessel in [5] is based on two different expressions for Stirling numbers of the second kind $S_2(n, k)$, Miki in [14] exploits a formula for the Fermat quotient $\frac{a^p - a}{p}$ modulo p^2, and Shiratani-Yokoyama in [16] employs p-adic analysis. As we can see, all of these proofs are quite involved. On the other hand, our proof of Miki's and Faber-Pandharipande-Zagier identities follow from the polynomial identity (1.1), which in turn follows immediately the Fourier series expansion results for $\gamma_m(<x>)$ in Theorems 4.3 and 4.4, together with the elementary manipulations outlined in (1.2)-(1.4). The obvious polynomial identities can be derived also for $\alpha_m(<x>)$ and $\beta_m(<x>)$ from (2.6) and (2.7), and (3.3) and (3.4), respectively. Some related works can be found in [10–13].

2. Fourier series of functions of the first type

In this section, we consider the function

$$\alpha_m(<x>) = \sum_{k=0}^{m} B_k(<x>)B_{m-k}(<x>), \tag{2.1}$$

defined on $(-\infty, \infty)$ which is periodic of period 1. The Fourier series of $\alpha_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m)} e^{2\pi i n x}, \tag{2.2}$$

where

$$\begin{aligned} A_n^{(m)} &= \int_0^1 \alpha_m(<x>)e^{-2\pi i n x}dx \\ &= \int_0^1 \alpha_m(x)e^{-2\pi i n x}dx. \end{aligned}$$

Then, we would like to determine the Fourier coefficients $A_n^{(m)}$ in (2.2)
Case 1 : $n \neq 0$,

$$\begin{aligned} A_n^{(m)} = &-\frac{1}{2\pi i n}\left[\alpha_m(x)e^{-2\pi i n x}\right]_0^1 \\ &+ \frac{1}{2\pi i n}\int_0^1 \alpha_m'(x)e^{-2\pi i n x}dx. \end{aligned}$$

We note that $\alpha_m'(x) = (m+1)\alpha_{m-1}(x)$. Indeed,

$$\begin{aligned} \alpha_m'(x) &= \sum_{k=0}^{m}(kB_{k-1}(x)B_{m-k}(x) + (m-k)B_k(x)B_{m-k-1}(x)) \\ &= \sum_{k=1}^{m} kB_{k-1}(x)B_{m-k}(x) + \sum_{k=0}^{m-1}(m-k)B_k(x)B_{m-k-1}(x) \\ &= \sum_{k=0}^{m-1}(k+1)B_k(x)B_{m-k-1}(x) + \sum_{k=0}^{m-1}(m-k)B_k(x)B_{m-k-1}(x) \\ &= (m+1)\sum_{k=0}^{m-1} B_k(x)B_{m-1-k}(x) \\ &= (m+1)\alpha_{m-1}(x). \end{aligned}$$

Hence

$$A_n^{(m)} = -\frac{1}{2\pi i n}(\alpha_m(1) - \alpha_m(0)) + \frac{m+1}{2\pi i n}A_n^{(m-1)}.$$

Observe here that

$$\alpha_m(1) - \alpha_m(0) = 2B_{m-1} + \delta_{m,2}. \tag{2.3}$$

Indeed,

$$\alpha_m(1) - \alpha_m(0) = \sum_{k=0}^{m}(B_k(1)B_{m-k}(1) - B_kB_{m-k})$$

$$
\begin{aligned}
&= \sum_{k=0}^{m} \left((B_k + \delta_{1,k})(B_{m-k} + \delta_{1,m-k}) - B_k B_{m-k}\right) \\
&= \sum_{k=0}^{m} \left(B_k \delta_{1,m-k} + \delta_{1,k} B_{m-k} + \delta_{1,k}\delta_{1,m-k}\right) \\
&= 2B_{m-1} + \delta_{m,2}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
A_n^{(m)} &= \frac{m+1}{2\pi i n} A_n^{(m-1)} - \frac{1}{2\pi i n}(2B_{m-1} + \delta_{m,2}) \\
&= \frac{m+1}{2\pi i n}\left(\frac{m}{2\pi i n} A_n^{(m-2)} - \frac{1}{2\pi i n}(2B_{m-2} + \delta_{m,3})\right) - \frac{1}{2\pi i n}(2B_{m-1} + \delta_{m,2}) \\
&= \frac{(m+1)m}{(2\pi i n)^2} A_n^{(m-2)} - \frac{m+1}{(2\pi i n)^2}(2B_{m-2} + \delta_{m,3}) - \frac{1}{2\pi i n}(2B_{m-1} + \delta_{m,2}) \\
&= \frac{(m+1)m}{(2\pi i n)^2}\left\{\frac{m-1}{2\pi i n} A_n^{(m-3)} - \frac{1}{2\pi i n}(2B_{m-3} + \delta_{m,4})\right\} \\
&\quad - \frac{m+1}{(2\pi i n)^2}(2B_{m-2} + \delta_{m,3}) - \frac{1}{2\pi i n}(2B_{m-1} + \delta_{m,2}) \\
&= \frac{(m+1)_3}{(2\pi i n)^3} A_n^{(m-3)} - \sum_{k=1}^{3} \frac{(m+1)_{k-1}}{(2\pi i n)^k}(2B_{m-k} + \delta_{m,k+1}) \\
&\ \ \vdots \\
&= \frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} A_n^{(1)} - \sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^k}(2B_{m-k} + \delta_{m,k+1}) \\
&= \frac{(m+1)_{m-1}}{(2\pi i n)^{m-1}} A_n^{(1)} - 2\sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^k} B_{m-k} - \frac{(m+1)_{m-2}}{(2\pi i n)^{m-1}},
\end{aligned}
$$

and

$$
\begin{aligned}
A_n^{(1)} &= \int_0^1 \alpha_1(x) e^{-2\pi i n x} dx \\
&= \int_0^1 2B_1(x) e^{-2\pi i n x} dx \\
&= \int_0^1 (2x-1) e^{-2\pi i n x} dx \\
&= 2\int_0^1 x e^{-2\pi i n x} dx - \int_0^1 e^{-2\pi i n x} dx \\
&= 2\left\{-\frac{1}{2\pi i n}\left[x e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 e^{-2\pi i n x} dx\right\} = -\frac{1}{\pi i n}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
A_n^{(m)} &= -\frac{2(m+1)_{m-1}}{(2\pi i n)^m} - 2\sum_{k=1}^{m-1} \frac{(m+1)_{k-1}}{(2\pi i n)^k} B_{m-k} - \frac{(m+1)_{m-2}}{(2\pi i n)^{m-1}} \\
&= -2 \sum_{k=1, k\neq m-1}^{m} \frac{(m+1)_{k-1}}{(2\pi i n)^k} B_{m-k}.
\end{aligned}
$$

Case 2: $n = 0$,

$$A_0^{(m)} = \int_0^1 \alpha_m(x)dx. \tag{2.4}$$

Then, we have the following theorem.

Theorem 2.1. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$\begin{aligned} A_0^{(m)} &= \begin{cases} \frac{2}{m+2}B_m, & \text{for} \quad m \geqslant 2, \\ 0 \quad, & \text{for} \quad m = 1, \end{cases} \\ &= \frac{2}{m+2}\left(B_m + \frac{1}{2}\delta_{m,1}\right), \quad \text{for} \quad m \geqslant 1. \end{aligned}$$

Proof. Let

$$I_{m,n} = \int_0^1 B_m(x)B_n(x)dx,$$

for $m, n \geqslant 0$. Then $I_{m,n} = I_{n,m}$, $I_{0,0} = 1$, $I_{m,0} = 0$, for $m \geqslant 1$. We can show, by integration by parts, that

$$I_{m,n} = (-1)^{m-1}\frac{B_{m+n}}{\binom{m+n}{m}}, \quad \text{for } m, n \geqslant 1.$$

Thus, from (2.1) and (2.4), we have

$$\begin{aligned} A_0^{(m)} &= \sum_{k=0}^{m} \int_0^1 B_k(x)B_{m-k}(x)dx, \\ &= I_{0,m} + \sum_{k=1}^{m-1} I_{k,m-k} + I_{m,0} \\ &= \sum_{k=1}^{m-1} (-1)^{k-1}\frac{B_m}{\binom{m}{k}} \\ &= B_m \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{\binom{m}{k}} \\ &= \frac{2}{m+2}B_m\delta_{m\equiv 0} \\ &= \frac{2}{m+2}\left(B_m + \frac{1}{2}\delta_{m,1}\right), \end{aligned}$$

where

$$\delta_{m\equiv 0} = \begin{cases} 0, & \text{if} \quad m \equiv 0 \pmod 2, \\ 1, & \text{if} \quad m \equiv 1 \pmod 2. \end{cases}$$

□

From (2.3), we observe that

$$\alpha_m(0) = \alpha_m(1) \Longleftrightarrow 2B_{m-1} + \delta_{m,2} = 0.$$

As $B_1 = -\frac{1}{2}$, $B_{2n+1} = 0$, for $n \geqslant 1$, and $(-1)^{n+1}B_{2n} > 0$, for $n \geqslant 1$, we see that

$$2B_{m-1} + \delta_{m,2} = 0 \ (2B_{m-1} + \delta_{m,2} \neq 0) \Longleftrightarrow m \text{ is an even positive integer} \\ (m \text{ is an odd positive integer}). \tag{2.5}$$

$\alpha_m(< x >)$ is piecewise C^∞. In addition, $\alpha_m(< x >)$ is continuous for all even positive integers m and discontinuous with jump discontinuities at integers for all odd positive integers m. We now recall the following facts about Bernoulli functions $B_m(< x >)$:

(a) for $m \geqslant 2$,

$$B_m(< x >) = -m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m};$$

(b) for $m = 1$,

$$-\sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{2\pi i n} = \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases}$$

where $\mathbb{Z}^c$ is $\mathbb{R} - \mathbb{Z}$.

Assume first that m is an even positive integer. Then $\alpha_m(1) = \alpha_m(0)$, and thus $\alpha_m(< x >)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\alpha_m(< x >)$ converges uniformly to $\alpha_m(< x >)$, and

$$\begin{aligned} \alpha_m(< x >) &= \frac{2}{m+2}B_m + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-2 \sum_{k=1, k\neq m-1}^{m} \frac{(m+1)_{k-1}}{(2\pi i n)^k} B_{m-k} \right) e^{2\pi i n x} \\ &= \frac{2}{m+2}B_m + 2 \sum_{k=1, k\neq m-1}^{m} \frac{(m+1)_{k-1}}{k!} B_{m-k} \left(-k! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right). \end{aligned}$$

If $m \geqslant 4$ and m is an even integer, then we have

$$\begin{aligned} \alpha_m(< x >) &= \frac{2}{m+2}B_m + 2 \sum_{k=2, k\neq m-1}^{m} \frac{(m+1)_{k-1}}{k!} B_{m-k} B_k(< x >) \\ &\quad + 2B_{m-1} \times \begin{cases} B_1(< x >), & \text{for} \quad x \in \mathbb{Z}^c, \\ 0, & \text{for} \quad x \in \mathbb{Z}, \end{cases} \\ &= \frac{2}{m+2}B_m + 2 \sum_{k=2, k\neq m-1}^{m} \frac{(m+1)_{k-1}}{k!} B_{m-k} B_k(< x >). \end{aligned}$$

For $m = 2$,

$$\alpha_2(< x >) = \frac{2}{2+2}B_2 + 2 \sum_{k=2}^{2} \frac{(2+1)_{k-1}}{k!} B_{2-k} B_k(< x >).$$

Thus we obtain the following theorem.

Theorem 2.2. *Let m be an even positive integer. Then we have the following.*

(a) $\sum_{k=0}^{m} B_k(< x >)B_{m-k}(< x >)$ *has the Fourier series expansion*

$$\begin{aligned} &\sum_{k=0}^{m} B_k(< x >)B_{m-k}(< x >) \\ &\qquad = \frac{2}{m+2}B_m + \sum_{n=-\infty, n\neq 0}^{\infty} \left(-2 \sum_{k=1, k\neq m-1}^{m} \frac{(m+1)_{k-1}}{(2\pi i n)^k} B_{m-k} \right) e^{2\pi i n x}, \end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}\sum_{k=0}^{m} B_k(<x>)B_{m-k}(<x>) &= \frac{2}{m+2}B_m + 2\sum_{k=2,k\neq m-1}^{m}\frac{(m+1)_{k-1}}{k!}B_{m-k}B_k(<x>)\\ &= (m+1)B_m(<x>) + \frac{2}{2+m}\sum_{k=0}^{m-2}\binom{m+2}{k}B_{m-k}B_k(<x>),\end{aligned} \tag{2.6}$$

for all $x \in (-\infty, \infty)$. *Here* $B_k(<x>)$ *is the Bernoulli function.*

Assume next that m is an odd positive integer. Then $\alpha_m(1) \neq \alpha_m(0)$, and hence $\alpha_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\alpha_m(<x>)$ converges pointwise to $\alpha_m(<x>)$, for $x \in \mathbb{Z}^c$, and converges to

$$\begin{aligned}\frac{1}{2}(\alpha_m(0)+\alpha_m(1)) &= \alpha_m(0) + B_{m-1} + \frac{1}{2}\delta_{m,2}\\ &= \alpha_m(0) + B_{m-1},\end{aligned}$$

for $x \in \mathbb{Z}$. Hence we get the following theorem.

Theorem 2.3. *Let* m *be an odd positive integer. Then we have the following:*

(a)

$$\begin{aligned}&\frac{2}{m+2}\left(B_m + \frac{1}{2}\delta_{m,1}\right) + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-2\sum_{k=1,k\neq m-1}^{m}\frac{(m+1)_{k-1}}{(2\pi i n)^k}B_{m-k}\right)e^{2\pi i n x}\\ &\quad= \begin{cases}\sum_{k=0}^{m} B_k(<x>)B_{m-k}(<x>), & \text{for} \quad x \in \mathbb{Z}^c,\\ \sum_{k=0}^{m} B_k B_{m-k} + B_{m-1}, & \text{for} \quad x \in \mathbb{Z}.\end{cases}\end{aligned}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned}&\frac{2}{m+2}\left(B_m + \frac{1}{2}\delta_{m,1}\right) + 2\sum_{k=1,k\neq m-1}^{m}\frac{(m+1)_{k-1}}{k!}B_{m-k}B_k(<x>)\\ &\quad= \sum_{k=0}^{m} B_k(<x>)B_{m-k}(<x>),\end{aligned} \tag{2.7}$$

for $x \in \mathbb{Z}^c$.

$$\begin{aligned}&\frac{2}{m+2}\left(B_m + \frac{1}{2}\delta_{m,1}\right) + 2\sum_{k=1,k\neq m-1}^{m}\frac{(m+1)_{k-1}}{k!}B_{m-k}B_k(<x>)\\ &\quad= \sum_{k=0}^{m} B_k B_{m-k} + B_{m-1},\end{aligned}$$

for $x \in \mathbb{Z}$.

3. Fourier series of functions of the second type

In this section, we consider the function

$$\beta_m(<x>)=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}B_k(<x>)B_{m-k}(<x>),$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\beta_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}B_n^{(m)}e^{2\pi inx},\tag{3.1}$$

where

$$\begin{aligned}B_n^{(m)}&=\int_0^1\beta_m(<x>)e^{-2\pi inx}dx\\&=\int_0^1\beta_m(x)e^{-2\pi inx}dx.\end{aligned}$$

Then, we would like to determine the Fourier coefficients $B_n^{(m)}$ in (3.1). For this, we observe that

$$\begin{aligned}\beta'_m(x)&=\sum_{k=0}^{m}\left\{\frac{k}{k!(m-k)!}B_{k-1}(x)B_{m-k}(x)+\frac{m-k}{k!(m-k)!}B_k(x)B_{m-k-1}(x)\right\}\\&=\sum_{k=1}^{m}\frac{1}{(k-1)!(m-k)!}B_{k-1}(x)B_{m-k}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}B_k(x)B_{m-k-1}(x)\\&=\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}B_k(x)B_{m-k-1}(x)+\sum_{k=0}^{m-1}\frac{1}{k!(m-k-1)!}B_k(x)B_{m-k-1}(x)=2\beta_{m-1}(x).\end{aligned}$$

Case 1 : $n\neq 0$,

$$\begin{aligned}B_n^{(m)}&=-\frac{1}{2\pi in}\left[\beta_m(x)e^{-2\pi inx}\right]_0^1+\frac{1}{2\pi in}\int_0^1\beta'_m(x)e^{-2\pi inx}dx\\&=-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)+\frac{1}{\pi in}\int_0^1\beta_{m-1}(x)e^{-2\pi inx}dx\\&=-\frac{1}{2\pi in}\left(\beta_m(1)-\beta_m(0)\right)+\frac{1}{\pi in}B_n^{(m-1)}.\end{aligned}$$

Observe that

$$\begin{aligned}\beta_m(1)-\beta_m(0)&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\left(B_k(1)B_{m-k}(1)-B_kB_{m-k}\right)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\{(B_k+\delta_{1,k})(B_{m-k}+\delta_{1,m-k})-B_kB_{m-k}\}\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}\{B_k\delta_{1,m-k}+B_{m-k}\delta_{1,k}+\delta_{1,k}\delta_{1,m-k}\}\\&=\frac{1}{(m-1)!}B_{m_1}+\frac{1}{(m-1)!}B_{m-1}+\frac{1}{(m-1)!}\delta_{1,m-1}=\frac{1}{(m-1)!}(2B_{m-1}+\delta_{m,2}).\end{aligned}\tag{3.2}$$

Hence,

$$
\begin{aligned}
B_n^{(m)} &= \frac{1}{\pi i n} B_n^{(m-1)} - \frac{1}{2\pi i n(m-1)!}(2B_{m-1}+\delta_{m,2}) \\
&= \frac{1}{\pi i n}\left(\frac{1}{\pi i n} B_n^{(m-2)} - \frac{1}{2\pi i n(m-2)!}(2B_{m-2}+\delta_{m,3})\right) \\
&\quad - \frac{1}{2\pi i n(m-1)!}(2B_{m-1}+\delta_{m,2}) \\
&= \frac{1}{(\pi i n)^2} B_n^{(m-2)} - \frac{1}{2(m-2)!}\frac{1}{(\pi i n)^2}(2B_{m-2}+\delta_{m,3}) \\
&\quad - \frac{1}{2(m-1)!}\frac{1}{\pi i n}(2B_{m-1}+\delta_{m,2}) \\
&= \frac{1}{(\pi i n)^2}\left(\frac{1}{\pi i n} B_n^{(m-3)} - \frac{1}{2\pi i n(m-3)!}(2B_{m-3}+\delta_{m,4})\right) \\
&\quad - \frac{1}{(m-2)!}\frac{1}{(\pi i n)^2}(2B_{m-2}+\delta_{m,3}) - \frac{1}{2(m-1)!}\frac{1}{\pi i n}(2B_{m-1}+\delta_{m,2}) \\
&= \frac{1}{(\pi i n)^3} B_n^{(m-3)} - \frac{1}{2(m-3)!}\frac{1}{(\pi i n)^3}(2B_{m-3}+\delta_{m,4}) \\
&\quad - \frac{1}{2(m-2)!}\frac{1}{(\pi i n)^2}(2B_{m-2}+\delta_{m,3}) - \frac{1}{2(m-1)!}\frac{1}{\pi i n}(2B_{m-1}+\delta_{m,2}) \\
&= \\
&\ \vdots \\
&= \frac{1}{(\pi i n)^{m-1}} B_n^{(1)} - \sum_{k=1}^{m-1}\frac{1}{2(m-k)!}\frac{1}{(\pi i n)^k}(2B_{m-k}+\delta_{m,k+1}),
\end{aligned}
$$

and

$$
\begin{aligned}
B_n^{(1)} &= \int_0^1 \beta_1(x) e^{-2\pi i n x}\,dx \\
&= \int_0^1 2B_1(x) e^{-2\pi i n x}\,dx \\
&= \int_0^1 (2x-1) e^{-2\pi i n x}\,dx \\
&= 2\int_0^1 x e^{-2\pi i n x}\,dx - \int_0^1 e^{-2\pi i n x}\,dx \\
&= -\frac{1}{\pi i n}\left[x e^{-2\pi i n x}\right]_0^1 + \frac{1}{\pi i n}\int_0^1 e^{-2\pi i n x}\,dx \\
&= -\frac{1}{\pi i n}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
B_n^{(m)} &= -\frac{1}{(\pi i n)^m} - \sum_{k=1}^{m-1}\frac{1}{2(m-k)!}\frac{1}{(\pi i n)^k}(2B_{m-k}+\delta_{m,k+1}) \\
&= -\frac{1}{(\pi i n)^m} - \sum_{k=1}^{m-1}\frac{1}{(m-k)!}\frac{1}{(\pi i n)^k}B_{m-k} - \frac{1}{2}\frac{1}{(\pi i n)^{m-1}} \\
&= -\sum_{k=1, k\neq m-1}^{m}\frac{1}{(m-k)!}\frac{1}{(\pi i n)^k}B_{m-k}.
\end{aligned}
$$

Case 2: $n = 0$,

$$B_0^{(m)} = \int_0^1 \beta_m(x)dx.$$

Then, we have the following theorem.

Theorem 3.1. *For* $n \in \mathbb{N} \cup \{0\}$, *we have*

$$\begin{aligned} B_0^{(m)} &= \begin{cases} \frac{B_m}{m!}, & \text{for} \quad m \equiv 0 \pmod 2, \\ 0 \quad , & \text{for} \quad m \equiv 1 \pmod 2, \end{cases} \\ &= \frac{B_m}{m!} + \frac{1}{2}\delta_{m,1}, \quad \text{for} \ \ m \geqslant 1. \end{aligned}$$

Proof. Let

$$\begin{aligned} B_0^{(m)} &= \int_0^1 \beta_m(x) \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} \int_0^1 B_k(x)B_{m-k}(x)dx \\ &= \sum_{k=0}^{m} \frac{1}{k!(m-k)!} I_{k,m-k} \\ &= \frac{2}{m!} I_{0,m} + \sum_{k=1}^{m-1} \frac{1}{k!(m-k)!} \frac{(-1)^{k-1}B_m}{\binom{m}{k}} \\ &= \frac{B_m}{m!} \sum_{k=1}^{m-1} (-1)^{k-1} \\ &= \begin{cases} \frac{B_m}{m!}, & \text{for} \quad m \equiv 0 \pmod 2, \\ 0 \quad , & \text{for} \quad m \equiv 1 \pmod 2. \end{cases} \\ &= \frac{B_m}{m!} + \frac{1}{2}\delta_{m,1}, \end{aligned}$$

□

From (3.2), we see that

$$\beta_m(0) = \beta_m(1) \Longleftrightarrow 2B_{m-1} + \delta_{m,2} = 0.$$

As we saw in (2.5),

$$2B_{m-1} + \delta_{m,2} = 0 \ (2B_{m-1} + \delta_{m,2} \neq 0) \Longleftrightarrow m \text{ is an even positive integer}$$
$$(m \text{ is an odd positive integer}).$$

$\beta_m(< x >)$ is piecewise C^∞. In addition, $\beta_m(< x >)$ is continuous for all even positive integers m and discontinuous with jump discontinuities at integers for all odd positive integers m.

Assume first that m is an even positive integer. Then $\beta_m(1) = \beta_m(0)$, and so $\beta_m(< x >)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\beta_m(< x >)$ converges uniformly to $\beta_m(< x >)$, and

$$\begin{aligned} \beta_m(< x >) &= \frac{B_m}{m!} + \sum_{n=-\infty, n\neq 0}^{\infty} \left(- \sum_{k=1, k\neq m-1}^{m} \frac{1}{(m-k)!} \frac{1}{(\pi i n)^k} B_{m-k} \right) e^{2\pi i n x} \\ &= \frac{B_m}{m!} + \sum_{k=1, k\neq m-1}^{m} \frac{2^k}{(m-k)!k!} B_{m-k} \left(-k! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right). \end{aligned}$$

If $m \geqslant 4$ and m is an even integer, then we have

$$\begin{aligned}\beta_m(<x>) &= \frac{B_m}{m!} + \frac{1}{m!}\sum_{k=2,k\neq m-1}^{m} 2^k \binom{m}{k} B_{m-k}B_k(<x>) \\ &\quad + \frac{1}{m!}2\binom{m}{1}B_{m-1}, \times \begin{cases} B_1(<x>), & \text{for } x \in \mathbb{Z}^c, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases} \\ &= \frac{B_m}{m!} + \frac{1}{m!}\sum_{k=2,k\neq m-1}^{m} 2^k \binom{m}{k} B_{m-k}B_k(<x>).\end{aligned}$$

For $m = 2$,

$$\beta_2(<x>) = \frac{B_2}{2!} + \frac{1}{2!}\sum_{k=2}^{2} 2^k \binom{2}{k} B_{2-k}B_k(<x>).$$

Now, we obtain the following theorem.

Theorem 3.2. *Let m be an even positive integer. Then we have the following.*

(a) $\sum_{k=0}^{m} \frac{1}{k!(m-k)!} B_k(<x>)B_{m-k}(<x>)$ *has the Fourier expansion*

$$\begin{aligned}&\sum_{k=0}^{m} \frac{1}{k!(m-k)!} B_k(<x>)B_{m-k}(<x>) \\ &= \frac{B_m}{m!} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{k=1,k\neq m-1}^{m} \frac{1}{(m-k)!}\frac{1}{(\pi i n)^k}B_{m-k}\right)e^{2\pi i n x},\end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\sum_{k=0}^{m} \frac{1}{k!(m-k)!} B_k(<x>)B_{m-k}(<x>) = \frac{B_m}{m!} + \frac{1}{m!}\sum_{k=2,k\neq m-1}^{m} 2^k \binom{m}{k} B_{m-k}B_k(<x>), \tag{3.3}$$

for all $x \in (-\infty, \infty)$. Here $B_k(<x>)$ is the Bernoulli function.

Assume next that m is an odd positive integer. Then $\beta_m(1) \neq \beta_m(0)$, and hence $\beta_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\beta_m(<x>)$ converges pointwise to $\beta_m(<x>)$, for $x \in \mathbb{Z}^c$ and converges to

$$\begin{aligned}\frac{1}{2}(\beta_m(0) + \beta_m(1)) &= \frac{1}{2}\left(\beta_m(0) + \beta_m(0) + \frac{1}{(m-1)!}(2B_{m-1} + \delta_{m,2})\right) \\ &= \beta_m(0) + \frac{1}{(m-1)!}B_{m-1},\end{aligned}$$

for $x \in \mathbb{Z}$. Hence we have the following theorem.

Theorem 3.3. *Let m be an odd positive integer. Then we have the following.*

(a)

$$\begin{aligned}&\frac{B_m}{m!} + \frac{1}{2}\delta_{m,1} + \sum_{n=-\infty,n\neq 0}^{\infty}\left(-\sum_{k=1,k\neq m-1}^{m} \frac{1}{(m-k)!}\frac{1}{(\pi i n)^k}B_{m-k}\right)e^{2\pi i n x} \\ &= \begin{cases} \sum_{k=0}^{m} \frac{1}{k!(m-k)!} B_k(<x>)B_{m-k}(<x>), & \text{for } x \in \mathbb{Z}^c, \\ \sum_{k=0}^{m} \frac{1}{k!(m-k)!} B_k B_{m-k} + \frac{1}{(m-1)!}B_{m-1}, & \text{for } x \in \mathbb{Z}. \end{cases}\end{aligned}$$

Here the convergence is pointwise.

(b)

$$\begin{aligned}&\frac{B_m}{m!}+\frac{1}{2}\delta_{m,1}+\frac{1}{m!}\sum_{k=1,k\neq m-1}^{m}2^k\binom{m}{k}B_{m-k}B_k(<x>)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}B_k(<x>)B_{m-k}(<x>),\end{aligned}\tag{3.4}$$

for $x\in\mathbb{Z}^c$.

$$\begin{aligned}&\frac{B_m}{m!}+\frac{1}{2}\delta_{m,1}+\frac{1}{m!}\sum_{k=2,k\neq m-1}^{m}2^k\binom{m}{k}B_{m-k}B_k(<x>)\\&=\sum_{k=0}^{m}\frac{1}{k!(m-k)!}B_kB_{m-k}+\frac{1}{(m-1)!}B_{m-1},\end{aligned}$$

for $x\in\mathbb{Z}$.

4. Fourier series of functions of the third type

In this section, we consider the function

$$\gamma_m(<x>)=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>),$$

defined on $(-\infty,-\infty)$ which is periodic of period 1. The Fourier series of $\gamma_m(<x>)$ is

$$\sum_{n=-\infty}^{\infty}C_n^{(m)}e^{2\pi inx},\tag{4.1}$$

where

$$C_n^{(m)}=\int_0^1\gamma_m(<x>)e^{-2\pi inx}dx=\int_0^1\gamma_m(x)e^{-2\pi inx}dx.$$

Next, we would like to determine the Fourier coefficients $C_n^{(m)}$ in (4.1). For this, we observe that

$$\begin{aligned}\gamma_m'(x)&=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}\{kB_{k-1}(x)B_{m-k}(x)+(m-k)B_k(x)B_{m-k-1}(x)\}\\&=\sum_{k=1}^{m-1}\frac{1}{m-k}B_{k-1}(x)B_{m-k}(x)+\sum_{k=1}^{m-1}\frac{1}{k}B_k(x)B_{m-k-1}(x)\\&=\sum_{k=0}^{m-2}\frac{1}{m-1-k}B_k(x)B_{m-1-k}(x)+\sum_{k=1}^{m-1}\frac{1}{k}B_k(x)B_{m-1-k}(x)\\&=\frac{2}{m-1}B_{m-1}(x)+\sum_{k=1}^{m-2}\left(\frac{1}{m-1-k}+\frac{1}{k}\right)B_k(x)B_{m-1-k}(x)\\&=\frac{2}{m-1}B_{m-1}(x)+(m-1)\gamma_{m-1}(x).\end{aligned}$$

Case 1 : $n \neq 0$,

$$
\begin{aligned}
C_n^{(m)} &= -\frac{1}{2\pi i n}\left[\gamma_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \gamma_m'(x)e^{-2\pi i n x}dx \\
&= -\frac{1}{2\pi i n}\left(\gamma_m(1)-\gamma_m(0)\right) + \frac{1}{\pi i n(m-1)}\int_0^1 B_{m-1}(x)e^{-2\pi i n x}dx + \frac{m-1}{2\pi i n}C_n^{(m-1)}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\gamma_m(1)-\gamma_m(0) &= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}(B_k(1)B_{m-k}(1)-B_kB_{m-k}) \\
&= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left((B_k+\delta_{1,k})(B_{m-k}+\delta_{1,m-k})-B_kB_{m-k}\right) \\
&= \sum_{k=1}^{m-1}\frac{1}{k(m-k)}\left(B_k\delta_{1,m-k}+\delta_{1,k}B_{m-k}+\delta_{1,k}\delta_{1,m-k}\right) \\
&= \frac{1}{m-1}B_{m-1}+\frac{1}{m-1}B_{m-1}+\frac{1}{m-1}\delta_{m,2} \\
&= \frac{2}{m-1}B_{m-1}+\frac{1}{m-1}\delta_{m,2} = \frac{1}{m-1}(2B_{m-1}+\delta_{m,2}),
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
\int_0^1 B_{m-1}(x)e^{-2\pi i n x}dx &= \frac{1}{m}\left[B_m(x)e^{-2\pi i n x}\right]_0^1 + \frac{2\pi i n}{m}\int_0^1 B_m(x)e^{-2\pi i n x}dx \\
&= \frac{1}{m}\left(B_m(1)-B_m(0)\right) + \frac{2\pi i n}{m}\int_0^1 B_m(x)e^{-2\pi i n x}dx.
\end{aligned}
$$

Denoting $\int_0^1 B_m(x)e^{-2\pi i n x}dx$ by $\rho_n^{(m)}$, we have

$$
\begin{aligned}
\rho_n^{(m)} &= \frac{m}{2\pi i n}\rho_n^{(m-1)} - \frac{1}{2\pi i n}\delta_{m,1} \\
&= \frac{m}{2\pi i n}\left(\frac{m-1}{2\pi i n}\rho_n^{(m-2)} - \frac{1}{2\pi i n}\delta_{m,2}\right) - \frac{1}{2\pi i n}\delta_{m,1} \\
&= \frac{m(m-1)}{(2\pi i n)^2}\rho_n^{(m-2)} - \frac{m}{(2\pi i n)^2}\delta_{m,2} - \frac{1}{2\pi i n}\delta_{m,1} \\
&= \frac{m(m-1)}{(2\pi i n)^2}\left(\frac{m-2}{2\pi i n}\rho_m^{(m-3)} - \frac{1}{2\pi i n}\delta_{m,3}\right) - \frac{m}{(2\pi i n)^2}\delta_{m,2} - \frac{1}{2\pi i n}\delta_{m,1} \\
&= \frac{m(m-1)(m-2)}{(2\pi i n)^3}\rho_m^{(m-3)} - \frac{m(m-1)}{(2\pi i n)^3}\delta_{m,3} - \frac{m}{(2\pi i n)^2}\delta_{m,2} - \frac{1}{2\pi i n}\delta_{m,1} \\
&\ \vdots \\
&= \frac{m!}{(2\pi i n)^{m-1}}\rho_n^{(1)} - \sum_{k=1}^{m-1}\frac{(m)_{k-1}}{(2\pi i n)^k}\delta_{m,k} = -\frac{m!}{(2\pi i n)^m},
\end{aligned}
$$

as one can see that $\rho_n^{(1)} = -\frac{1}{2\pi i n}$. Thus, for $n \neq 0$,

$$
C_n^{(m)} = \frac{m-1}{2\pi i n}C_n^{(m-1)} - \frac{1}{2\pi i n(m-1)}(2B_{m-1}+\delta_{m,2}) - \frac{2(m-2)!}{(2\pi i n)^m}
$$

$$
\begin{aligned}
&= \frac{m-1}{2\pi in}\left(\frac{m-2}{2\pi in}C_n^{(m-2)} - \frac{1}{2\pi in(m-2)}(2B_{m-2}+\delta_{m,3}) - \frac{2(m-3)!}{(2\pi in)^{m-1}}\right)\\
&\quad - \frac{1}{2\pi in(m-1)}(2B_{m-1}+\delta_{m,2}) - \frac{2(m-2)!}{(2\pi in)^m}\\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}C_n^{(m-2)} - \frac{m-1}{m-2}\frac{1}{(2\pi in)^2}(2B_{m-2}+\delta_{m,3})\\
&\quad - \frac{1}{m-1}\frac{1}{2\pi in}(2B_{m-1}+\delta_{m,2}) - \frac{2(m-1)(m-3)!}{(2\pi in)^m} - \frac{2(m-2)!}{(2\pi in)^m}\\
&= \frac{(m-1)(m-2)}{(2\pi in)^2}\left(\frac{m-3}{2\pi in}C_n^{(m-3)} - \frac{1}{2\pi in(m-3)}(2B_{m-3}+\delta_{m,4}) - \frac{2(m-4)!}{(2\pi in)^{m-2}}\right)\\
&\quad - \frac{m-1}{m-2}\frac{1}{(2\pi in)^2}(2B_{m-2}+\delta_{m,3}) - \frac{1}{m-1}\frac{1}{2\pi in}(2B_{m-1}+\delta_{m,2}) - \frac{2(m-1)(m-3)!}{(2\pi in)^m}\\
&\quad - \frac{2(m-2)!}{(2\pi in)^m}\\
&= \frac{(m-1)(m-2)(m-3)}{(2\pi in)^3}C_n^{(m-3)} - \frac{(m-1)(m-2)}{(m-3)(2\pi in)^3}(2B_{m-3}+\delta_{m,4})\\
&\quad - \frac{m-1}{(m-2)(2\pi in)^2}(2B_{m-2}+\delta_{m,3}) - \frac{1}{(m-1)2\pi in}(2B_{m-1}+\delta_{m,2})\\
&\quad - \frac{2(m-1)(m-2)(m-4)!}{(2\pi in)^m} - \frac{2(m-1)(m-3)!}{(2\pi in)^m} - \frac{2(m-2)!}{(2\pi in)^m}\\
&=\\
&\vdots\\
&= \frac{(m-1)!}{(2\pi in)^{m-2}}C_n^{(2)} - \sum_{k=1}^{m-2}\frac{(m-1)_{k-1}}{(m-k)(2\pi in)^k}(2B_{n-k}+\delta_{m,k+1}) - \frac{2(m-1)!}{(2\pi in)^m}\sum_{k=1}^{m-2}\frac{1}{m-k},
\end{aligned}
$$

and

$$
\begin{aligned}
c_n^{(2)} &= \int_0^1 \gamma_2(x)e^{-2\pi inx}dx\\
&= \int_0^1 B_1(x)^2 e^{-2\pi inx}dx = \int_0^1\left(x^2-x+\frac{1}{4}\right)e^{-2\pi inx}dx\\
&= -\frac{1}{2\pi in}\left[\left(x^2-x+\frac{1}{4}\right)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1(2x-1)e^{-2\pi inx}dx\\
&= \frac{1}{2\pi in}\left(-\frac{1}{2\pi in}\left[(2x-1)e^{-2\pi inx}\right]_0^1 + \frac{1}{2\pi in}\int_0^1 2e^{-2\pi inx}dx\right)\\
&= -\frac{2}{(2\pi in)^2}.
\end{aligned}
$$

We note that

$$\sum_{k=1}^{m-2}\frac{1}{m-k} = \frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m-1} = H_{m-1}-1,$$

where $H_m = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{m}$ is the harmonic number. Thus we have the following theorem.

Theorem 4.1. *For* $n \neq 0$,

$$C_n^{(m)} = -\frac{2(m-1)!}{(2\pi in)^m}H_{m-1} - \sum_{k=1}^{m-2}\frac{(m-1)_{k-1}}{(m-k)(2\pi in)^k}(2B_{m-k}+\delta_{m,k+1}).$$

Case 2: $n = 0$,

$$C_0^{(m)} = \int_0^1 \gamma_m(x)dx.$$

Then, we have the following theorem.

Theorem 4.2. *For $m \in \mathbb{N} \cup \{0\}$, we have*

$$C_0^{(m)} = \begin{cases} \frac{2}{m^2}B_m, & \text{for} \quad m \equiv 0 \pmod 2 \\ 0 \quad , & \text{for} \quad m \equiv 1 \pmod 2 \end{cases} = \frac{2}{m^2}\left(B_m + \frac{1}{2}\delta_{m,1}\right), \quad \text{for} \quad m \geqslant 1.$$

Proof. Let

$$\begin{aligned} C_0^{(m)} &= \int_0^1 \gamma_m(x)dx \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \int_0^1 B_k(x)B_{m-k}(x)dx \\ &= \sum_{k=1}^{m-1} \frac{1}{k(m-k)} I_{k,m-k} = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} \frac{(-1)^{k-1}B_m}{\binom{m}{k}} \\ &= B_m \sum_{k=1}^{m-1} \frac{(-1)^{k-1}}{k(m-k)\binom{m}{k}} \\ &= \begin{cases} \frac{2}{m^2}B_m, & \text{for} \quad m \equiv 0 \pmod 2, \\ 0 \quad , & \text{for} \quad m \equiv 1 \pmod 2, \end{cases} \\ &= \frac{2}{m^2}\left(B_m + \frac{1}{2}\delta_{m,1}\right). \end{aligned}$$

□

From (4.2), we see that

$$\gamma_m(1) - \gamma_m(0) = \frac{1}{m-1}\left(2B_{m-1} + \delta_{m,2}\right).$$

As we saw in (2.5),

$$2B_{m-1} + \delta_{m,2} = 0 \ (2B_{m-1} + \delta_{m,2} \neq 0) \iff m \text{ is an even positive integer} \\ (m \text{ is an odd positive integer}).$$

$\gamma_m(< x >)$ is piecewise C^∞. In addition, $\gamma_m(< x >)$ is continuous for all even positive integers m and discontinuous with jump discontinuities at integers for all odd positive integers m. Assume first that m is an even positive integer. Then $\gamma_m(1) = \gamma_m(0)$, and thus $\gamma_m(< x >)$ is piecewise C^∞, and continuous. Hence the Fourier series of $\gamma_m(< x >)$ converges uniformly to $\gamma_m(< x >)$, and

$$\begin{aligned} \gamma_m(< x >) &= \frac{2}{m^2}B_m + \sum_{n=-\infty, n\neq 0}^{\infty} \left\{ -\frac{2(m-1)!}{(2\pi i n)^m} H_{m-1} - \sum_{k=1}^{m-2} \frac{2(m-1)_{k-1}}{(m-k)(2\pi i n)^k} B_{m-k} \right\} e^{2\pi i n x} \\ &= \frac{2}{m^2}B_m + \frac{2}{m}H_{m-1} \times \left(-m! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^m} \right) \\ &\quad + \frac{2}{m} \sum_{k=1}^{m-2} \frac{1}{m-k} \binom{m}{k} B_{m-k} \left(-k! \sum_{n=-\infty, n\neq 0}^{\infty} \frac{e^{2\pi i n x}}{(2\pi i n)^k} \right). \end{aligned}$$

If $m \geqslant 4$ and m is an even integer, then we have

$$\begin{aligned}\gamma_m(<x>) &= \frac{2}{m^2}B_m + \frac{2}{m}H_{m-1}B_m(<x>) + \frac{2}{m}\sum_{k=2}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(<x>)\\ &\quad + \frac{2}{m}\frac{1}{m-1}\binom{m}{1}B_{m-1} \times \begin{cases} B_1(<x>), & \text{for} \quad x \in \mathbb{Z}^c,\\ 0, & \text{for} \quad x \in \mathbb{Z},\end{cases}\\ &= \frac{2}{m^2}B_m + \frac{2}{m}H_{m-1}B_m(<x>) + \frac{2}{m}\sum_{k=2}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(<x>),\end{aligned}$$

for all $x \in (-\infty, \infty)$. For $m = 2$,

$$\gamma_2(<x>) = \frac{2}{2^2}B_2 + H_1B_2(<x>).$$

Hence, we obtain the following theorem.

Theorem 4.3. *Let m be an even positive integer. Then we have the following.*

(a) $\sum_{k=0}^{m}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>)$ *has the Fourier expansion*

$$\begin{aligned}&\sum_{k=0}^{m}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>)\\ &= \frac{2}{m^2}B_m + \sum_{n=-\infty, n\neq 0}^{\infty}\left\{-\frac{2(m-1)!}{(2\pi i n)^m}H_{m-1} - \sum_{k=1}^{m-2}\frac{2(m-1)_{k-1}}{(m-k)(2\pi i n)^k}B_{m-k}\right\}e^{2\pi i n x},\end{aligned}$$

for all $x \in (-\infty, \infty)$, where the convergence is uniform.

(b)

$$\begin{aligned}&\sum_{k=0}^{m}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>)\\ &= \frac{2}{m^2}B_m + \frac{2}{m}H_{m-1}B_m(<x>) + \frac{2}{m}\sum_{k=2}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(<x>),\end{aligned} \tag{4.3}$$

for all $x \in (-\infty, \infty)$. Here $B_k(<x>)$ is the Bernoulli function.

Assume next that m is an odd positive integer. Then $\gamma_m(1) \neq \gamma_m(0)$, and hence $\gamma_m(<x>)$ is piecewise C^∞ and discontinuous with jump discontinuities at integers. Thus the Fourier series of $\gamma_m(<x>)$ converges pointwise to $\gamma_m(<x>)$, for $x \in \mathbb{Z}^c$ and converges to

$$\begin{aligned}\frac{1}{2}(\gamma_m(0) + \gamma_m(1)) &= \frac{1}{2}\left(\gamma_m(0) + \gamma_m(0) + \frac{2}{m-1}B_{m-1} + \frac{1}{m-1}\delta_{m,2}\right)\\ &= \gamma_m(0) + \frac{1}{m-1}B_{m-1},\end{aligned}$$

for $x \in \mathbb{Z}$. Hence we have the following theorem.

Theorem 4.4. *Let m be an odd positive integer. Then we have the following.*

(a)

$$\frac{2}{m^2}\left(B_m+\frac{1}{2}\delta_{m,1}\right)+\sum_{n=-\infty,n\neq 0}^{\infty}\left\{-\frac{2(m-1)!}{(2\pi in)^m}H_{m-1}-\sum_{k=1}^{m-2}\frac{2(m-1)_{k-1}}{(m-k)(2\pi in)^k}B_{m-k}\right\}e^{2\pi inx}$$
$$=\begin{cases}\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>), & \text{for } x\in\mathbb{Z}^c,\\ \sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_kB_{m-k}+\frac{1}{m-1}B_{m-1}, & \text{for } x\in\mathbb{Z}.\end{cases}$$

Here the convergence is pointwise.

(b)

$$\frac{2}{m^2}(B_m+\frac{1}{2}\delta_{m,1})+\frac{2}{m}H_{m-1}B_m(<x>)+\frac{2}{m}\sum_{k=1}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_k(<x>)B_{m-k}(<x>), \tag{4.4}$$

for $x\in\mathbb{Z}^c$.

$$\frac{2}{m^2}\left(B_m+\frac{1}{2}\delta_{m,1}\right)+\frac{2}{m}H_{m-1}B_m(<x>)+\frac{2}{m}\sum_{k=2}^{m-2}\frac{1}{m-k}\binom{m}{k}B_{m-k}B_k(<x>)$$
$$=\sum_{k=1}^{m-1}\frac{1}{k(m-k)}B_kB_{m-k}+\frac{1}{m-1}B_{m-1},$$

for $x\in\mathbb{Z}$.

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