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Oscillation of nonlinear second-order neutral delay differential equations

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Abstract

By using a couple of Riccati substitutions, we establish several new oscillation criteria for a class of second-order nonlinear neutral delay differential equations. These results complement and improve the related contributions reported in the literature.

Keywords: Oscillation, neutral differential equation, nonlinear equation, delayed argument, Riccati substitution.
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1. Introduction

The neutral differential equations find a number of applications in the natural sciences and technology. For instance, they are frequently used in the study of distributed networks containing lossless transmission lines. On the basis of these background details, this paper is concerned with the oscillatory properties of solutions to the second-order neutral delay differential equation

$$[a(t)\phi_1(z'(t))]' + q(t)f(\phi_2(x(\delta(t)))) = 0, \tag{1.1}$$

where $t \geqslant t_0 > 0$, $z(t) = x(t) + p(t)x(\tau(t))$, $\phi_1(u) = |u|^{\lambda-1}u$, $\phi_2(u) = |u|^{\beta-1}u$, λ and β are positive constants. Throughout, we assume that the following hypotheses are satisfied.

(H_1) $a \in C^1([t_0, +\infty), (0, +\infty))$, $p, q \in C([t_0, +\infty), \mathbb{R})$, $q(t) \geqslant 0$, and $q(t)$ is not identically zero for large t;

(H_2) $\tau, \delta \in C^1([t_0, +\infty), \mathbb{R})$, $\tau(t) \leqslant t$, $\delta(t) \leqslant t$, $\lim_{t\to+\infty} \tau(t) = \lim_{t\to+\infty} \delta(t) = +\infty$, $\tau \circ \delta = \delta \circ \tau$, $\tau'(t) \geqslant \tau_0 > 0$, $\delta'(t) > 0$, and $\delta(t) \leqslant \tau(t)$, where τ_0 is a constant;

(H_3) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(u)/u \geqslant L > 0$ for all $u \neq 0$, where L is a constant.

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By a solution of (1.1), we mean a function $x \in C([T, +\infty), \mathbb{R})$ for some $T \geqslant t_0$ which has the properties that $z, a\phi_1(z') \in C^1([T, +\infty), \mathbb{R})$ and satisfies (1.1) on the interval $[T, +\infty)$. A solution x of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be nonoscillatory. Equation (1.1) is called oscillatory if all its nontrivial solutions oscillate.

In recent years, there has been an increasing interest in obtaining sufficient conditions for the oscillation and nonoscillation of solutions to different classes of differential equations, we refer the reader to [1–28] and the references cited therein. The qualitative analysis of neutral differential equations is more difficult in comparison with that of ordinary differential equations; see, for instance, the monograph [1] and the paper [17]. Most oscillation criteria reported in the literature for equation (1.1) and its particular cases have been obtained under the assumptions that

$$\int_{t_0}^{+\infty} a^{-1/\lambda}(t)dt = +\infty \tag{1.2}$$

and

$$0 \leqslant p(t) < 1, \tag{1.3}$$

although the papers [6–8, 11, 13, 14, 21, 26, 28] were concerned with the oscillation of equation (1.1) and its particular cases in the cases where

$$0 \leqslant p(t) \leqslant p_0 < +\infty, \tag{1.4}$$

$$p(t) \geqslant 1, \quad p(t) \not\equiv 1 \text{ eventually}, \tag{1.5}$$

or

$$0 \leqslant p(t) = p_0 \neq 1, \tag{1.6}$$

where p_0 is a constant.

The objective of this paper is to establish new oscillation results for equation (1.1) without imposing assumptions (1.3), (1.5), and (1.6). As is customary, all functional inequalities considered in the sequel are assumed to be satisfied for all t large enough.

2. Auxiliary inequalities

To prove the main results, we need the following auxiliary inequalities.

Lemma 2.1 ([21]). *If $X \geqslant 0$, $Y \geqslant 0$, and $0 < \lambda \leqslant 1$, then $X^\lambda + Y^\lambda \geqslant (X+Y)^\lambda$.*

Lemma 2.2 ([7]). *If $X \geqslant 0$, $Y \geqslant 0$, and $\lambda \geqslant 1$, then $X^\lambda + Y^\lambda \geqslant 2^{1-\lambda}(X+Y)^\lambda$.*

Lemma 2.3 ([13]). *If $B > 0$ and $\alpha > 0$, then*

$$Au - Bu^{\frac{\alpha+1}{\alpha}} \leqslant \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} \frac{A^{\alpha+1}}{B^\alpha}.$$

3. Oscillation criteria

To present the main results, we use the notation

$$L_0 = \begin{cases} L, & 0 < \beta \leqslant 1, \\ 2^{1-\beta}L, & \beta > 1, \end{cases} \qquad Q(t) = \min\{q(t), q(\tau(t))\},$$

$$\varphi'_+(t) = \max\{0, \varphi'(t)\}, \qquad A(t) = \int_{t_0}^{t} a^{-1/\lambda}(s)ds,$$

where the meaning of φ will be explained later.

Theorem 3.1. *Assume that conditions* (H_1)-(H_3), (1.2), *and* (1.4) *are satisfied. If there exists a function* $\varphi \in C^1([t_0, +\infty), (0, +\infty))$ *such that, for all constants* $\eta > 0$,

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \varphi(s) \left[L_0 Q(s) - \left(1 + \frac{p_0^\beta}{\tau_0}\right) \frac{(\lambda/\beta)^\lambda a(\delta(s))}{(\lambda+1)^{\lambda+1}\eta^{\beta-\lambda}(\delta'(s))^\lambda} \left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\lambda+1} \right] ds = +\infty, \quad \lambda \leqslant \beta \tag{3.1}$$

or

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \varphi(s) \left[L_0 Q(s) - \left(1 + \frac{p_0^\beta}{\tau_0}\right) \frac{\eta^{(\lambda-\beta)/\lambda} a^{\beta/\lambda}(\delta(s))}{(\beta+1)^{\beta+1}(\delta'(s))^\beta} \left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\beta+1} \right] ds = +\infty, \quad \lambda > \beta, \tag{3.2}$$

then equation (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a $t_1 \geqslant t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for $t \geqslant t_1$. Then, using the definition of z, we have $z(t) > 0$ and $z(t) \geqslant x(t)$, $t \geqslant t_1$. From (1.1), we have

$$[a(t)\phi_1(z'(t))]' \leqslant -Lq(t)x^\beta(\delta(t)) \leqslant 0. \tag{3.3}$$

By (1.2) and (3.3), one can easily obtain $z'(t) > 0$. It follows from (3.3) that

$$\frac{[a(\tau(t))\phi_1(z'(\tau(t)))]'}{\tau'(t)} + Lq(\tau(t))x^\beta(\delta(\tau(t))) \leqslant 0. \tag{3.4}$$

Combining (3.3) and (3.4), we conclude that

$$[a(t)\phi_1(z'(t))]' + p_0^\beta \frac{[a(\tau(t))\phi_1(z'(\tau(t)))]'}{\tau'(t)} + Lq(t)x^\beta(\delta(t)) + p_0^\beta Lq(\tau(t))x^\beta(\delta(\tau(t))) \leqslant 0. \tag{3.5}$$

Assume first that $0 < \beta \leqslant 1$. Applications of conditions $\tau'(t) \geqslant \tau_0 > 0$, $\tau \circ \delta = \delta \circ \tau$, $z(t) \leqslant x(t) + p_0 x(\tau(t))$, (3.5), and Lemma 2.1 yield

$$\begin{aligned} [a(t)\phi_1(z'(t))]' + \frac{p_0^\beta}{\tau_0}[a(\tau(t))\phi_1(z'(\tau(t)))]' &\leqslant -Lq(t)x^\beta(\delta(t)) - p_0^\beta Lq(\tau(t))x^\beta(\delta(\tau(t))) \\ &\leqslant -LQ(t)\left[x^\beta(\delta(t)) + p_0^\beta x^\beta(\delta(\tau(t)))\right] \\ &\leqslant -LQ(t)\left[x(\delta(t)) + p_0 x(\tau(\delta(t)))\right]^\beta \\ &\leqslant -LQ(t)z^\beta(\delta(t)). \end{aligned} \tag{3.6}$$

Suppose now that $\beta > 1$. Similarly, in view of Lemma 2.2, we have

$$\begin{aligned} [a(t)\phi_1(z'(t))]' + \frac{p_0^\beta}{\tau_0}[a(\tau(t))\phi_1(z'(\tau(t)))]' &\leqslant -LQ(t)\left[x^\beta(\delta(t)) + p_0^\beta x^\beta(\delta(\tau(t)))\right] \\ &\leqslant -2^{1-\beta}LQ(t)\left[x(\delta(t)) + p_0 x(\tau(\delta(t)))\right]^\beta \\ &\leqslant -2^{1-\beta}LQ(t)z^\beta(\delta(t)). \end{aligned} \tag{3.7}$$

It follows now from (3.6) and (3.7) that

$$[a(t)\phi_1(z'(t))]' + \frac{p_0^\beta}{\tau_0}[a(\tau(t))\phi_1(z'(\tau(t)))]' \leqslant -L_0 Q(t)z^\beta(\delta(t)). \tag{3.8}$$

Case (i): $\lambda \leqslant \beta$. For $t \geqslant t_1$, we define a Riccati substitution

$$w(t) = \varphi(t)\frac{a(t)\phi_1(z'(t))}{\phi_2(z(\delta(t)))} = \varphi(t)\frac{a(t)(z'(t))^\lambda}{z^\beta(\delta(t))}. \tag{3.9}$$

Then $w(t) > 0$ for $t \geqslant t_1$. Differentiating (3.9), we get

$$\begin{aligned} w'(t) &= \varphi'(t)\frac{a(t)(z'(t))^\lambda}{z^\beta(\delta(t))} + \varphi(t)\frac{[a(t)(z'(t))^\lambda]'z^\beta(\delta(t)) - \beta a(t)(z'(t))^\lambda z^{\beta-1}(\delta(t))z'(\delta(t))\delta'(t)}{z^{2\beta}(\delta(t))} \\ &= \frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]' + \frac{\varphi'(t)}{\varphi(t)}w(t) - \beta\frac{\varphi(t)a(t)\delta'(t)(z'(t))^\lambda z'(\delta(t))}{z^{\beta+1}(\delta(t))}. \end{aligned} \tag{3.10}$$

From (3.3) and $z'(t) > 0$, we have $a(\delta(t))(z'(\delta(t)))^\lambda \geqslant a(t)(z'(t))^\lambda$, i.e.,

$$z'(\delta(t)) \geqslant \left(\frac{a(t)}{a(\delta(t))}\right)^{1/\lambda} z'(t). \tag{3.11}$$

Since $z(t) > 0$ and $z'(t) > 0$, there exists a constant $\eta_1 > 0$ such that $z(t) \geqslant z(\delta(t)) \geqslant \eta_1$. Combining (3.9)-(3.11) and using Lemma 2.3, we obtain

$$\begin{aligned} w'(t) &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]' + \frac{\varphi'(t)}{\varphi(t)}w(t) - \beta\frac{\varphi(t)a(t)\delta'(t)(z'(t))^\lambda}{z^{\beta+1}(\delta(t))}\left(\frac{a(t)}{a(\delta(t))}\right)^{1/\lambda} z'(t) \\ &= \frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]' + \frac{\varphi'(t)}{\varphi(t)}w(t) - \beta\frac{\delta'(t)z^{(\beta-\lambda)/\lambda}(\delta(t))}{[\varphi(t)a(\delta(t))]^{1/\lambda}}w^{(\lambda+1)/\lambda}(t) \\ &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]' + \frac{\varphi'_+(t)}{\varphi(t)}w(t) - \beta\frac{\eta_1^{(\beta-\lambda)/\lambda}\delta'(t)}{[\varphi(t)a(\delta(t))]^{1/\lambda}}w^{(\lambda+1)/\lambda}(t) \\ &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]' + \frac{\lambda^\lambda\varphi(t)a(\delta(t))}{(\lambda+1)^{\lambda+1}\beta^\lambda\eta_1^{\beta-\lambda}(\delta'(t))^\lambda}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\lambda+1}. \end{aligned} \tag{3.12}$$

Similarly, for $t \geqslant t_1$, we introduce another Riccati transformation

$$v(t) = \varphi(t)\frac{a(\tau(t))\phi_1(z'(\tau(t)))}{\phi_2(z(\delta(t)))} = \varphi(t)\frac{a(\tau(t))(z'(\tau(t)))^\lambda}{z^\beta(\delta(t))}. \tag{3.13}$$

Then $v(t) > 0$ for $t \geqslant t_1$. Differentiating (3.13), by $a(\delta(t))(z'(\delta(t)))^\lambda \geqslant a(\tau(t))(z'(\tau(t)))^\lambda > 0$ and $z(\delta(t)) \geqslant \eta_1$, we get

$$\begin{aligned} v'(t) &= \varphi'(t)\frac{a(\tau(t))(z'(\tau(t)))^\lambda}{z^\beta(\delta(t))} \\ &\quad + \varphi(t)\frac{[a(\tau(t))(z'(\tau(t)))^\lambda]'z^\beta(\delta(t)) - \beta a(\tau(t))(z'(\tau(t)))^\lambda z^{\beta-1}(\delta(t))z'(\delta(t))\delta'(t)}{z^{2\beta}(\delta(t))} \\ &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}[a(\tau(t))(z'(\tau(t)))^\lambda]' + \frac{\varphi'(t)}{\varphi(t)}v(t) \\ &\quad - \beta\frac{\varphi(t)a(\tau(t))(z'(\tau(t)))^\lambda\delta'(t)}{z^{\beta+1}(\delta(t))}\left(\frac{a(\tau(t))}{a(\delta(t))}\right)^{1/\lambda} z'(\tau(t)) \\ &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}[a(\tau(t))(z'(\tau(t)))^\lambda]' + \frac{\varphi'_+(t)}{\varphi(t)}v(t) - \beta\frac{\eta_1^{(\beta-\lambda)/\lambda}\delta'(t)}{[\varphi(t)a(\delta(t))]^{1/\lambda}}v^{(\lambda+1)/\lambda}(t) \\ &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}[a(\tau(t))(z'(\tau(t)))^\lambda]' + \frac{\lambda^\lambda\varphi(t)a(\delta(t))}{(\lambda+1)^{\lambda+1}\beta^\lambda\eta_1^{\beta-\lambda}(\delta'(t))^\lambda}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\lambda+1}. \end{aligned} \tag{3.14}$$

By virtue of (3.8), (3.12), and (3.14), we deduce that

$$\begin{aligned}
w'(t)+\frac{p_0^\beta}{\tau_0}v'(t) &\leqslant \frac{\varphi(t)}{z^\beta(\delta(t))}\left([a(t)(z'(t))^\lambda]'+\frac{p_0^\beta}{\tau_0}[a(\tau(t))(z'(\tau(t)))^\lambda]'\right)\\
&\quad+\frac{\left(1+\frac{p_0^\beta}{\tau_0}\right)\lambda^\lambda\varphi(t)a(\delta(t))}{(\lambda+1)^{\lambda+1}\beta^\lambda\eta_1^{\beta-\lambda}(\delta'(t))^\lambda}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\lambda+1}\\
&\leqslant -L_0\varphi(t)Q(t)+\frac{\left(1+\frac{p_0^\beta}{\tau_0}\right)\lambda^\lambda\varphi(t)a(\delta(t))}{(\lambda+1)^{\lambda+1}\beta^\lambda\eta_1^{\beta-\lambda}(\delta'(t))^\lambda}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\lambda+1}.
\end{aligned}$$

Integrating the latter inequality from t_1 to t, we see that

$$\int_{t_1}^t\varphi(s)\left[L_0Q(s)-\left(1+\frac{p_0^\beta}{\tau_0}\right)\frac{(\lambda/\beta)^\lambda a(\delta(s))}{(\lambda+1)^{\lambda+1}\eta_1^{\beta-\lambda}(\delta'(s))^\lambda}\left(\frac{\varphi'_+(s)}{\varphi(s)}\right)^{\lambda+1}\right]ds\leqslant w(t_1)+\frac{p_0^\beta}{\tau_0}v(t_1),$$

which contradicts our assumption (3.1).

Case (ii): $\lambda>\beta$. Define the function w by (3.9). Then (3.10) holds. Using (3.3) and $z'(t)>0$, we have (3.11). Furthermore, there exists a constant $\eta_2>0$ such that $a(t)(z'(t))^\lambda\leqslant a(\tau(t))(z'(\tau(t)))^\lambda\leqslant\eta_2$, i.e.,

$$\frac{1}{(z'(t))^{(\lambda-\beta)/\beta}}\geqslant\frac{a^{(\lambda-\beta)/(\beta\lambda)}(t)}{\eta_2^{(\lambda-\beta)/(\beta\lambda)}}\quad\text{and}\quad\frac{1}{(z'(\tau(t)))^{(\lambda-\beta)/\beta}}\geqslant\frac{a^{(\lambda-\beta)/(\beta\lambda)}(\tau(t))}{\eta_2^{(\lambda-\beta)/(\beta\lambda)}}.\tag{3.15}$$

By virtue of (3.9)-(3.11) and (3.15), we conclude that

$$\begin{aligned}
w'(t)&\leqslant\frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]'+\frac{\varphi'(t)}{\varphi(t)}w(t)-\beta\frac{\varphi(t)a(t)\delta'(t)(z'(t))^\lambda}{z^{\beta+1}(\delta(t))}\left(\frac{a(t)}{a(\delta(t))}\right)^{1/\lambda}z'(t)\\
&\leqslant\frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]'+\frac{\varphi'_+(t)}{\varphi(t)}w(t)-\beta\frac{\delta'(t)(w(t))^{(\beta+1)/\beta}}{\varphi^{1/\beta}(t)a^{1/\lambda}(\delta(t))a^{(\lambda-\beta)/(\beta\lambda)}(t)(z'(t))^{(\lambda-\beta)/\beta}}\\
&\leqslant\frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]'+\frac{\varphi'_+(t)}{\varphi(t)}w(t)-\beta\frac{\delta'(t)(w(t))^{(\beta+1)/\beta}}{\varphi^{1/\beta}(t)a^{1/\lambda}(\delta(t))\eta_2^{(\lambda-\beta)/(\beta\lambda)}}\\
&\leqslant\frac{\varphi(t)}{z^\beta(\delta(t))}[a(t)(z'(t))^\lambda]'+\frac{\eta_2^{(\lambda-\beta)/\lambda}\varphi(t)a^{\beta/\lambda}(\delta(t))}{(\beta+1)^{\beta+1}(\delta'(t))^\beta}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\beta+1}.
\end{aligned}\tag{3.16}$$

On the other hand, define the function v as in (3.13). Similarly, we have

$$v'(t)\leqslant\frac{\varphi(t)}{z^\beta(\delta(t))}[a(\tau(t))(z'(\tau(t)))^\lambda]'+\frac{\eta_2^{(\lambda-\beta)/\lambda}\varphi(t)a^{\beta/\lambda}(\delta(t))}{(\beta+1)^{\beta+1}(\delta'(t))^\beta}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\beta+1}.\tag{3.17}$$

Applications of (3.8), (3.16), and (3.17) imply that

$$\begin{aligned}
w'(t)+\frac{p_0^\beta}{\tau_0}v'(t)&\leqslant\frac{\varphi(t)}{z^\beta(\delta(t))}\left([a(t)(z'(t))^\lambda]'+\frac{p_0^\beta}{\tau_0}[a(\tau(t))(z'(\tau(t)))^\lambda]'\right)\\
&\quad+\left(1+\frac{p_0^\beta}{\tau_0}\right)\frac{\eta_2^{(\lambda-\beta)/\lambda}\varphi(t)a^{\beta/\lambda}(\delta(t))}{(\beta+1)^{\beta+1}(\delta'(t))^\beta}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\beta+1}\\
&\leqslant -L_0\varphi(t)Q(t)+\left(1+\frac{p_0^\beta}{\tau_0}\right)\frac{\eta_2^{(\lambda-\beta)/\lambda}\varphi(t)a^{\beta/\lambda}(\delta(t))}{(\beta+1)^{\beta+1}(\delta'(t))^\beta}\left(\frac{\varphi'_+(t)}{\varphi(t)}\right)^{\beta+1}.
\end{aligned}$$

Consequently,

$$\int_{t_1}^{t}\varphi(s)\left[L_0Q(s)-\left(1+\frac{p_0^{\beta}}{\tau_0}\right)\frac{\eta_2^{(\lambda-\beta)/\lambda}a^{\beta/\lambda}(\delta(s))}{(\beta+1)^{\beta+1}(\delta'(s))^{\beta}}\left(\frac{\varphi_+'(s)}{\varphi(s)}\right)^{\beta+1}\right]ds\leqslant w(t_1)+\frac{p_0^{\beta}}{\tau_0}v(t_1),$$

which contradicts our assumption (3.2). The proof is complete. □

Choosing $\varphi(t)=A(\delta(t))$, by Theorem 3.1, we have the following results.

Corollary 3.2. *Let $\beta=\lambda$. Assume that conditions (H_1)-(H_3), (1.2), and (1.4) are satisfied. If*

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\left[L_0A(\delta(s))Q(s)-\frac{(1+p_0^{\lambda}/\tau_0)\delta'(s)}{(\lambda+1)^{\lambda+1}A^{\lambda}(\delta(s))a^{1/\lambda}(\delta(s))}\right]ds=+\infty,\tag{3.18}$$

then equation (1.1) is oscillatory.

Corollary 3.3. *Let $\beta=\lambda\leqslant 1$. Assume that conditions (H_1)-(H_3), (1.2), and (1.4) are satisfied. If*

$$\liminf_{t\to+\infty}\frac{1}{A^{1-\lambda}(\delta(t))}\int_{t_0}^{t}A(\delta(s))Q(s)ds>\frac{1+p_0^{\lambda}/\tau_0}{L(\lambda+1)^{\lambda+1}(1-\lambda)},\quad \lambda<1\tag{3.19}$$

or

$$\liminf_{t\to+\infty}\frac{1}{\ln A(\delta(t))}\int_{t_0}^{t}A(\delta(s))Q(s)ds>\frac{1+p_0/\tau_0}{4L},\quad \lambda=1,\tag{3.20}$$

then equation (1.1) is oscillatory.

Proof.

Case (i): $\lambda<1$. Condition (3.19) yields existence of a constant $\varepsilon>0$ such that

$$\frac{1}{A^{1-\lambda}(\delta(t))}\int_{t_0}^{t}A(\delta(s))Q(s)ds\geqslant\frac{1+p_0^{\lambda}/\tau_0}{L(\lambda+1)^{\lambda+1}(1-\lambda)}+\varepsilon,$$

that is,

$$\int_{t_0}^{t}A(\delta(s))Q(s)ds-\frac{1+p_0^{\lambda}/\tau_0}{L(\lambda+1)^{\lambda+1}(1-\lambda)}A^{1-\lambda}(\delta(t))\geqslant\varepsilon A^{1-\lambda}(\delta(t)),$$

which implies that

$$\begin{aligned}&\int_{t_0}^{t}\left[A(\delta(s))Q(s)-\frac{(1+p_0^{\lambda}/\tau_0)\delta'(s)}{L(\lambda+1)^{\lambda+1}A^{\lambda}(\delta(s))a^{1/\lambda}(\delta(s))}\right]ds\\&=\int_{t_0}^{t}A(\delta(s))Q(s)ds-\frac{1+p_0^{\lambda}/\tau_0}{L(\lambda+1)^{\lambda+1}(1-\lambda)}A^{1-\lambda}(\delta(t))+\frac{1+p_0^{\lambda}/\tau_0}{L(\lambda+1)^{\lambda+1}(1-\lambda)}A^{1-\lambda}(\delta(t_0))\\&\geqslant\varepsilon A^{1-\lambda}(\delta(t))+\frac{1+p_0^{\lambda}/\tau_0}{L(\lambda+1)^{\lambda+1}(1-\lambda)}A^{1-\lambda}(\delta(t_0))\to+\infty\quad\text{as } t\to+\infty.\end{aligned}$$

An application of Corollary 3.2 yields oscillation of equation (1.1).

Case (ii): $\lambda=1$. Note that $[\ln A(\delta(s))]'=\delta'(s)/(A(\delta(s))a(\delta(s)))$. Similar as in the proof of case (i), an application of condition (3.20) implies that (3.18) is satisfied. By Corollary 3.2, equation (1.1) is oscillatory. This completes the proof. □

Remark 3.4. Define the Riccati substitutions

$$w(t)=\varphi(t)\frac{a(t)\phi_1(z'(t))}{\phi_2(z(\tau(t)))}\quad\text{and}\quad v(t)=\varphi(t)\frac{a(\tau(t))\phi_1(z'(\tau(t)))}{\phi_2(z(\tau(t)))}.$$

Proceeding similarly as above, one can easily establish oscillation results for equation (1.1) in the case when $\delta(t)\geqslant\tau(t)$.

Remark 3.5. The results obtained in this paper complement those reported in [6, 8, 11, 13, 21, 26, 28] since our criteria can be applied to the case where $\beta \neq \lambda$ and do not require assumptions (1.3), (1.5), and (1.6). Furthermore, our criteria solve a problem posed in [7, Summary] because these results can be applied to equation (1.1) for $\beta > \lambda$.

4. Example

Example 4.1. For $t \geqslant 1$ and $\alpha > 0$, consider the second-order neutral delay differential equation

$$\left(x(t) + \frac{9}{10}x\left(\frac{t}{4}\right)\right)'' + \frac{\alpha}{t^2}x\left(\frac{t}{5}\right) = 0. \tag{4.1}$$

Let $\beta = \lambda = 1$, $a(t) = 1$, $p(t) = p_0 = 9/10$, $\tau(t) = t/4$, $q(t) = \alpha/t^2$, $\delta(t) = t/5$, $f(u) = u$, $L = L_0 = 1$, and $\varphi(t) = t$. Then $\tau_0 = 1/4$, $Q(t) = \min\{q(t), q(\tau(t))\} = \alpha/t^2$, and so

$$\begin{aligned}
&\limsup_{t\to+\infty} \int_{t_0}^{t} \varphi(s)\left[L_0 Q(s) - \left(1 + \frac{p_0^\beta}{\tau_0}\right)\frac{(\lambda/\beta)^\lambda a(\delta(s))}{(\lambda+1)^{\lambda+1}\eta^{\beta-\lambda}(\delta'(s))^\lambda}\left(\frac{\varphi_+'(s)}{\varphi(s)}\right)^{\lambda+1}\right] ds \\
&= \left(\alpha - \frac{23}{4}\right)\limsup_{t\to+\infty}\int_1^t \frac{1}{s} ds = +\infty,
\end{aligned}$$

provided that $\alpha > 23/4 = 5.75$. Hence, by Theorem 3.1, equation (4.1) is oscillatory for any $\alpha > 5.75$, whereas applications of [7, Corollary 2] and [25, Theorem 2.1] (let $\rho(t) = t$) yield oscillation of (4.1) for any $\alpha > 23/(e \ln 1.25) \approx 37.92$ and $\alpha > 12.5$, respectively. Hence, Theorem 3.1 improves [7, Corollary 2] and [25, Theorem 2.1] in some cases.

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