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# Identities for Korobov-type polynomials arising from functional equations and p-adic integrals

Ahmet Yardimci[a], Yilmaz Simsek[b,*]

[a] *Department of Biostatistics and Medical Informatics, Faculty of Medicine, University of Akdeniz, TR-07058 Antalya, Turkey.*

[b] *Department of Mathematics, Faculty of Science, University of Akdeniz, TR-07058 Antalya, Turkey.*

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**Abstract**

By using generating functions and their functional equations for the special numbers and polynomials, we derive various identities and combinatorial sums including the Korobov-type polynomials, the Bernoulli numbers, the Stirling numbers, the Daehee numbers and the Changhee numbers. Furthermore, by using the Volkenborn integral and the fermionic p-adic integral, we also derive combinatorial sums associated with the Korobov-type polynomials, the Lah numbers, the Changhee numbers and the Daehee numbers. Finally, we give a conclusion on our results. 

Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Daehee numbers and polynomials, Changhee numbers and polynomials, Lah numbers, Apostol-Daehee numbers, Korobov polynomials, Stirling numbers, generating functions, functional equation, p-adic integral.
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## 1. Introduction

The Korobov-type polynomials have been used in mechanical characteristics of molecular dynamics model ([5, 15, 21]). Polynomials and their derivatives, integrals and algebraic properties are easily computed. Because of these easily computational advantages, polynomials and their generating functions have been used by many scientists. In this paper, we give various identities, formulas and relations related to the Korobov-type polynomials and the other special numbers and polynomials. Firstly, we glance at the p-adic integrals and their properties. By using the bosonic and the fermionic p-adic integrals with their integral equations, many families of generating functions for the special numbers and polynomials have been constructed. We know that there are many valuable applications of the generating functions and the p-adic integrals not only in mathematics, but also in other science ([8, 9, 25, 28]; see also the references cited in each of these earlier works). Recently, Kim [8] constructed the p-adic q-Volkenborn integral and their integral equations (see also [9, 11, 13]).

Some notations are given as follows:

*Corresponding author
*Email addresses:* `yardimci@akdeniz.edu.tr` (Ahmet Yardimci), `ysimsek@akdeniz.edu.tr` (Yilmaz Simsek)

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$\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. $\mathbb{K} \subseteq \mathbb{C}_p$, $C^1(\mathbb{Z}_p \to \mathbb{K})$ denotes a set of continuous derivative functions from $\mathbb{Z}_p \to \mathbb{K}$.

By using the p-adic q-Volkenborn integral, which was found by Kim [8], one can derive two kinds of integrals which are so-called the p-adic bosonic integral or Volkenborn integral and the p-adic fermionic integral. These are given, respectively, as follows:

$$I_1(f(x)) = \int_{\mathbb{Z}_p} f(x)\, d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),$$

where

$$\mu_1(x) = \mu_1\left(x + p^N \mathbb{Z}_p\right) = \frac{1}{p^N}$$

([25], see also [9, 11, 13]) and

$$I_{-1}(f(x)) = \int_{\mathbb{Z}_p} f(x)\, d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x f(x),$$

where ([9])

$$\mu_{-1}\left(x + p^N \mathbb{Z}_p\right) = \frac{(-1)^x}{p^N}.$$

Especially, p-adic integrals are of many applications in almost all branches of mathematics, physics, engineering and also other sciences. That is, these integrals appear in analytic number theory, in the quantum groups, in cohomology groups, in q-deformed oscillator and also in p-adic models.

Secondly, in order to give main results of this paper, we also glance at the rising factorial polynomials, the falling factorial polynomials, special numbers, and polynomials with their generating functions. The rising factorial polynomials $x^{(n)}$ and the falling factorial polynomials $x_{(n)}$ are defined as ([1, 32])

$$x^{(n)} = x(x+1)(x+2)\dots(x+n-1) x^{(0)} = 1,$$

and

$$x_{(n)} = x(x-1)(x-2)\dots(x-n+1) x_{(0)} = 1,$$

respectively.

The generating function for the Bernoulli polynomials is given by

$$F_A(t,x) = \frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.1}$$

From the above equation, one has

$$B_n = B_n(0),$$

where denotes the Bernoulli numbers ([1–33]; see also the references cited in each of these earlier works).

The Euler polynomials are defined by

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

Substituting $x = 0$ into the above equation, one has ([1–33])

$$E_n = E_n(0).$$

Generating functions for the Stirling numbers of the first kind are given by

$$F_{S1}(t,k)=\frac{(\log(1+t))^k}{k!}=\sum_{n=0}^{\infty}S_1(n,k)\frac{t^n}{n!}.$$

By using the above function, we have $S_1(0,0)=1$, $S_1(0,k)=0$ if $k>0$, $S_1(n,0)=0$ if $n>0$, and $S_1(n,k)=0$ if $k>n$ ([1–33]).

Generating functions for the Stirling numbers of the second kind are given by

$$F_S(t,k)=\frac{\left(e^t-1\right)^k}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{t^n}{n!}, \tag{1.2}$$

where $k$ is nonnegative integer. From this generating function, we have $S_2(n,n)=1$, $S_2(0,k)=0$ if $k>0$, $S_2(n,0)=0$ if $n>0$, and $S_2(n,k)=0$ if $k>n$ ([1], [24, p. 116], [31, 32]).

Generating function for the Bernoulli polynomials of the second kind is given by ([24, pp. 113-117])

$$F_{b2}(t,x)=\frac{t(1+t)^x}{\log(1+t)}=\sum_{n=0}^{\infty}b_n(x)\frac{t^n}{n!}. \tag{1.3}$$

Substituting $x=0$ into (1.3), we get the second kind Bernoulli numbers (the Cauchy numbers of the first kind), which are defined as follows ([24, p. 116]; see also the references cited in each of these earlier works)

$$F_{b2}(t)=\frac{t}{\log(1+t)}=\sum_{n=0}^{\infty}b_n(0)\frac{t^n}{n!}.$$

The Lah numbers are defined by means of the following generating function ([1, 23, 28]; and see also the references cited in each of these earlier works)

$$\left(-\frac{t}{1+t}\right)^k=\sum_{n=0}^{\infty}k!L(n,k)\frac{t^n}{n!}.$$

By using the above equation, we have

$$L(n,k)=(-1)^n\frac{n!}{k!}\binom{n-1}{k-1}$$

and the unsigned Lah numbers are given by

$$|L(n,k)|=\frac{n!}{k!}\binom{n-1}{k-1}.$$

A relation between the Lah numbers and the first and the second kind Stirling numbers is given as follows ([23, p. 44])

$$L(n,k)=\sum_{j=0}^{n}(-1)^j s_1(n,j)S_2(j,k).$$

Relations between the Lah numbers, the falling, and rising factorial polynomials are given by

$$(-x)_{(n)}=\sum_{k=0}^{n}L(n,k)x_{(k)},$$

$$x_{(n)}=\sum_{k=0}^{n}L(n,k)\,(-x)_{(k)},$$

and

$$x^{(n)} = \sum_{k=1}^{n} |L(n,k)|\, x_{(k)}.$$

( cf. [1], [23, p. 43]).

The first and the second kinds of the Daehee numbers are given respectively as ([14])

$$F_D(t) = \frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!}$$

and

$$\frac{(1+t)\log(1+t)}{t} = \sum_{n=0}^{\infty} \widehat{D}_n \frac{t^n}{n!}.$$

The generating functions for the Daehee polynomials of the first kind are given by the following formula ([14])

$$F_D(x,t) = F_D(t)(1+t)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}. \tag{1.4}$$

The first kind Daehee numbers are given by the formula ([4, 14], [23, p. 45]; see also the references cited in each of these earlier works)

$$D_n = \sum_{k=0}^{n} S_1(n,k) B_k = (-1)^n \frac{n!}{n+1}.$$

Generating function for the Apostol-type Daehee polynomials is given by ([26, p 560, Eq-(9)]; see also the references cited in each of these earlier works)

$$F_{AD}(x,t;\lambda) = \frac{\log(1+\lambda t)}{t\lambda^{x+1}}(1+t)^x = \sum_{n=0}^{\infty} D_n(x;\lambda) \frac{t^n}{n!}. \tag{1.5}$$

Setting $x = 0$ in (1.5), we have the Apostol-type Daehee numbers ([26]; see also the references cited in each of these earlier works)

$$D_n(\lambda) = D_n(0;\lambda).$$

The first and the second kinds of the Changhee numbers are given respectively as ([18])

$$\frac{2}{t+2} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!}$$

and

$$\frac{2(1+t)}{t+2} = \sum_{n=0}^{\infty} \widehat{Ch}_n \frac{t^n}{n!}.$$

The Changhee numbers are given by the following formula ([18]; see also the references cited in each of these earlier works)

$$Ch_n = \sum_{k=0}^{n} S_1(n,k) E_k = (-1)^n \frac{n!}{2^n}.$$

The Korobov-type polynomials of the first kind have been studied in mathematics and in other sciences such as analytic number theory and algebra especially in mechanical characteristics of molecular dynamics model. These polynomials are defined by means of the following generating functions ([5, 15, 21, 22]; and see also the references cited in each of these earlier works)

$$F_{K1}(x,t;\lambda) = \frac{\lambda t(1+t)^x}{(1+t)^\lambda - 1} = \sum_{n=0}^{\infty} K_n(x;\lambda) \frac{t^n}{n!}. \tag{1.6}$$

In [15], Kim et al. defined the Korobov polynomials of the third and fourth kinds, respectively, as

$$F_{K3}(x,t;\lambda)=\frac{\log(1+\lambda t)}{\lambda\log(1+t)}(1+t)^x=\sum_{n=0}^{\infty}K_{n,3}(x;\lambda)\frac{t^n}{n!} \tag{1.7}$$

and

$$F_{K4}(x,t;\lambda)=\frac{\log(1+\lambda t)}{(1+t)^{\lambda}-1}(1+t)^x=\sum_{n=0}^{\infty}K_{n,4}(x;\lambda)\frac{t^n}{n!}. \tag{1.8}$$

Observe that substituting $x=0$ into (1.7) and (1.8), one has the Korobov numbers of the third and fourth kinds, respectively as

$$K_{n,3}(\lambda)=K_{n,3}(0;\lambda)\quad\text{and}\quad K_{n,4}(\lambda)=K_{n,4}(0;\lambda).$$

## 2. Identities and relations for Korobov-type polynomials

In this section, by using generating functions for the Korobov-type polynomials, we derive various identities and relations including the Korobov-type polynomials of the third and fourth kinds, the Bernoulli numbers, the Cauchy numbers, the Stirling numbers, the Apostol-type Daehee numbers, and also combinatorial sums.

By using (1.7), we get

$$(1+t)^x\log(1+\lambda t)=\lambda\log(1+t)\sum_{n=0}^{\infty}K_{n,3}(x;\lambda)\frac{t^n}{n!}.$$

By using the Taylor series of the function $\log(1+t)$ and combining with the Cauchy product rule in the above equation, we obtain

$$(1+t)^x\log(1+\lambda t)=\sum_{n=0}^{\infty}\left(\lambda\sum_{j=0}^{n-1}\frac{n!(-1)^j}{(j+1)(n-j-1)!}K_{n-j-1,3}(x;\lambda)\right)\frac{t^n}{n!}. \tag{2.1}$$

By using (1.8), we get

$$(1+t)^x\log(1+\lambda t)=\sum_{n=0}^{\infty}\binom{\lambda}{n}t^n\sum_{n=0}^{\infty}K_{n,4}(x;\lambda)\frac{t^n}{n!}-\sum_{n=0}^{\infty}K_{n,4}(x;\lambda)\frac{t^n}{n!}.$$

By using the Cauchy product rule in the above equation, we obtain

$$(1+t)^x\log(1+\lambda t)=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\lambda_{(j)}K_{n-j,4}(x;\lambda)-K_{n,4}(x;\lambda)\right)\frac{t^n}{n!}. \tag{2.2}$$

Combining (2.1) with (2.2), after some elementary calculations, we get

$$\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\lambda_{(j)}K_{n-j,4}(x;\lambda)-K_{n,4}(x;\lambda)\right)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\lambda\sum_{j=0}^{n-1}\frac{n!(-1)^j}{(j+1)(n-j-1)!}K_{n-j-1,3}(x;\lambda)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

**Theorem 2.1.** *Let $n$ be a positive integer. Then we have*

$$\lambda\sum_{j=0}^{n-1}\frac{n!(-1)^j}{(j+1)(n-j-1)!}K_{n-j-1,3}(x;\lambda)=\sum_{j=0}^{n}\binom{n}{j}\lambda_{(j)}K_{n-j,4}(x;\lambda)-K_{n,4}(x;\lambda).$$

By combining (1.4) and (1.3) with (1.6), we get the following functional equation

$$F_{b2}(t,x)F_D(x,t)F_{K1}(x,t;\lambda) = F_{K1}(3x,t;\lambda).$$

By using the above functional equation, we have

$$\sum_{n=0}^{\infty} K_n(3x;\lambda)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!}\sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} K_n(x;\lambda)\frac{t^n}{n!}\right).$$

Therefore

$$\sum_{n=0}^{\infty} K_n(3x;\lambda)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j} b_j(x)D_{m-j}(x)K_{n-m}(x;\lambda)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the following theorem.

**Theorem 2.2.**

$$K_n(3x;\lambda) = \sum_{m=0}^{n}\sum_{j=0}^{m}\binom{n}{m}\binom{m}{j} b_j(x)D_{m-j}(x)K_{n-m}(x;\lambda).$$

Substituting $t = e^t - 1$ into (1.8), combining with (1.1) and (1.5), we get the following functional equations

$$\lambda t F_{K4}(x, e^t - 1;\lambda) = \log\left(1+\lambda\left(e^t-1\right)\right) F_A\left(\lambda t, \frac{x}{\lambda}\right) \tag{2.3}$$

$$F_{K4}(x, e^t - 1;\lambda) = F_{AD}(0, e^t - 1;\lambda)F_A\left(\lambda t, \frac{x}{\lambda}\right). \tag{2.4}$$

Using (2.3), we get

$$\lambda t \sum_{n=0}^{\infty} K_{n,4}(x;\lambda)\frac{\left(e^t-1\right)^n}{n!} = \sum_{n=0}^{\infty} B_n\left(\frac{x}{\lambda}\right)\frac{(\lambda t)^n}{n!}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\lambda^n\left(e^t-1\right)^n}{n}.$$

Combining the above equation with (1.2), we obtain

$$\lambda\sum_{m=0}^{\infty} m\sum_{n=0}^{m-1} K_{n,4}(x;\lambda)S_2(m-1,n)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{n=0}^{j}(-1)^n\binom{m}{j} n!\lambda^{n+m-j}S_2(j,n+1)B_{m-j}\left(\frac{x}{\lambda}\right)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem.

**Theorem 2.3.**

$$m\sum_{n=0}^{m-1} K_{n,4}(x;\lambda)S_2(m-1,n) = \sum_{j=0}^{m}\sum_{n=0}^{j}(-1)^n\binom{m}{j} n!\lambda^{n+m-j-1}\times S_2(j,n+1)B_{m-j}\left(\frac{x}{\lambda}\right).$$

By using (2.4), we obtain

$$\sum_{n=0}^{\infty} K_{n,4}(x;\lambda)\frac{\left(e^t-1\right)^n}{n!} = \sum_{n=0}^{\infty} B_n\left(\frac{x}{\lambda}\right)\frac{(\lambda t)^n}{n!}\sum_{n=0}^{\infty} D_n(\lambda)\frac{\left(e^t-1\right)^n}{n!}.$$

Hence

$$\sum_{m=0}^{\infty}\sum_{n=0}^{m} K_{n,4}(x;\lambda)S_2(m,n)\frac{t^m}{m!} = \sum_{m=0}^{\infty}\sum_{j=0}^{m}\sum_{n=0}^{j}\binom{m}{j}\lambda^{m-j}B_{m-j}\left(\frac{x}{\lambda}\right)S_2(j,n)D_n(\lambda)\frac{t^m}{m!}.$$

Comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation, we arrive at the following theorem.

**Theorem 2.4.**

$$\sum_{n=0}^{m} K_{n,4}(x;\lambda)S_2(m,n) = \sum_{j=0}^{m}\sum_{n=0}^{j}\binom{m}{j}\lambda^{m-j}B_{m-j}\left(\frac{x}{\lambda}\right)S_2(j,n)D_n(\lambda).$$

### 2.1. *Applications of the p-adic integrals to the Korobov-type polynomials*

In this section, by applying the bosonic and the fermionic p-adic integrals on p-adic integers, we derive some formulas and identities associated with the Bernoulli numbers, the Euler numbers, the Korobov-type polynomials, and combinatorial sums.

In order to give the results of this section, we need the following formulae.

The p-adic bosonic integral representation of Bernoulli numbers and polynomials are respectively given by ([8, 9, 25])

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = B_n$$

and

$$\int_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z).$$

**Theorem 2.5.**

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_1(x) = \frac{(-1)^j}{j+1} \tag{2.5}$$

*(*[14, 18, 25]*).*

The p-adic fermionic integral representation of Euler numbers and polynomials are respectively given by ([6, 9]; see also the references cited in each of these earlier works)

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = E_n$$

and

$$\int_{\mathbb{Z}_p} (z+x)^n d\mu_{-1}(x) = E_n(z).$$

The p-adic bosonic integral representation for the Daehee numbers of the second kind was given by Kim et al. [14] as

$$\widehat{D_n} = \int_{\mathbb{Z}_p} t^{(n)} d\mu_1(t). \tag{2.6}$$

Kim et al. [18] gave the p-adic fermionic integral representation of the second kind Changhee numbers as

$$\widehat{Ch}_n = \int_{\mathbb{Z}_p} x^{(n)} d\mu_{-1}(x). \tag{2.7}$$

**Theorem 2.6.**

$$\int_{\mathbb{Z}_p} \binom{x}{j} d\mu_{-1}(x) = \frac{(-1)^j}{2^j}. \tag{2.8}$$

Proof of this theorem was given by Kim et al. [18].

A p-adic bosonic integral representation for the polynomials $K_{n,3}(x;1)$ is given by the following theorem.

**Theorem 2.7.**

$$\int_{\mathbb{Z}_p} K_{n,3}(x;1) d\mu_1(x) = \frac{(-1)^n}{n+1} n!. \tag{2.9}$$

*Proof.* Substituting $\lambda = 1$ into (1.7), we get

$$(1+t)^x = \sum_{n=0}^{\infty} K_{n,3}(x;1)\frac{t^n}{n!}.$$

From the above equation, we have

$$\sum_{n=0}^{\infty} x_{(n)}\frac{t^n}{n!} = \sum_{n=0}^{\infty} K_{n,3}(x;1)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, and applying the bosonic integral with (2.5), we arrive at the desired result. □

A p-adic fermionic integral representation for the polynomials $K_{n,3}(x;1)$ is given by the following theorem.

**Theorem 2.8.**

$$\int_{\mathbb{Z}_p} K_{n,3}(x;1)d\mu_{-1}(x) = (-1)^n\frac{n!}{2^n}. \tag{2.10}$$

By using (2.8) and using the same method in proof of (2.10), we easily get the proof of (2.9). So we omit it.

*Remark* 2.9.

$$\int_{\mathbb{Z}_p} x_{(n)}d\mu_1(x) = \frac{(-1)^n}{n+1}n!. \tag{2.11}$$

Proof of the above integral was given by Kim et al. [14]. This integral value is also known as the Daehee numbers of the first kind ([4, 14], [23, p. 117], [28, 29]). Consequently, the bosonic integral of the polynomials $K_{n,3}(x;1)$ is related to the Daehee numbers.

*Remark* 2.10.

$$\int_{\mathbb{Z}_p} x_{(n)}d\mu_{-1}(x) = (-1)^n\frac{n!}{2^n}. \tag{2.12}$$

Proof of the above integral was given by Kim et al. [18]. This integral value is also known as the Changhee numbers of the first kind ([14, 18, 28, 29]). Consequently, the fermionic integral of the polynomials $K_{n,3}(x;1)$ is related to the Changhee numbers.

In [15, Eq-(2.57)], Kim et al. gave the following identity:

$$K_{n,3}(x;\lambda) = \sum_{m=0}^{n} C_{n,m}(x,\lambda)x^{(m)},$$

where

$$C_{n,j}(x,\lambda) = \sum_{l=0}^{n-m}\sum_{k=0}^{n-m-l}(-1)^l\binom{m+l-1}{l}\binom{n}{m+l}\binom{m+l}{m} \times\binom{n-m-l}{k}l!\lambda^k b_{n-m-l-k}(0)D_k.$$

By applying the p-adic bosonic integral and the fermionic integral to the above equation, we obtain

$$\int_{\mathbb{Z}_p} K_{n,3}(x;\lambda)d\mu_1(x) = \sum_{m=0}^{n} C_{n,m}(x,\lambda)\int_{\mathbb{Z}_p} x^{(m)}d\mu_1(x)$$

and

$$\int_{\mathbb{Z}_p} K_{n,3}(x;\lambda)d\mu_{-1}(x) = \sum_{m=0}^{n} C_{n,m}(x,\lambda)\int_{\mathbb{Z}_p} x^{(m)}d\mu_{-1}(x).$$

Combining the above equations with (2.6) and (2.7), we get the following integral formulas for the polynomials $K_{n,3}(x;\lambda)$.

**Theorem 2.11.**

$$\int_{\mathbb{Z}_p} K_{n,3}(x;\lambda)d\mu_1(x) = \sum_{m=0}^{n} C_{n,m}(x,\lambda)\widehat{D_m} \quad \text{and} \quad \int_{\mathbb{Z}_p} K_{n,3}(x;\lambda)d\mu_{-1}(x) = \sum_{m=0}^{n} C_{n,m}(x,\lambda)\widehat{Ch}_m.$$

Simsek [28] also gave some $p$-adic fermionic integral formulas including the rising factorials, combinatorial sums, and special numbers.

**Theorem 2.12** ([28]).

$$y_2(n:E) = \sum_{k=1}^{n}(-1)^k |L(n,k)|\, k!2^{-k}. \tag{2.13}$$

We give another formula for the numbers $Y_2(n:B)$ by the following theorem.

**Theorem 2.13.**

$$Y_2(n:B) = \sum_{k=0}^{n} C(n,k)B_k, \tag{2.14}$$

*where* $C(n,k) = |s_1(n,k)|$ *and* $B_k$ *denotes the Bernoulli numbers.*

For detail proofs of (2.13) and (2.14) see [14, 18, 28].

By using the $p$-adic integral formulas including the rising factorials, Simsek [28] gave the following relations

$$Y_2(m:B) = \widehat{D_m} \quad \text{and} \quad y_2(m:E) = \widehat{Ch}_m.$$

Combining the above relations with Theorem 2.11, we arrive at the following corollary.

**Corollary 2.14.**

$$\sum_{m=0}^{n} C_{n,m}(x,\lambda)\widehat{D_m} = \sum_{m=0}^{n}\sum_{k=0}^{m} C_{n,m}(x,\lambda)C(m,k)B_k$$

*and*

$$\sum_{m=0}^{n} C_{n,m}(x,\lambda)\widehat{Ch}_m = \sum_{m=0}^{n}\sum_{k=1}^{m}(-1)^k C_{n,m}(x,\lambda)\,|L(m,k)|\, k!2^{-k}.$$

In [15, Eq-(2.23)], Kim gave the following identity

$$K_{n,4}(x;\lambda) = \sum_{j=0}^{n}\binom{n}{j}K_{n-j,4}(\lambda)x_{(j)}.$$

By applying the $p$-adic bosonic and fermionic integral to the above equation, and combining with (2.11) and (2.12), we get the following integral formulas for the polynomials $K_{n,4}(x;\lambda)$.

**Theorem 2.15.**

$$\int_{\mathbb{Z}_p} K_{n,4}(x;\lambda)d\mu_1(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}K_{n-j,4}(\lambda)\frac{j!}{j+1}$$

*and*

$$\int_{\mathbb{Z}_p} K_{n,4}(x;\lambda)d\mu_{-1}(x) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}K_{n-j,4}(\lambda)\frac{j!}{2^j}.$$

## 3. Conclusion

In this paper, we combine various generating functions and their functional equations of the Bernoulli numbers, the Stirling numbers, the Lah numbers, the Daehee numbers, the Changhee numbers, and the Korobov-type polynomials. By using these numbers and polynomials with their generating functions, we give various identities, relations and combinatorial sums. Our results are applicable to many areas such as almost all branches of mathematics, mathematical physics, engineering problems including modelling, and also the others. On the other hand, it is well-known that the p-adic integrals can be used to construct generating functions for the new families of the special numbers and polynomials. These are also used in problems of the quantum mechanics and p-adic analysis problems. For these reasons, by applying the bosonic and the fermionic p-adic integrals to the Korobov-type polynomials, we obtain some novel integral formulas including aforementioned numbers, polynomials, and combinatorial sums.

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