



The exact controllability of Euler-Bernoulli beam systems with small delays in the boundary feedback controls

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Communicated by A. Atangana

Abstract

This work is concerned with the exact controllability of an Euler-Bernoulli beam system with small delays in the boundary feedback controls

$$\begin{aligned}w_{tt}(x, t) + w_{xxxx}(x, t) &= 0, \quad x \in (0, 1), \quad t > 0, \\w(0, t) = w_x(0, t) &= 0, \quad t \geq 0, \\w_{xx}(1, t - \varepsilon) &= -k_2^2 w_{tx}(1, t) - c_2 w_t(1, t - \varepsilon), \quad \varepsilon > 0, \quad k_1^2 + k_2^2 \neq 0, \\w_{xxx}(1, t) &= k_1^2 w_t(1, t - \varepsilon) - c_1 w_{tx}(1, t - \varepsilon), \quad k_i, c_i \in \mathbb{R}, \quad (i = 1, 2),\end{aligned}$$

with boundary conditions

$$w(x, t) = \varphi(x, t), \quad w_t(x, t) = \psi(x, t), \quad -\varepsilon \leq t \leq 0.$$

Our analysis relies on the exact controllability on Hilbert space M and state space H . Our results based on formulating the original system as a state linear system. We formulate the system as the state feedback control systems $\sum(A, B, C)$, and we get the generalized eigenvectors of the operator A . Then we prove that they can form a Riesz basis for the state space H . In the end, the system is proved to be exactly controllable on H . ©2017 All rights reserved.

Keywords: Euler-Bernoulli beam, delay, boundary feedback control, exact controllability.
2010 MSC: 35J30, 35J35.

1. Introduction

Euler-Bernoulli beam systems arise in many fields such as physics, mechanics, chemistry, economics, engineering and biological science, etc., see [1–5, 10, 12] for example. In these papers, the authors considered the exact controllability of the Euler-Bernoulli plate equation with variable coefficients and mixed boundary conditions. By using the methods of basis perturbation and Weiss regular system, the papers in [7–9] showed that the generalized eigenfunction of a flexible Euler-Bernoulli beam system with boundary feedback consisting of velocity and angular velocity form a Riesz basis for the state space H and they gave the computational formula exponential decay rate of the system, which solved the problems considered in engineering, such as exponential decay rate, eigenvalues and so on. But there is always time-delay in engineering practice. So the research of it is also active up recently, see [6, 11, 13] and the

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doi:[10.22436/jnsa.010.05.44](https://doi.org/10.22436/jnsa.010.05.44)

Received 2017-02-15

references therein. In these papers, many authors investigated the robust stability of time-delay system with time-varying uncertainties via HPLKF. They gave a sufficient condition to demonstrate that the system is asymptotically stable. A new class of Lyapunov-Krasovskii function is introduced, whose main feature is that the conservativeness due to uncertainties is reduced. Numerical examples were given to illustrate the effectiveness of their method.

In this paper, we discuss the following flexible Euler-Bernoulli beam system with small delays in boundary feedback controls

$$\begin{aligned} w_{tt}(x, t) + w_{xxxx}(x, t) &= 0, \quad x \in (0, 1), \quad t > 0, \\ w(0, t) = w_x(0, t) &= 0, \quad t \geq 0, \\ w_{xx}(1, t - \varepsilon) &= -k_2^2 w_{tx}(1, t) - c_2 w_t(1, t - \varepsilon), \quad \varepsilon > 0, \quad k_1^2 + k_2^2 \neq 0, \\ w_{xxx}(1, t) &= k_1^2 w_t(1, t - \varepsilon) - c_1 w_{tx}(1, t - \varepsilon), \quad k_i, c_i \in \mathbb{R}, \quad (i = 1, 2), \end{aligned} \tag{1.1}$$

with boundary conditions

$$w(x, t) = \varphi(x, t), \quad w_t(x, t) = \psi(x, t), \quad -\varepsilon \leq t \leq 0.$$

Different from the works mentioned above, we formulate the system (1.1) as the state feedback control systems $\sum(A, B, C)$, where A is the generator of a C_0 -semigroup on H , B is a control operator and C is an observation operator. Moreover, B and C are unbounded operators. And we formulate the system without time-delay as abstract differential functions on the Hilbert space X of the form

$$\begin{cases} \frac{d\omega(t)}{dt} = A_1 \omega(t), \\ \omega(0) = (u(x, 0), u_t(x, 0))^T. \end{cases}$$

We get the eigenvalues and eigenvectors of the operator A through studying the eigenvalues and eigenvectors of the operator A_1 . Then we prove that $\sum(A, B, C)$ is exactly controllable on H , which is generalized by the eigenvectors of the operator A . Since the system (1.1) and the system $\sum(A, B, C)$ are equivalent, we only need to study the exact controllability of $\sum(A, B, C)$. In the end, (1.1) is proved to be exactly controllable on H .

2. Riesz basis of auxiliary system

To begin with, let

$$H^{(n)}(0, 1) = \{\varphi : (0, 1) \rightarrow \mathbb{C} \mid \varphi^{(k)} \text{ is absolutely continuous } (k = 1, 2, \dots, n - 1), \varphi^{(n)} \in L^2(0, 1)\},$$

$$H_0^2(0, 1) = \{\varphi \in H^2(0, 1) \mid \varphi(0) = \varphi'(0) = 0\}.$$

We denote

$$X = H_0^2(0, 1) \times L^2(0, 1),$$

where

$$((f, g)^T, (u, v)^T)_X = \int_0^1 f''(x) \overline{u''(x)} dx + \int_0^1 g(x) \overline{v(x)} dx. \tag{2.1}$$

Then, X is a Hilbert space under (2.1). Let

$$M = X \times L^2([-\varepsilon, 0], X),$$

where

$$((p, f(\cdot))^T, (q, g(\cdot))^T)_M = (p, q)_X + \int_{-\varepsilon}^0 (f(\theta), g(\theta))_X d\theta. \tag{2.2}$$

Then M is a Hilbert space under (2.2). Construct the system

$$\begin{aligned} w_{tt}(x, t) + w_{xxxx}(x, t) &= 0, \quad x \in (0, 1), \quad t > 0, \\ w(0, t) = w_x(0, t) &= 0, \quad t \geq 0, \\ w_{xx}(1, t) &= -k_2^2 w_{tx}(1, t) - c_2 w_t(1, t), \quad k_1^2 + k_2^2 \neq 0, \\ w_{xxx}(1, t) &= k_1^2 w_t(1, t) - c_1 w_{tx}(1, t), \quad k_i, c_i \in \mathbb{R}, \quad (i = 1, 2), \\ w(x, 0) = \varphi(x, 0), \quad w_t(x, 0) &= \psi(x, 0). \end{aligned}$$

Let

$$A_1 = \begin{pmatrix} 0 & I \\ -\frac{d^4}{dx^4} & 0 \end{pmatrix}, \tag{2.3}$$

$$\begin{aligned} D(A_1) = \{(u, v)^T \in X \mid u \in H^4(0, 1) \cap H_0^2(0, 1), v \in H_0^2(0, 1), u'''(1) = k_1^2 v_1(1) - c_1 v'(1), \\ u''(1) = -k_2^2 v'(1) - c_2 v(1)\}. \end{aligned}$$

Lemma 2.1 ([9]). *Let A_1 be as in definition (2.3), then*

(i) $\mu_{nk} = \tau_{nk}^2$ ($k = 1, 2, n \geq 1$) is the eigenvalue of A_1 ,

$$\tau_{nk} = \rho_{nk} + o\left(\frac{1}{\rho_{nk}}\right);$$

(ii) we can denote generalized eigenvectors of A_1 by $\{F_0, F_{nk} \mid k = 1, 2, n \geq 1\}$ (see [7]), where

$$F_{nk}(x) = \frac{F(x, \tau_{nk})}{\|G(x, \rho_{nk})\|_X}, \quad (k = 1, 2, n \geq N).$$

For the proof we refer the reader to [9, Theorem 3.2], where ρ_{nk} ($k = 1, 2$), $F(x, \tau_{nk})$, $G(x, \rho_{nk})$ can be seen [9, Eqs. (3.3), (4.2)]. The generalized eigenvectors of A_1 , i.e. $\{F_0, F_{nk} \mid k = 1, 2, n \geq 1\}$ form a Riesz basis for the state space X (this result is contained in [8]), so A_1 is the infinitesimal generator of a C_0 -semigroup on the state space.

In order to formulate (1.1) as a state linear system, let H be a Hilbert space, A be the generator of a strongly continuous semigroup T on H . For any $\mu \in \rho(A)$, we denote by H_{-1} the completion of H with respect to the norm $\|x\|_{-1} = \|(\mu I - A^{-1})x\|$. This space is isomorphic to $D(A^*)^*$, and we have

$$D(A) = H_1 \subseteq H \subseteq H_{-1} = D(A^*)^*, \quad A \in L(H_1, H) \cap L(H, H_{-1}).$$

So $\tilde{A} : H \rightarrow H_{-1}$ is the extension of A :

$$\left(\tilde{A} \begin{pmatrix} p \\ f(\cdot) \end{pmatrix}, \begin{pmatrix} q \\ g(\cdot) \end{pmatrix} \right)_{D(A^*)^* \times D(A^*)} = \left(\begin{pmatrix} p \\ f(\cdot) \end{pmatrix}, A^* \begin{pmatrix} q \\ g(\cdot) \end{pmatrix} \right)_X,$$

where

$$\begin{pmatrix} p \\ f(\cdot) \end{pmatrix} \in H, \quad \begin{pmatrix} q \\ g(\cdot) \end{pmatrix} \in D(A^*).$$

Hence $\tilde{A} \in \mathcal{L}(H, H_{-1})$. For convenience, we denote \tilde{A} by A in the following. Let

$$B = \begin{pmatrix} B_0 \\ 0 \end{pmatrix}, \quad B_0^* \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -k_2^{-2} f_{xx}(1) \\ g(1) \end{pmatrix}, \tag{2.4}$$

$$C\xi(x, \cdot) = C \begin{pmatrix} p(x) \\ f(x, \cdot) \end{pmatrix} = \begin{pmatrix} -k_2^2 f_x(1, t) \\ f_{xxx}(1, t) \end{pmatrix}. \tag{2.5}$$

Output function:

$$y(t) = \begin{pmatrix} -k_2^2 w_{tx}(1, t) \\ w_{xxx}(1, t) \end{pmatrix}.$$

Input function:

$$u(t) = \begin{pmatrix} w_{xx}(1, t - \varepsilon) + c_2 w_t(1, t - \varepsilon) \\ k_1^2 w_t(1, t - \varepsilon) - c_1 w_{tx}(1, t - \varepsilon) \end{pmatrix}.$$

3. The exact controllability of system (1.1)

Let

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & \frac{d}{d\theta} I \end{pmatrix}, \tag{3.1}$$

$$D(A) = \{(p, f(\cdot))^\top \in D(A_1) \times H^1([-\varepsilon, 0], X) \mid f(0) = p\},$$

where $H^1([-\varepsilon, 0], X) = \{f : [-\varepsilon, 0] \rightarrow X \mid f \text{ is absolutely continuous on } [-\varepsilon, 0], f'(\cdot) \in L^2([-\varepsilon, 0], X)\}$. Since $D(A) \hookrightarrow M$ is a compact embedding, A is a discrete operator.

Theorem 3.1. *Let A, B, C be as in definitions (3.1), (2.4), (2.5), then system (1.1) can be formulated as the state linear systems $\Sigma(A, B, C)$:*

$$\begin{cases} \frac{dz(t)}{dt} = Az(t) + Bu(t), & (t > 0), \\ y(t) = Cz(t), \\ z(0) = \begin{pmatrix} x(0) \\ f(\cdot) \end{pmatrix}, \end{cases} \tag{3.2}$$

where $z(t) = \begin{pmatrix} x(t) \\ x(t+\cdot) \end{pmatrix}$, $x(t) = \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}$, and $w(\cdot, t)$ is the solution of (1.1).

Proof. Take $\begin{pmatrix} q \\ g(\cdot) \end{pmatrix} \in D(A)$, where $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in D(A_1)$, $g(\cdot) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in D(F)$, then

$$\begin{aligned} \left(\frac{dz(t)}{dt}, \begin{pmatrix} q \\ g(\cdot) \end{pmatrix} \right)_H &= \left(\frac{dx(t)}{dt}, q \right)_X + (Fx(t+\cdot), g(\cdot))_{X[-\varepsilon, 0]} \\ &= \left(\begin{pmatrix} w_t(\cdot, t) \\ w_{tt}(\cdot, t) \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right) + (Fx(t+\cdot), g(\cdot))_{X[-\varepsilon, 0]}. \end{aligned}$$

Let us prove the first equation now:

$$\begin{aligned} \left(\begin{pmatrix} w_t(\cdot, t) \\ w_{tt}(\cdot, t) \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right)_X &= \left(\begin{pmatrix} -w_{txx} \\ -w_{xxx} \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right) \\ &= \int_0^1 w_{txx} \overline{q_1}'' dx - \int_0^1 w_{xxx} \overline{q_2} dx \\ &= w_{tx} \overline{q_1}''|_0^1 - w_t \overline{q_1}'''|_0^1 + \int_0^1 w_t \overline{q_1}'''' dx - w_{xxx} \overline{q_2}|_0^1 + w_{xx} \overline{q_2}'|_0^1 - \int_0^1 w_{xx} \overline{q_2}'' dx \\ &= w_{tx}(1, t) \overline{q_1}''(1) - w_{xxx}(1, t) \overline{q_2}(1) + \left(A_0 \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right) \\ &= [-k_2^{-2} w_{xx}(1, t - \varepsilon) - k_2^{-2} c_2 w_t(1, t - \varepsilon)] \overline{q_1}''(1) - [k_1^2 w_t(1, t - \varepsilon) - c_1 w_{tx}(1, t - \varepsilon)] \\ &\quad \overline{q_2}(1) + \left(A_0 \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right) \\ &= \left(A_0 \begin{pmatrix} w(\cdot, t) \\ w_t(\cdot, t) \end{pmatrix}, \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right) + \left(u(t), B_0^* \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \right) \\ &= (A_0 x(t) + B_0 u(t)) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{dz(t)}{dt}, \begin{pmatrix} q \\ g(\cdot) \end{pmatrix} \right)_H &= (A_0 x(t) + B_0 u(t)) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + (Fx(t+\cdot), g(\cdot))_{X[-\varepsilon, 0]} \\ &= (A_0 x(t) + B_0 u(t)) q + (Fx(t+\cdot), g(\cdot))_{X[-\varepsilon, 0]} \\ &= (Az(t) + Bu(t)) \begin{pmatrix} q \\ g(\cdot) \end{pmatrix}. \end{aligned}$$

From the above, we can see that these families of operators constitute an alternative description of the system. □

Now we are going to consider the eigenvalue problem of the operator A to research the exact controllability of system (1.1) in the following part.

Theorem 3.2. *With the above notations, the following assertions hold true:*

- (i) *if (λ, r) is a pair of eigenvalues of A_1 , then $(\lambda, (e^{\lambda \cdot} r))$ is a pair of eigenvalues of A . On the other hand, if $(\frac{p}{f(\cdot)})$ is the eigenvectors of A corresponding to λ , then $(\frac{p}{f(\cdot)}) = (e^{\lambda \cdot} p)$;*
- (ii) *there exists a constant N , when $n \geq N$, the eigenvalues of A : $\mu_{nk}, (k = 1, 2)$ (see Lemma 2.1) are simple.*

Proof.

(1) Firstly, for any $\lambda \in \sigma(A_1)$, $\lambda \neq 0$, since A_1 is discrete, we know that $\lambda \in \sigma_p(A_1)$, take $r \neq 0$, $r \in D(A_1)$ and $(\lambda I - A_1)r = 0$, then $(e^{\lambda \cdot} r) \in D(A)$. Further, $(\lambda I - A)(e^{\lambda \cdot} r) = 0$. So $\sigma_p(A_1) \subset \sigma_p(A)$, moreover, $(e^{\lambda \cdot} r)$ is the eigenvectors of A corresponding to the λ . Set $\lambda \neq 0$, $\lambda \in \sigma_p(A)$, take $(\frac{p}{f(\theta)})$ as the eigenvectors of A corresponding to the λ , then $p \in D(A_1)$ and

$$(\lambda I - A) \begin{pmatrix} p \\ f(\theta) \end{pmatrix} = \begin{pmatrix} (\lambda I - A_1)p \\ \lambda f(\theta) - \frac{d}{d\theta} f(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From $(\lambda I - A_1)p = 0$, we obtain $\lambda \in \sigma(A_1)$ and $\sigma_p(A) \subset \sigma_p(A_1)$. So $\sigma_p(A) = \sigma_p(A_1)$. Solving this equation

$$\begin{cases} \frac{df(\theta)}{d\theta} = \lambda f(\theta), \\ f(0) = p, \end{cases}$$

we have $f(\theta) = e^{\lambda \theta} p$.

(2) In the following, we will prove that there exists a constant N , when $n \geq N$, the eigenvalues of A , $\mu_{nk}, (k = 1, 2)$ are simple.

Actually, if $(\lambda I - A)^2 (\frac{p}{f(\theta)}) = 0$, denotes

$$(\lambda I - A) \begin{pmatrix} p \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} q \\ g(\cdot) \end{pmatrix},$$

then $A_1 q = \lambda q$, $g(\cdot) = e^{\lambda \cdot} q$. For

$$(\lambda I - A)^2 \begin{pmatrix} p \\ f(\cdot) \end{pmatrix} = [\lambda(\lambda I - A) - A(\lambda I - A)] \begin{pmatrix} p \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} \lambda q \\ \lambda e^{\lambda \cdot} q \end{pmatrix} - A \begin{pmatrix} q \\ e^{\lambda \cdot} q \end{pmatrix} = \begin{pmatrix} \lambda q \\ \lambda e^{\lambda \cdot} q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

we have $\lambda q = 0$, $\lambda \neq 0$. It is easy to know that $q = 0$. So

$$(\lambda I - A) \begin{pmatrix} p \\ f(\cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \dim((\lambda I - A)M) = 1.$$

Noticing $\mu_{nk}, k = 1, 2, n \geq N$ is the simple eigenvalues of A_1 (see [9, Theorem 3.2]). Evidently, it is also a simple eigenvalue of A .

It follows from the above theorem and Lemma 2.1 that generalized eigenvectors of A are:

$$\{\Phi_{nk} = \begin{pmatrix} F_{nk} \\ e^{\mu_{nk} \cdot} F_{nk} \end{pmatrix} \mid k = 1, 2, n \geq N\}.$$

Because A is discrete, all of the generalized eigenvectors can be set as

$$\{\Phi_0, \Phi_{nk} \mid k = 1, 2, n \geq 1\}. \tag{3.3}$$

Evidently, (3.3) is not complete on M . Let

$$H = \overline{\text{span}}\{\Phi_0, \Phi_{nk} \mid k = 1, 2, n \geq 1\}.$$

It is easy to see $H \neq M$ and H is the A -invariant subspace of M . Using that H is closed, we know H is a Hilbert space with the inner product (2.2). □

For the sake of convenience, we denote by A the $A|_H$. So we have the following conclusions.

Lemma 3.3 ([8]). $\{\Phi_0, \Phi_{nk} \mid k = 1, 2, n \geq 1\}$ forms a Riesz basis for the state space H . When $t \geq 0$, let $z(t) = \begin{pmatrix} x(t) \\ x(t^+.) \end{pmatrix}$, $x(t) = \begin{pmatrix} w(\cdot, t) \\ w_i(\cdot, t) \end{pmatrix}$, where $w(\cdot, t)$ is the solution of system (1.1). Denote

$$x(s) = f(\cdot, s) = \begin{pmatrix} \varphi(\cdot, s) \\ \psi(\cdot, s) \end{pmatrix} \quad (-\varepsilon \leq s \leq 0),$$

then (1.1) can be rewritten as an abstract equation

$$\begin{cases} \frac{dz(t)}{dt} = Az(t), & t \geq 0, \\ z(0) = (x(0), f(\cdot))^T \in H. \end{cases}$$

This implies that A is the infinitesimal generator of the C_0 -semigroup $\{T(t) \mid t \geq 0\}$. So the following results hold true.

Theorem 3.4 ([8]). Let A be given as in (3.1) and assume that $\{T(t) \mid t \geq 0\}$ is the C_0 -semigroup, then we have the following conclusions:

- (i) A is a Riesz spectral operator on H ;
- (ii) $A\xi = \sum_{n=N}^{+\infty} \sum_{k=1,2} \mu_{nk} (\xi, \Phi_{nk})_H \Psi_{nk}$, $\forall \xi \in H$;
- (iii) $T(t)\xi = e^{\mu_0 t} (\xi, \Psi_0)_H \Phi_0 + \sum_{n=N}^{+\infty} \sum_{k=1,2} e^{\mu_{nk} t} (\xi, \Phi_{nk})_H \Psi_{nk}$, $\forall \xi \in H$, where μ_0 is the eigenvalue of A for Φ_0 , $\{\Psi_0, \Psi_{nk} \mid k = 1, 2, n \geq 1\}$ is the biorthogonal column of $\{\Phi_0, \Phi_{nk} \mid k = 1, 2, n \geq 1\}$.

We are going to give the exact controllability of the system (3.2) in the space H , which is the main result of this paper.

Theorem 3.5. Let B be given as in definition (2.4), then B is an admissible control operator for H .

Proof. We only need to prove that for any step function $u(\tau) \in L^2([0, t])$, there exist $t > 0$ and $M > 0$ such that

$$\left\| \int_0^t T(t-\tau)Bu(\tau)d\tau \right\|_H \leq M \|u\|_{L^2([0,t],U)},$$

holds true. Actually,

$$\begin{aligned} \left\| \int_0^t T(t-\tau)Bu(\tau)d\tau \right\|_H &= \left\| \int_0^t \sum_{k=1}^2 \sum_{n \in \mathbb{N}} e^{\lambda_{nk}(t-\tau)} (Bu(\tau), \Phi_{nk})_U \Psi_{nk} d\tau \right\|_H \\ &= \left\| \int_0^t \sum_{k=1}^2 \sum_{n \in \mathbb{N}} e^{\lambda_{nk}(t-\tau)} (u(\tau), B^* \Phi_{nk})_U \Psi_{nk} d\tau \right\|_H \\ &= \left\| \int_0^t \sum_{k=1}^2 \sum_{n \in \mathbb{N}} e^{\lambda_{nk}(t-\tau)} \begin{pmatrix} (u_1(\tau), (-k_2^{-2} \mu_{nk}^{-1} \varphi''(1, \tau_{nk}))) \\ (u_2(\tau), \varphi(1, \tau_{nk})) \end{pmatrix}_U \Psi_{nk} d\tau \right\|_H \\ &\leq \left\{ \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left| \int_0^t e^{\lambda_{nk}(t-\tau)} [-k_2^{-2} \mu_{nk}^{-1} u_1(\tau) \overline{\varphi''(1, \tau_{nk})} + u_2(\tau) \overline{\varphi(1, \tau_{nk})}] d\tau \right|^2 \right\}^{\frac{1}{2}} \\ &\leq \left\{ \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left[\int_0^t e^{-\lambda_{nk} \tau} |u_1(\tau)| d\tau + |\varphi(1, \tau_{nk})| \int_0^t e^{-\lambda_{nk} \tau} |u_2(\tau)| d\tau \right]^2 \right\}^{\frac{1}{2}} \\ &\quad \times \left\{ \int_0^t e^{-\lambda_{nk} \tau} |u_2(\tau)| d\tau \right\}^{\frac{1}{2}}. \end{aligned}$$

Set $u_1(\tau) = \sum_{i=1}^m c_i \chi_{[x_{i-1}, x_i]}(\tau)$, then

$$\begin{aligned} \left| \int_0^t e^{-\lambda_{nk}\tau} u_1(\tau) d\tau \right| &= \left| \sum_{i=1}^m c_i \int_{x_{i-1}}^{x_i} e^{-\lambda_{nk}\tau} d\tau \right| \\ &= \left| \sum_{i=1}^m \frac{c_i}{-\lambda_{nk}} (e^{-\lambda_{nk}x_i} - e^{-\lambda_{nk}x_{i-1}}) \right| \\ &\leq \sum_{i=1}^m \frac{|c_i|}{|\lambda_{nk}|} M_1 \\ &= \frac{M_1}{|\lambda_{nk}|} \int_0^t |u_1(\tau)| d\tau. \end{aligned}$$

Therefore,

$$\left| \int_0^t e^{-\lambda_{nk}\tau} u_2(\tau) d\tau \right| \leq \frac{M_1}{|\lambda_{nk}|} \int_0^t |u_2(\tau)| d\tau,$$

which is the same as the previous proof. Where

$$\begin{aligned} M_1 &= |e^{-\lambda_{nk}x_i} - e^{-\lambda_{nk}x_{i-1}}|, \\ M_2 &= \max\{M_1, | -k_2^{-2} \mu_{nk}^{-1} \varphi''(1, \tau_{nk}) |, |e^{\lambda_{nk}t}|, |\varphi(1, \tau_{nk})|\}. \end{aligned}$$

Then

$$\begin{aligned} \left\| \int_0^t \Gamma(t-\tau) B u(\tau) d\tau \right\|_H &\leq \left\{ \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \frac{M_2^4}{|\lambda_{nk}|^2} \left[\int_0^t |u_1(\tau)| d\tau + \int_0^t |u_2(\tau)| d\tau \right]^2 \right\}^{\frac{1}{2}} \\ &\leq M_3 \left[\int_0^t |u_1(\tau)| d\tau + \int_0^t |u_2(\tau)| d\tau \right], \end{aligned}$$

where $M_3 = \left[\sum_{k=1}^2 \sum_{n \in \mathbb{N}} \frac{M_2^4}{|\lambda_{nk}|^2} \right]^{\frac{1}{2}}$. Furthermore,

$$\begin{aligned} \|u\|_{L^2([0,t],U)} &= \left(\int_0^t \|u(\tau)\|_U^2 d\tau \right)^{\frac{1}{2}} \\ &= \left(\int_0^t |u_1(\tau)|^2 + |u_2(\tau)|^2 d\tau \right)^{\frac{1}{2}} \\ &\geq \left[\left(\int_0^t |u_1(\tau)| d\tau \right)^2 + \left(\int_0^t |u_2(\tau)| d\tau \right)^2 \right]^{\frac{1}{2}} \\ &\geq \frac{\sqrt{2}}{2} \left[\int_0^t (|u_1(\tau)| + |u_2(\tau)|) d\tau \right]. \end{aligned}$$

Let $M = \sqrt{2}M_3$, we have

$$\left\| \int_0^t \Gamma(t-\tau) B u(\tau) d\tau \right\|_H \leq M \|u\|_{L^2([0,t],U)}.$$

□

Theorem 3.6. Let A, B, C and $\Gamma(t)$ be given as in definitions (3.1), (2.4), (2.5) and Theorem 3.4 (iii), set $z = \begin{pmatrix} p(x) \\ f(x, \cdot) \end{pmatrix}$, where $p(x) = \begin{pmatrix} p_1(x) \\ p_2(x) \end{pmatrix}$, $f(x, \cdot) = \begin{pmatrix} f_1(x, \cdot) \\ f_2(x, \cdot) \end{pmatrix}$, then system (3.2) is exact controllable on $[0, \varepsilon]$.

Proof. In order to prove that system (3.2) is exactly controllable on $[0, \varepsilon]$, we only need to prove for any $\varepsilon > 0$, there exists $\gamma > 0$ such that $\int_0^\varepsilon \|B^* \Gamma^*(t) z\|_U^2 dt \geq \gamma \|z\|_H^2$ holds true. If we give an equivalent norm,

$\{\phi_{nk} | k = 1, 2, n \in \mathbb{N}\}$ will become an orthonormal basis of H . So, let us assume that $\{\phi_{nk} | k = 1, 2, n \in \mathbb{N}\}$ are orthogonal for writing convenience. Actually,

$$\begin{aligned} \|z\|_H^2 &= \left\| \sum_{k=1}^2 \sum_{n \in \mathbb{N}} (z, \phi_{nk})_H \phi_{nk} \right\|_H^2 \\ &= \left\| \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left(\begin{pmatrix} p(x) \\ f(x, \cdot) \end{pmatrix}, \begin{pmatrix} F(x, \tau_{nk}) \\ e^{\lambda_{nk} \cdot F(x, \tau_{nk})} \end{pmatrix} \right)_H \begin{pmatrix} F(x, \tau_{nk}) \\ e^{\lambda_{nk} \cdot F(x, \tau_{nk})} \end{pmatrix} \right\|_H^2 \\ &= \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left| \left(\begin{pmatrix} p(x) \\ f(x, \cdot) \end{pmatrix}, \begin{pmatrix} F(x, \tau_{nk}) \\ e^{\lambda_{nk} \cdot F(x, \tau_{nk})} \end{pmatrix} \right)_H \right|^2 \\ &\leq M \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left| \int_0^1 [p_1''(x) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + p_2(x) \overline{\varphi(x, \tau_{nk})}] dx + e^{\overline{\lambda_{nk} t}} \int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \right. \\ &\quad \left. \times \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + f_2(x, \theta) \overline{\varphi(x, \tau_{nk})}] dx d\theta \right|^2 \\ &= M \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left[\left(\int_0^1 [p_1''(x) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + p_2(x) \overline{\varphi(x, \tau_{nk})}] dx \right)^2 + 2e^{\text{Re} \lambda_{nk} t} \right. \\ &\quad \times \int_0^1 [p_1''(x) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + p_2(x) \overline{\varphi(x, \tau_{nk})}] dx \int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + f_2(x, \theta) \\ &\quad \left. \times \overline{\varphi(x, \tau_{nk})}] dx d\theta + e^{2\text{Re} \lambda_{nk} t} \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + f_2(x, \theta) \overline{\varphi(x, \tau_{nk})}] dx d\theta \right)^2 \right]. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_0^\varepsilon \|B^* T^*(t)z\|_U^2 dt &= \int_0^\varepsilon \|B^* \left(\sum_{k=1}^2 \sum_{n \in \mathbb{N}} e^{\overline{\lambda_{nk} t}} (z, \phi_{nk})_H \phi_{nk} \right)\|_U^2 dt \\ &= \int_0^\varepsilon \left\| \sum_{k=1}^2 \sum_{n \in \mathbb{N}} e^{\overline{\lambda_{nk} t}} \begin{pmatrix} p(x) \\ f(x, \cdot) \end{pmatrix}, \begin{pmatrix} F(x, \tau_{nk}) \\ e^{\lambda_{nk} t F(x, \tau_{nk})} \end{pmatrix} \right)_H \begin{pmatrix} -k_2^{-2} \lambda_{nk}^{-1} \varphi''(\tau_{nk}) \\ \varphi(1, \tau_{nk}) \end{pmatrix} \right\|_U^2 dt \\ &= \int_0^\varepsilon \sum_{k=1}^2 \sum_{n \in \mathbb{N}} |e^{2\overline{\lambda_{nk} t}}| \left| \int_0^1 [p_1''(x) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + p_2(x) \overline{\varphi(x, \tau_{nk})}] dx \right. \\ &\quad \left. + e^{\overline{\lambda_{nk} t}} \int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} \right. \\ &\quad \left. + f_2(x, \theta) \overline{\varphi(x, \tau_{nk})}] dx d\theta \right|^2 [k_2^{-4} \lambda_{nk}^{-2} |\varphi''(1, \tau_{nk})|^2 + |\varphi(1, \tau_{nk})|^2] dt \\ &\geq m \int_0^\varepsilon \sum_{k=1}^2 \sum_{n \in \mathbb{N}} e^{2\text{Re} \lambda_{nk} t} \left(\left(\int_0^1 [p_1''(x) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + p_2(x) \overline{\varphi(x, \tau_{nk})}] dx \right)^2 + 2e^{\text{Re} \lambda_{nk} t} \right. \\ &\quad \times \left(\int_0^1 [p_1''(x) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + p_2(x) \overline{\varphi(x, \tau_{nk})}] dx \right) \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} \right. \\ &\quad \left. \left. + f_2(x, \theta) \overline{\varphi(x, \tau_{nk})}] dx d\theta \right) + e^{2\text{Re} \lambda_{nk} t} \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \overline{\lambda_{nk}^{-1} \varphi''(x, \tau_{nk})} + f_2(x, \theta) \right. \right. \end{aligned}$$

$$\begin{aligned}
 & \times \overline{\varphi(x, \tau_{nk})}] dx d\theta \Big)^2 dt \\
 = & m \sum_{k=1}^2 \sum_{n \in \mathbb{N}} \left[\frac{1}{2\text{Re}\lambda_{nk}} (e^{2\text{Re}\lambda_{nk}\varepsilon} - 1) \left(\int_0^1 [p_1''(x)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} + p_2(x)\overline{\varphi(x, \tau_{nk})}] dx \right)^2 \right. \\
 & + \frac{2}{3\text{Re}\lambda_{nk}} (e^{3\text{Re}\lambda_{nk}\varepsilon} - 1) \left(\int_0^1 [p_1''(x)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} + p_2(x)\overline{\varphi(x, \tau_{nk})}] dx \right) \\
 & \times \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} \right. \\
 & \left. \left. + f_2(x, \theta)\overline{\varphi(x, \tau_{nk})}] dx d\theta \right) + \frac{1}{4\text{Re}\lambda_{nk}} (e^{4\text{Re}\lambda_{nk}\varepsilon} - 1) \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta) \right. \\
 & \left. \times \overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} + f_2(x, \theta)\overline{\varphi(x, \tau_{nk})}] dx d\theta \right)^2 \Big] \\
 \geq & m \min\left\{ \frac{1}{2\text{Re}\lambda_{nk}} (e^{2\text{Re}\lambda_{nk}\varepsilon} - 1), \frac{2}{3\text{Re}\lambda_{nk}} (e^{3\text{Re}\lambda_{nk}\varepsilon} - 1), \frac{1}{4\text{Re}\lambda_{nk}} (e^{4\text{Re}\lambda_{nk}\varepsilon} - 1), 1, \right. \\
 & \left. 2e^{\text{Re}\lambda_{nk}t}, e^{2\text{Re}\lambda_{nk}t} \right\} \left(\left(\int_0^1 [p_1''(x)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} + p_2(x)\overline{\varphi(x, \tau_{nk})}] dx \right)^2 + 2e^{\text{Re}\lambda_{nk}t} \right) \\
 & \times \left(\int_0^1 [p_1''(x)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} + p_2(x)\overline{\varphi(x, \tau_{nk})}] dx \right) \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} \right. \\
 & \left. + f_2(x, \theta)\overline{\varphi(x, \tau_{nk})}] dx d\theta \right) + e^{2\text{Re}\lambda_{nk}t} \left(\int_{-\varepsilon}^0 \int_0^1 [f_{1xx}(x, \theta)\overline{\lambda_{nk}^{-1}\varphi''(x, \tau_{nk})} \right. \\
 & \left. + f_2(x, \theta)\overline{\varphi(x, \tau_{nk})}] dx d\theta \right)^2 \Big) \\
 \geq & \frac{m}{M} \min\left\{ \frac{1}{2\text{Re}\lambda_{nk}} (e^{2\text{Re}\lambda_{nk}\varepsilon} - 1), \frac{2}{3\text{Re}\lambda_{nk}} (e^{3\text{Re}\lambda_{nk}\varepsilon} - 1), \frac{1}{4\text{Re}\lambda_{nk}} (e^{4\text{Re}\lambda_{nk}\varepsilon} - 1), 1, \right. \\
 & \left. 2e^{\text{Re}\lambda_{nk}t}, e^{2\text{Re}\lambda_{nk}t} \right\} \|z\|_H^2,
 \end{aligned}$$

where

$$m = \inf_{k=1,2, n \in \mathbb{N}} [k_2^{-4}\lambda_{nk}^{-2}|\varphi''(1, \tau_{nk})|^2 + |\varphi(1, \tau_{nk})|^2].$$

Take

$$\gamma = \frac{m}{M} \min\left\{ \frac{1}{2\text{Re}\lambda_{nk}} (e^{2\text{Re}\lambda_{nk}\varepsilon} - 1), \frac{2}{3\text{Re}\lambda_{nk}} (e^{3\text{Re}\lambda_{nk}\varepsilon} - 1), \frac{1}{4\text{Re}\lambda_{nk}} (e^{4\text{Re}\lambda_{nk}\varepsilon} - 1), 1, 2e^{\text{Re}\lambda_{nk}t}, e^{2\text{Re}\lambda_{nk}t} \right\}.$$

Therefore,

$$\int_0^\varepsilon \|B^*T^*(t)z\|_U^2 dt \geq \gamma \|z\|_H^2.$$

Since systems (1.1) and (3.2) are equivalent, we get the exact controllability of system (1.1). □

Conclusion

In this paper we discuss the flexible Euler-Bernoulli beam system with small delays in boundary feedback controls (1.1) with boundary conditions. Different from the works in literature, we formulate

the system (1.1) as the state feedback control systems $\Sigma(A, B, C)$. And then we formulate the system without time-delay as abstract differential functions on the Hilbert space X . We get the eigenvalues and eigenvectors of the operator A through studying the eigenvalues and eigenvectors of the operator A_1 . Then we prove that $\Sigma(A, B, C)$ is exactly controllable on H , which is generalized by the eigenvectors of the operator A . Since the system (1.1) and the system $\Sigma(A, B, C)$ are equivalent, we only need to study the exact controllability of $\Sigma(A, B, C)$. In the end, (1.1) is proved to be exactly controllable on H .

Acknowledgment

The paper was supported by the Science Foundation of Shanxi Province (2013011003-3).

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